The Annals of Probability
1999, Vol. 26, No. 4, 1575–1613

WAVEFRONT PROPAGATION FOR REACTION–DIFFUSION SYSTEMS AND BACKWARD SDES

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We first show a large deviation principle for degenerate diffusion–transmutation processes and study the Riemannian metric associated with the action functional under a Hörmander-type assumption. Then we study the behavior of the solution u^ε of a system of strongly coupled scaled KPP equations. Using backward stochastic differential equations and the theory of Hamilton–Jacobi equations, we show that, when the parabolic operator satisfies a Hörmander-type hypothesis or when the nonlinearity depends on the gradient, the wavefront location is given by the same formula as that in Freidlin and Lee or Barles, Evans and Souganidis. We obtain the exact logarithmic rates of convergence to the unstable equilibrium state in the general case and to the stable equilibrium state when the equations are uniformly positively coupled.

1. Introduction. Many probabilistic methods have been developed to study parabolic partial differential equations (PDEs) since we know that such equations are connected to Markov processes. Reaction–diffusion equations and, in particular, KPP equations [18] have been extensively studied in that way during the last few years: [8], [9], [12], [13], [21], [22] and [25] for example. The original KPP equation is

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}\Delta u(t,x)\\ &\qquad + c(x)u(t,x)(1-u(t,x)), \qquad t>0,\ x\in\mathbb{R},\\ u(0,x) &= \mathbb{1}_{x<0}, \qquad x\in\mathbb{R}.\end{aligned}$$

It is well known that u looks like a running wave when t and x are far from the origin. This type of result was extended by Freidlin [13], using large deviations, and by Evans and Souganidis [10], developing an analytical method, to the nonhomogeneous case of scaled KPP equations

$$\begin{aligned}\frac{\partial u^\varepsilon}{\partial t}(t,x) &= \varepsilon L u^\varepsilon(t,x)\\ &\qquad + \frac{1}{\varepsilon}c(x)u^\varepsilon(t,x)(1-u^\varepsilon(t,x)), \qquad t>0,\ x\in\mathbb{R}^d,\\ u^\varepsilon(0,x) &= g(x), \qquad x\in\mathbb{R}^d,\end{aligned}$$

Received May 1996; revised February 1998.

AMS 1991 *subject classifications*. 35K57, 35K65, 49L25, 60F10, 60H30.

Key words and phrases. Large deviations, degenerate diffusion–transmutation process, sub-Riemannian metric, backward stochastic differential equations, systems of reaction–diffusion equations, Hamilton–Jacobi equations, viscosity solutions.

where L is a second-order uniform elliptic operator, $0 < \underline{c} \le c(x) \le \bar{c} < \infty$ and g is a bounded positive function.

Barles, Evans and Souganidis [5] on one side and Freidlin and Lee [14, 16] on the other one gave generalizations of their results for systems of strongly coupled KPP equations, typically:

$$
(1)\qquad
\begin{aligned}
\frac{\partial u_l^\varepsilon}{\partial t} &= \varepsilon L_l u_l^\varepsilon + \frac{1}{\varepsilon}\Bigg[c_u(x) u_l^\varepsilon (e_l - u_l^\varepsilon) \\
&\qquad\qquad + \sum_{i \neq l} c_{li}(x)(u_i^\varepsilon - u_l^\varepsilon)\Bigg], \qquad t > 0,\ x \in \mathbb{R}^d,\\
u_l^\varepsilon(0, x) &= g_l(x), \qquad 1 \le l \le k,\ x \in \mathbb{R}^d,
\end{aligned}
$$

where L_l is an elliptic operator, $0 < \underline{c} \le c_{li}(x) \le \bar{c} < \infty$, g_l is a bounded positive function and $e_l > 0$. One can also cite the work of Zhao [29].

There is wavefront propagation with the same speed on each component of u^ε and the exponential rate of convergence to the unstable equilibrium state is computed. In fact, for all components of u^ε, $\varepsilon \ln u_l^\varepsilon$ converges uniformly on compact sets to V^*. Moreover, if $E = \{V^* < 0\}$ and $M = \{V^* = 0\}$, u^ε converges uniformly to 0 on compact subsets of E, and $\liminf u_l^\varepsilon > 0$ uniformly on compact subsets of $\mathring{M}$ for all $1 \le l \le k$ which means that the wavefront is located on $\partial M = \partial E$.

In [23] Pardoux and Peng show that backward stochastic differential equations (BSDEs) driven by a Brownian motion provide a representation of the viscosity solution of semilinear parabolic PDEs. We show in [25] that it allows us to consider the hypoelliptic and gradient dependent [i.e., $c(x, \nabla u^\varepsilon)$] cases. More recently, with Pardoux and Rao [24], we give a link between BSDEs driven by a Brownian motion and Poisson processes and the viscosity solution of a system of semilinear parabolic PDEs. Let W be a Brownian motion and let $N(l)$, $1 \le l \le k-1$, be independent Poisson processes and independent of W. Let $(\nu^n, X^{x,n}, Y^{t,x,n}, H^{t,x,n}, Z^{t,x,n})$ be the solution of

$$
(1)\qquad
\begin{aligned}
\nu_t^n &= n + \sum_{l=1}^{k-1} l N(l)_t \qquad \operatorname{mod}[k],\\
X_s^{x,n} &= x + \int_0^s b(X_r^{x,n}, \nu_r^n)\, dr + \int_0^s \sigma(X_r^{x,n}, \nu_r^n)\, dWr,\\
Y_s^{t,x,n} &= g(X_t^{x,n}, \nu_t^n) + \int_s^t f\big(\nu_r^n, X_r^{x,n}, Y_r^{t,x,n}, H_r^{t,x,n}, Z_r^{t,x,n}\big)\, dr\\
&\qquad - \int_s^t Z_r^{t,x,n}\, dWr - \int_s^t \sum_{l=1}^{k-1} H_r^{t,x,n}(l)\, dN(l)_r.
\end{aligned}
$$

Then, on assumptions recalled later, $u(t, x) = (Y_0^{t,x,l}, 1 \le l \le k)$ is the unique viscosity solution of

$$
\begin{aligned}
\frac{\partial u_l}{\partial t} &= L_l u_l + f\big(l, x, u_l, u_{l+1} - u_l, \dots, u_{l-1} - u_l, \nabla u_l \sigma(x, l)\big),\\
&\qquad\qquad\qquad\qquad t > 0,\ x \in \mathbb{R}^d,\\
u_l(0, x) &= g(x, l), \qquad 1 \le l \le k,\ x \in \mathbb{R}^d,
\end{aligned}
$$

where L_l is the operator associated with $(b(\cdot, l), \sigma(\cdot, l))$. As in [25], we use this representation to solve the classical problem under the hypoelliptic hypothesis or when $(c_{l,i})_i$ depends on ∇u_l^ε. The case where $(c_{l,i})_i$ depends on ∇u_m^ε with $m \neq l$ is not studied since there exists no definition for viscosity solutions of such systems for the moment. Our approach is the following. We show that $(\varepsilon \ln u_l^\varepsilon)_l$ converges uniformly on compact sets of $]0, \infty[\times \mathbb{R}^d$ by technics developed in [5]. Then, using the BSDE representation and the probabilistic method [16], we identify the limit and show the wavefront propagation. Moreover, we give an example where the convergence to the stable equilibrium state can be proved and we compute the exponential rate of convergence to this state. However, we need a large deviation principle for diffusion−transmutation processes with degenerated diffusion. We prove it in Section 2. Then, Section 3 is dedicated to the study of the hypoelliptic case and the "stabilization" for some particular systems and Section 4 is dedicated to systems which are nonlinear in ∇u^ε.

2. A theorem of large deviations. The aim of this chapter is to show a large deviation principle for a diffusion−transmutation process $(X_s^\varepsilon, \nu_s^\varepsilon)_{0 \le s \le T}$ of $\mathbb{R}^d \times [\![1, k]\!]$ without nondegeneracy assumption on the diffusion. The generator of this process is given by: for all $v \in (C^{1,2}([0, T] \times \mathbb{R}^d))^k$,

$$\begin{aligned}\frac{\partial v_l}{\partial t} &= \frac{\varepsilon}{2} \sum_{i,j=1}^d a^{ij}(x, l) \frac{\partial^2 v_l}{\partial x^i \partial x^j}\\ &\quad + (b_\varepsilon(x, l), \nabla v_l) + \frac{1}{\varepsilon} \sum_{i=1}^k c(x, l, i)(v_i - v_l)\end{aligned}$$

for all $l \in [\![1, k]\!] = \{l, 1 \le l \le k\}$, $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$ with $v_n = v_l$ if $n = l \bmod[k]$. In fact, following the idea of Freidlin and Lee [15], we replace ν_t^ε by its occupation-time vector U which is defined by

$$U_t^\varepsilon = (U_t^\varepsilon(l))_{1 \le l \le k} = \left(\int_0^t \mathbb{1}_l(\nu_s^\varepsilon)\, ds\right)_{1 \le l \le k} \tag{2}$$

since, when c is independent of x, ν^ε is a speeded-up Markov process and so, in general, ν_t^ε has no limit when ε tends to 0. Such a process can be built as follows: let (Ω, F, P) be a probability space and $(W^l, 1 \le l \le k, N(i), 1 \le i \le k-1)$ be $2k - 1$ independent processes defined on (Ω, F, P) where W^l is a d-dimensional Brownian motion and $N(i)$ is a standard Poisson process. Let (X^ε, ν) be the unique solution of

$$\nu_t^n = n + \sum_{l=1}^{k-1} l N(l)_t \qquad \bmod[k],$$

$$\begin{aligned}X_s^{\varepsilon,x,n} &= x + \sum_{l=1}^k \int_0^s b_\varepsilon(X_r^{\varepsilon,x,n}, l) \dot{U}_r^{\varepsilon,x,n}(l)\, dr\\ &\quad + \sqrt{\varepsilon} \int_0^s \sum_{l=1}^k \sigma(X_r^{\varepsilon,x,n}, l) \dot{U}_r(l)\, dW_r^l.\end{aligned} \tag{3}$$

where $\dot{U}_r(l) = \mathbb{1}_{\{\nu_r^n = l\}}$. Now, we define the probability P^ε by

$$\frac{dP^\varepsilon}{dP}(X^\varepsilon, \nu) = \exp\left\{\int_0^T \left[k - 1 - \frac{\hat{c}(X_s^\varepsilon, \nu_s)}{\varepsilon}\right] ds + \sum_{l,\, m} \sum_{\tau \in \tau_{l,\, m}} \ln \frac{c(X_\tau^\varepsilon, l, m)}{\varepsilon}\right\},$$

where $\tau_{l,\, m}$ is the set of times lower than T when ν jumps from l to $l + m \operatorname{mod}[k]$. On our assumptions, P^ε and P are equivalent and, under P^ε, (X^ε, ν) is the required process and we will work with the triplet $(P^\varepsilon, X^\varepsilon, U)$. Using this notation, $X^{\varepsilon,\, x,\, n}$ is the solution of

$$X_s^{\varepsilon,\, x,\, n} = x + \sum_{l=1}^{k} \left(\int_0^s b_\varepsilon(X_r^{\varepsilon,\, x,\, n}, l) \dot{U}_r^{\varepsilon,\, x,\, n}(l)\, dr \right.$$
$$\left. + \sqrt{\varepsilon} \int_0^s \sigma(X_r^{\varepsilon,\, x}, l) \dot{U}_r^{\varepsilon,\, x,\, n}(l)\, dW_r^\varepsilon \right).$$

In fact, the large deviation theorem we prove deals with $(P^\varepsilon, X^\varepsilon, U)_\varepsilon$ as was done by Freidlin and Lee [15] on a uniform elliptic assumption.

2.1. *Statement of the main theorem.* Our assumptions are of two types: each transition of ν^ε is always possible but with finite intensity and, for all l, the diffusion family associated with $(b_\varepsilon(\cdot, l), \sqrt{\varepsilon}\, \sigma(\cdot, l))$ satisfies a uniform large deviation principle. In other words:

ASSUMPTION 2.1. For all $(l, m) \in [\![1, k]\!] \times [\![1, k]\!]$, $(b_\varepsilon(\cdot, l))_{0 < \varepsilon} \subset C(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma(\cdot, l) \in C(\mathbb{R}^d, \mathbb{R}^{d \times d})$, $c(\cdot, l, m) \in C(\mathbb{R}^d, \mathbb{R})$ and

(a) there exists $K \in]0, \infty[$ such that for all $x, x' \in \mathbb{R}^d \times \mathbb{R}^d$, for all $l, m \in [\![1, k]\!]$, for all $\varepsilon > 0$:

$$\begin{aligned} |b_\varepsilon(x, l) - b_\varepsilon(x, l')| &\le K|x - x'|, \qquad |b_\varepsilon(x, l)| \le K, \\ \|\sigma(x, l) - \sigma(x', l)\| &\le K|x - x'|, \qquad \|\sigma(x, l)\| \le K, \\ |c(x, l, m) - c(x', l, m)| &\le K|x - x'|, \qquad \frac{1}{K} \le c(x, l, m); \end{aligned}$$

(b) $(b_\varepsilon)_{0 < \varepsilon}$ converges uniformly to b when ε tends to 0; b is uniformly Lipschitz continuous and bounded by K.

Let us now introduce some notation in order to define the action of (X^ε, U):

(a) for all $l, i \in [\![1, k]\!]$, $l \neq i$, set $c_{li}(x) = \hat{c}_{li}(x) = c(x, l, m)$ if $m = l + i \operatorname{mod}[k]$ and $\hat{C}_{ll}(x) = -\hat{c}(x, l) = -\sum_{i \neq l} c_{li}(x)$;

(b) $x, p \in \mathbb{R}^d$, $\alpha \in \mathbb{R}^k$,

$$\Pi(x, p, \alpha) = \hat{C}(x) + \operatorname{Diag}\left(\frac{1}{2}|\sigma^*(x, l)p|^2 + (b(x, l), p) + \alpha_l, 1 \le l \le k\right);$$

(c) $\lambda(x, p, \alpha)$ is the Hamiltonian of $\Pi(x, p, \alpha)$, which means that $\lambda(x, p, \alpha)$ is an eigenvalue and, for any eigenvalue λ, $\operatorname{Re}(\lambda) \le \lambda(x, p, \alpha)$;

(d) $\eta(x, q, \beta)$ is the Legendre transform of $\lambda(x, p, \alpha)$ with respect to (p, α);

(e) $D = \{\beta \in \mathbb{R}^k,\ \beta(l) \ge 0,\ \sum \beta(l) = 1\}$;

(f) $\mu \in C_+$ if $\mu \in C([0, T], [0, T]^k)$ and is absolutely continuous, for all $l \in [\![1, k]\!]$, $\mu(l)$ is nondecreasing with $\mu(l)_0 = 0$ and, for all $t \in [0, T]$, $\sum \mu_t(l) = t$ or, in other words, $\dot{\mu}_t \in D$;

(g) $I = \{\psi,\ \int_0^T |\dot{\psi}_s^l|^2\, ds < \infty\}$ and $I_a = \{\psi,\ \int_0^T |\dot{\psi}_s^l|^2\, ds \le a\}$:

$$S_{0T}^{WU}(\psi, \mu) = \begin{cases} \displaystyle\int_0^T \eta(\varphi_s, 0, \dot{\mu}_s)\, ds + \sum_{l=1}^k \frac{1}{2}\int_0^T |\dot{\psi}_s^l|^2\, ds, & \text{if } \mu \in C_+ \text{ and } \psi \in I^k, \\ +\infty, & \text{otherwise.} \end{cases}$$

Finally, if $z \in \mathbb{R}^d$, F_z: $I^k \times C_+ \to I$ is defined by $F_z(\psi, \mu) = \varphi$ where φ is the unique solution of

$$\varphi_t = z + \int_0^t \sum_{l=1}^k \left[b(\varphi_s, l) + \sigma(\varphi_s, l) \cdot \dot{\psi}^l_{\mu_s(l)} \right] \dot{\mu}_s(l)\, ds. \tag{4}$$

Now we state the theorem.

THEOREM 2.1. *The triplet* $(P^\varepsilon, X^\varepsilon, U)$ *satisfies a uniform large deviation principle on* $\mathbb{R}^d \times [\![1, k]\!]$. *The action functional is* $\varepsilon^{-1} S_{0T}$ *with* $\forall\, (\varphi, \mu) \in C([0, T], \mathbb{R}^d) \times C_+$:

$$\begin{aligned} S_{0T}(\varphi, \mu) &= \inf_\psi \{S_{0T}^{WU}(\psi, \mu), F_x(\psi, \mu) = \varphi\} \\ &= \begin{cases} \displaystyle\int_0^T \eta(\varphi_s, \dot{\varphi}_s, \dot{\mu}_s)\, ds, & \textit{if } \mu \in C_+ \textit{ and } \varphi \in I, \\ +\infty, & \textit{otherwise,} \end{cases} \end{aligned}$$

where $X_0^\varepsilon = x$.

2.2. *The proof.* Our strategy is strongly based on a contraction principle which is the method used by Azencott [2] and Priouret [27]. We first show that $(P^\varepsilon, \sqrt{\varepsilon} W^{\varepsilon, l}, 1 \le l \le k, U)$ satisfies a uniform large deviation principle, the action of which is given by S_{0T}^{WU} and then we deduce Theorem 2.1. To achieve this, we show that X^ε is close to $F_x(\psi, \mu)$ with P^ε-probability greater than $1 - \exp(-R/\varepsilon)$, R being arbitrarily large, when $(\sqrt{\varepsilon} W^l, 1 \le l \le k, U)$ is close to $(\psi^l, 1 \le l \le k, \mu)$. In fact, although we may consider a different drift, we only fix the case $\psi \equiv 0$ thanks to a change of probability. This is the purpose of:

PROPOSITION 2.2. *Let $a \geq 0$. We assume that, for a family $(W^{\varepsilon,l}, 1 \leq l \leq k)$ of Brownian motions under a probability Q^ε and $((\psi^l)_{1 \leq l \leq k}, \mu) \in (I_a)^k \times C_+$, $\varphi = F_z(\psi, \mu)$ and X^ε is the solution of*

$$X_t^\varepsilon = x + \int_0^t \sum_{l=1}^k \dot{U}_s(l)\Big(\big[b_\varepsilon(X_s^\varepsilon, l) + \sigma(X_s^\varepsilon, l) \cdot \dot{\psi}^l_{U_s(l)}\big]\,ds + \sqrt{\varepsilon}\,\sigma(X_s^\varepsilon, l)\,dW_s^{\varepsilon,l}\Big). \tag{5}$$

Then, for all $R > 0$, $\eta > 0$, there exist $\varepsilon_0 > 0$, $\alpha > 0$, $\beta > 0$, $r > 0$ such that, if $\varepsilon \leq \varepsilon_0$ and $|x - z| \leq r$, we have

$$Q^\varepsilon\left[\|X^\varepsilon - \varphi\| \geq \eta,\ \sup_{l \in [\![1,k]\!]} \|\sqrt{\varepsilon}\,W^{\varepsilon,l}\| < \alpha,\ \|U - \mu\| < \beta\right] \leq \exp\left(-\frac{R}{\varepsilon}\right)$$

and ε_0, α, β, r are independent of (ψ, μ) and depend only on a.

PROOF. We will write W for $W^\varepsilon = \sum_{l=1}^k W_U^{\varepsilon,l}$. As Priouret did, we discretize X^ε. Let $n > 0$. Then

$$x_t^{\varepsilon,n} = X_{t_i}^\varepsilon \quad \text{if } t \in \Big[t_i, t_{i+1}\Big) = \left[\frac{iT}{n}, \frac{(i+1)T}{n}\right)$$

for $i \in [\![0, n[\![$ and $x_T^{\varepsilon,n} = X_T^\varepsilon$. We need the following lemmas, the proofs of which are slight generalizations of Priouret's results.

LEMMA 2.3. *For all $R > 0$, $\gamma > 0$, there exist $\varepsilon_0 > 0$ and $n > 0$ such that, for $\varepsilon \leq \varepsilon_0$,*

$$Q^\varepsilon(\|X^\varepsilon - x^{\varepsilon,n}\| > \gamma) \leq \exp\left(-\frac{R}{\varepsilon}\right).$$

Denote $V_t^\varepsilon = \sqrt{\varepsilon}\int_0^t \sigma(X_s^\varepsilon, \nu_s)\,dW_s$. We have:

LEMMA 2.4. *For all $R > 0$, $\gamma > 0$, there exist $\varepsilon_0 > 0$, $\alpha > 0$ such that, if $\varepsilon \leq \varepsilon_0$, then*

$$Q^\varepsilon\left[\|V^\varepsilon\| > \gamma,\ \sup_{l \in [\![1,k]\!]} \|\sqrt{\varepsilon}\,W^{\varepsilon,l}\| < \alpha\right] \leq \exp\left(-\frac{R}{\varepsilon}\right).$$

If we denote

$$\varphi_t^\varepsilon = z + \int_0^t \left[b(\varphi_s^\varepsilon, \nu) + \sum_{l=1}^k \sigma(\varphi_s^\varepsilon, l) \cdot \dot{\psi}^l_{U_s(l)}\right] \dot{U}_s(l)\,ds,$$

we show Proposition 2.2 for X^ε and φ^ε.

LEMMA 2.5. *For all $R > 0$, $\eta > 0$, there exist $\varepsilon_0 > 0$, $\alpha > 0$, $r > 0$ such that, if $\varepsilon \leq \varepsilon_0$ and $|x - z| \leq r$, we have*

$$Q^{\varepsilon}\left[\|X^{\varepsilon} - \varphi^{\varepsilon}\| \geq \eta,\ \sup_{l \in [\![1, k]\!]} \|\sqrt{\varepsilon}\, W^{\varepsilon, l}\| < \alpha\right] \leq \exp\left(-\frac{R}{\varepsilon}\right)$$

and ε_0, α, β and r are independent of $(\psi^l)_{1 \leq l \leq k}$ and only depend on a.

It remains now to compare φ and φ^{ε} with respect to $\|U - \mu\|$.

LEMMA 2.6. *For all $\eta > 0$, there exists $\beta > 0$ such that, for all $\mu \in C_+$, if $\|U - \mu\| < \beta$, then $\|\varphi^{\varepsilon} - \varphi\| \leq \eta$.*

PROOF. We also introduce a discretized version $\varphi^{\varepsilon, n}$ of φ^{ε}:

$$\varphi_t^{\varepsilon, n} = \varphi_{t_i}^{\varepsilon} \quad \text{if } t \in \left[\ t_i, t_{i+1}\right) = \left[\frac{iT}{n}, \frac{(i+1)T}{n}\right)$$

for $i \in [\![0, n[\![$ and $\varphi_T^{\varepsilon, n} = \varphi_T^{\varepsilon}$. As in Lemma 2.3, we have: for all $\gamma > 0$, there exists $n(\gamma, a)$ such that, if $n \geq n(\gamma, a)$, then $\|\varphi^{\varepsilon} - \varphi^{\varepsilon, n}\| < \gamma$. Let $\gamma > 0$ and n be such that $\|\varphi^{\varepsilon} - \varphi^{\varepsilon, n}\| < \gamma$. A short computation leads to

$$|\varphi_t^{\varepsilon} - \varphi_t| \leq K\int_0^t \Psi_s |\varphi_s^{\varepsilon} - \varphi_s|\, ds + 2K\gamma T + 2Kn \sum_{l=1}^{k} \|U(l) - \mu(l)\|$$

$$+ 2kK\gamma\sqrt{aT} + 2Kn \sum_{l=1}^{k} \sqrt{a\|U(l) - \mu(l)\|},$$

where $\Psi_s = 1 + \sum_{l=1}^{k} |\dot{\psi}_{\mu_s(l)}^{l}|\, \dot{\mu}_s(l)$. If $K(a)$ is such that $K\int_0^T \Psi_s\, ds \leq K(a)$, Gronwall's lemma leads to

$$\|\varphi^{\varepsilon} - \varphi\| \leq 2kK\left(n\|U - \mu\| + \gamma T + \gamma\sqrt{aT} + n\sqrt{a\|U - \mu\|}\right)e^{K(a)}.$$

Then we choose

$$\gamma \leq \frac{\eta e^{-K(a)}}{4kK\left(T + \sqrt{Ta}\right)},$$

and $n \geq n(\gamma, a)$. Hence, there exists $\beta(\eta, a)$ such that

$$\|U - \mu\| < \beta(\eta, a) \Rightarrow \|\varphi^{\varepsilon} - \varphi\| \leq \eta. \qquad \square$$

Now, we can prove Proposition 2.2. We only have to choose α according to Lemma 2.5 such that

$$Q^{\varepsilon}\left[\|X^{\varepsilon} - \varphi^{\varepsilon}\| \geq \frac{\eta}{2},\ \sup_{l \in [\![1, k]\!]} \|\sqrt{\varepsilon}\, W^{\varepsilon, l}\| < \alpha\right] \leq \exp\left(-\frac{R}{\varepsilon}\right)$$

and β according to Lemma 2.6 such that $\|\varphi^{\varepsilon} - \varphi\| \leq \eta/2$. Then, we have

$$Q^{\varepsilon}\left[\|X^{\varepsilon} - \varphi\| \geq \eta,\ \sup_{l \in [\![1, k]\!]} \|\sqrt{\varepsilon}\, W^{\varepsilon, l}\| < \alpha,\ \|U - \mu\| < \beta\right] \leq \exp\left(-\frac{R}{\varepsilon}\right). \qquad \square$$

Now, we come back to the initial problem. Let $((\psi^l)_{1\le l\le k},\mu)\in(I_a)^k\times C_+$. We define the probability $Q^{\varphi,\varepsilon}$ by

$$\frac{dQ^{\varphi,\varepsilon}}{dP^\varepsilon}=\prod_{l=1}^k\exp\left\{\frac{1}{\sqrt{\varepsilon}}\int_0^T\dot\psi_s^l\,dW_s^l-\frac{1}{2\varepsilon}\int_0^T|\dot\psi_s^l|^2\,ds\right\}.$$

Under $Q^{\varphi,\varepsilon}$, $\overline{W}^l=W^l-\psi^l/\sqrt{\varepsilon}$ and $W^\varepsilon=\sum_{l=1}^k\overline{W}^l_{U(l)}$ are Brownian motions and X^ε is a solution of

$$X_t^\varepsilon=x+\int_0^t\sum_{l=1}^k\Big[b_\varepsilon(X_s^\varepsilon,l)+\sigma(X_s^\varepsilon,l)\cdot\dot\psi^l_{U_s(l)}\Big]\dot U_s(l)\,ds$$
$$+\sqrt{\varepsilon}\int_0^t\sigma(X_s^\varepsilon,\nu_s)\,dW_s^\varepsilon.$$

THEOREM 2.7. *If φ is defined by* (4), *for all $R>0$, $\rho>0$, there exist $\varepsilon_0>0$, $\alpha>0$, $\beta>0$ and $r>0$ such that, if $\varepsilon\le\varepsilon_0$ and $|x-z|\le r$,*

$$P^\varepsilon\left[\|X^\varepsilon-\varphi\|\ge\rho,\ \sup_{l\in[\![1,k]\!]}\|\sqrt{\varepsilon}W^l-\psi^l\|\le\alpha,\ \|U-\mu\|\le\beta\right]\le\exp\left(-\frac{R}{\varepsilon}\right),$$

and, for a given a, ε_0, α, β, r are independent of (ψ,μ).

PROOF. Denote

$$A=\left\{\|X^\varepsilon-\varphi\|\ge\rho,\ \sup_{l\in[\![1,k]\!]}\|\sqrt{\varepsilon}W^l-\psi^l\|\le\alpha,\ \|U-\mu\|\le\beta\right\}$$

and

$$\xi=\prod_{l=1}^k\exp\left\{-\frac{1}{\sqrt{\varepsilon}}\int_0^T\dot\psi_s^l\,dW_s^l\right\}.$$

The exponential inequality leads to

$$P^\varepsilon\left[\xi>\exp\left(\frac{k\lambda}{\varepsilon}\right)\right]\le\sum_{l=1}^kP^\varepsilon\left[\left|\int_0^T\dot\psi_s^l\,dW_s^l\right|>\frac{\lambda}{\sqrt{\varepsilon}}\right]\le 2k\exp\left(-\frac{\lambda^2}{a\varepsilon}\right).$$

Moreover,

$$P^\varepsilon\left[A,\xi\le\exp\left(\frac{k\lambda}{\varepsilon}\right)\right]\le\exp\left(k\frac{\lambda+a/2}{\varepsilon}\right)Q^{\varphi,\varepsilon}[A].$$

But, under $Q^{\varphi,\varepsilon}$, A is the event we considered in Proposition 2.2. Once we have chosen λ large enough in order that $\lambda^2/2a\ge R$, we only have to apply Proposition 2.2. □

Let us now assume that we are under a probability Q^ε such that the jump intensities are only time dependent which means that $c(x,l,m)$, $\hat C(x)$ and $\Pi(x,p,\alpha)$ become $c(t,l,m)$, $\hat C(t)$ and $\Pi(t,p,\alpha)$. For example,

$$\frac{dQ^\varepsilon}{dP}=\exp\left\{\int_0^T\left[k-1-\frac{\hat c(s,\nu_s)}{\varepsilon}\right]ds+\sum_{l,m}\sum_{\tau\in\tau_{l,m}}\ln\frac{c(\tau,l,m)}{\varepsilon}\right\},$$

where $\tau_{l,m}$ is the set of times T when ν jumps from l to $l+m$ mod$[k]$. Then, $(W^{\varepsilon,l})_{1\le l\le k}$ is still a family of independent Brownian motions and independent of U. Set $\lambda(t,\alpha)\equiv\lambda(t,0,\alpha)$ and $\eta(t,\beta)$ its Legendre transform with respect to α.

$$S^1_{0T}(\psi,\mu)=\begin{cases}\displaystyle\int_0^T\eta(s,\dot\mu_s)\,ds+\sum_{l=1}^k\frac12\int_0^T|\dot\psi^l_s|^2\,ds, & \text{if } \mu\in C_+ \text{ and } \psi\in I^k,\\ +\infty, & \text{otherwise.}\end{cases}$$

PROPOSITION 2.8. *The family $(Q^\varepsilon,(\sqrt{\varepsilon}W^l)_{1\le l\le k},U)$ satisfies a uniform large deviation principle in the uniform topology. The action functional is $\varepsilon^{-1}S^1_{0T}$.*

PROOF. It is a direct consequence of the Schilder theorem and Proposition 2.2 of [17]. □

We know that we can associate $(\psi^l)_{1\le l\le k}$ with φ defined by (4). Let $P^{\varphi,\varepsilon}$ be defined by

$$\frac{dP^{\varphi,\varepsilon}}{dP}=\exp\left\{\int_0^T\left[k-1-\frac{\hat c(\varphi_s,\nu_s)}{\varepsilon}\right]ds+\sum_{l,m}\sum_{\tau\in\tau_{l,m}}\ln\frac{c(\varphi_\tau,l,m)}{\varepsilon}\right\},$$

or, equivalently,

$$\frac{dP^{\varphi,\varepsilon}}{dP^\varepsilon}=\exp\left\{\frac1\varepsilon\int_0^T[\hat c(X^\varepsilon_s,\nu_s)-\hat c(\varphi_s,\nu_s)]\,ds-\sum_{l,m}\sum_{\tau\in\tau_{l,m}}\ln\frac{c(X^\varepsilon_\tau,l,m)}{c(\varphi_\tau,l,m)}\right\}.$$

Under $P^{\varphi,\varepsilon}$, X^ε satisfies the same SDE but the matrix $\hat C$ is time dependent and no more space dependent. Then, we are in the previous case.

PROPOSITION 2.9. *The family $(P^\varepsilon,(\sqrt{\varepsilon}W^l)_{1\le l\le k},U)$ satisfies a uniform large deviation principle in the uniform topology. The action functional is $\varepsilon^{-1}S^{WU}_{0T}$ with*

$$S^{WU}_{0T}(\psi,\mu)=\begin{cases}\displaystyle\int_0^T\eta(\varphi_s,0,\dot\mu_s)\,ds+\sum_{l=1}^k\frac12\int_0^T|\dot\psi^l_s|^2\,ds, & \text{if } \mu\in C_+ \text{ and } \psi\in I^k,\\ +\infty, & \text{otherwise.}\end{cases}$$

PROOF. Our approach is deeply inspired by the work of Freidlin and Lee [15]. We just have to show that, if $S^{WU}_{0T}(\psi,\mu)<\infty$, then

$$\lim_{\delta\to0}\lim_{\varepsilon\to0}\varepsilon\ln P^\varepsilon\left[\sup_{l\in[\![1,k]\!]}\|\sqrt{\varepsilon}W^{\varepsilon,l}-\psi^l\|<\delta,\ \|U-\mu\|<\delta\right]=-S^{WU}_{0T}(\psi,\mu)$$

and $(P^\varepsilon,(\sqrt{\varepsilon}W^l)_{1\le l\le k},U)$ is exponentially tight; that is, there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets $I^k\times C_+$ endowed with the uniform topology such that

$$\lim_{n\to\infty}\lim_{\varepsilon\to0}\varepsilon\ln P^\varepsilon\Big(\big((\sqrt{\varepsilon}W^{\varepsilon,l})_{1\le l\le k},U\big)\notin K_n\Big)=-\infty.$$

We just have to notice that C_+ is compact and that, for $(\sqrt{\varepsilon}\, W^{\varepsilon, l})_{1 \le l \le k}$, we can choose

$$K_n^W = \left\{\psi \colon \psi_0 = x,\ |\psi_t^l - \psi_s^l| \le n|t-s|^{1/4},\ l \in [\![1, k]\!],\ 0 \le s \le t \le T\right\},$$

according to a result of Ben Arous and Ledoux [7]. Let $\hat{C}$ be uniformly Lipschitz continuous. Therefore, there exists $\gamma(\eta)$ such that $\gamma(\eta)$ tends to 0 when η tends to 0 and, if $\|X^\varepsilon - \varphi\| < \eta$, $|\int_0^T [\hat{c}(\varphi_s, \nu_s) - \hat{c}(X_s^\varepsilon, \nu_s)]\, ds| \le \gamma(\eta)$ since $\hat{c}(x, l) = \sum_{m=1}^{k-1} c(x, l, m)$ and

$$\sup_{l, m} \|\ln c(\varphi_s, l, m) - \ln c(X_s^\varepsilon, l, m)\| \le \gamma(\delta).$$

Set

$$A = \left\{\sup_{l \in [\![1, k]\!]} \|\sqrt{\varepsilon}\, W^l - \psi^l\| < \delta,\ \|U - \mu\| < \delta\right\},$$

$$B = \left\{\|X^\varepsilon - \varphi\| < \eta,\ \sup_{l \in [\![1, k]\!]} \|\sqrt{\varepsilon}\, W^l - \psi^l\| < \delta,\ \|U - \mu\| < \delta\right\}.$$

Finally, denote by $\|\tau\|$ the number of jumps of ν before time T. We have

$$\begin{aligned} P^\varepsilon[A] &= P^{\varphi, \varepsilon}\left[\frac{dP^\varepsilon}{dP^{\varphi, \varepsilon}}(X^\varepsilon, \nu); B\right] + P^\varepsilon[A \setminus B] \\ &\le \exp\left\{\frac{\gamma(\eta)}{\varepsilon}\right\} P^{\varphi, \varepsilon}\big[\exp\{\gamma(\eta)\|\tau\|\}; B\big] + P^\varepsilon[A \setminus B]. \end{aligned}$$

The Hölder inequality leads to

$$P^{\varphi, \varepsilon}\big[\exp\{\gamma(\eta)\|\tau\|\};\ B\big] \le P^{\varphi, \varepsilon}[A]^{1 - 1/q} \cdot P^{\varphi, \varepsilon}\big[\exp\{q\gamma(\eta)\|\tau\|\}\big]^{1/q}$$

for all $q > 1$. Notice that $\max_{l, m} \|c(\varphi, l, m)\| \le K$, which implies

$$P^{\varphi, \varepsilon}\big[\exp\{q\gamma(\eta)\|\tau\|\}\big] \le \exp\left\{\frac{T(k-1)K}{\varepsilon}(e^{q\gamma(\eta)} - 1)\right\}.$$

But, under $P^{\varphi, \varepsilon}$, Proposition 2.8 applies. Hence,

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \varepsilon \ln P^{\varphi, \varepsilon}\big[\exp\{\gamma(\eta)\|\tau\|\}; B\big] \\ &\qquad \le -\left(1 - \frac{1}{q}\right) S_{0T}^{WU}(\psi, \mu) + T(k-1)K(e^{q\gamma(\eta)} - 1) \end{aligned}$$

for all $q > 1$ and all $\eta > 0$. Moreover, for a given η, for all $R > 0$, there exists $\delta > 0$ such that $P^\varepsilon[A \setminus B] \le \exp(-R/\varepsilon)$ according to Theorem 2.7. Then

$$\begin{aligned} &\limsup_{\delta \to 0} \limsup_{\varepsilon \to 0} \varepsilon \ln P^\varepsilon[A] \\ &\qquad \le -\left(1 - \frac{1}{q}\right) S_{0T}^{WU}(\psi, \mu) + T(k-1)K(e^{q\gamma(\eta)} - 1) + \gamma(\eta) \end{aligned}$$

for arbitrary η and q. In fact,

$$\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0}\varepsilon\ln P^{\varepsilon}[A]\le -S_{0T}^{WU}(\psi,\mu).$$

Using the arguments developed previously and the following inequality

$$P^{\varepsilon}[A]\ge \exp\left\{-\frac{\gamma(\eta)}{\varepsilon}\right\}\left(P^{\varphi,\varepsilon}[A]-P^{\varphi,\varepsilon}[A\setminus B]\right)^2$$
$$\times\left(P^{\varphi,\varepsilon}\left[\exp\{\gamma(\eta)\|\tau\|\};B\right]\right)^{-1},$$

we get the lower bound. $\square$

We are now ready to prove Theorem 2.1. We go back to the method used by Priouret. We notice that in the definition of S_{0T} the infimum is realized for a vector $(\psi_l,\ 1\le l\le k)$: for a given μ, we have the same kind of problem as in the case of the action functional of a small perturbation of a dynamical system and Azencott ([2], Proposition 3.2.10) showed that the infimum is actually achieved. Indeed,

$$\begin{aligned}S_{0T}(\varphi,\mu)&=\int_0^T\eta(\varphi_s,0,\dot\mu_s)\,ds+\inf_{\psi}\left\{\sum_{l=1}^k\frac12\int_0^T|\dot\psi_s^l|^2\,ds,\ F_x(\psi,\mu)=\varphi\right\}\\&=\int_0^T\eta(\varphi_s,0,\dot\mu_s)\,ds+\inf_{\psi}\left\{\sum_{l=1}^k\frac12\int_0^T|\dot\psi^l_{\mu_s(l)}|^2\dot\mu_s(l)\,ds,\ F_x(\psi,\mu)=\varphi\right\}.\end{aligned}$$

But if we denote $\Sigma(x,\beta)=(\sigma(x,1)\sqrt{\beta(1)},\dots,\sigma(x,k)\sqrt{\beta(k)})\in\mathbb{R}^{d\times kd}$ and $\overline{b}(x,\beta)=\Sigma b(x,l)\beta(l)$, then

$$\dot\varphi_s=\overline{b}(\varphi_s,\dot\mu_s)+\Sigma(\varphi_s,\dot\mu_s)\left(\dot\psi^l_{\mu_s(1)}\sqrt{\dot\mu_s(1)},\dots,\dot\psi^l_{\mu_s(k)}\sqrt{\dot\mu_s(k)}\right)^*,$$

which is, for a given μ, the problem studied by Azencott even though the diffusion coefficient depends on time. Hence,

$$\begin{aligned}&S_{0T}(\varphi,\mu)\\&\quad=\int_0^T\left[\eta(\varphi_s,0,\dot\mu_s)+\sup_p\left(p\cdot\left(\dot\varphi_s-\overline{b}(\varphi_s,\dot\mu_s)\right)-\frac12\|\Sigma^*(x,\dot\mu_s)p\|^2\right)\right]ds.\end{aligned}$$

A short computation leads to

$$S_{0T}(\varphi,\mu)=\int_0^T\eta(\varphi_s,\dot\varphi_s,\dot\mu_s)\,ds.$$

Denote $\Phi_x(s_0)=\{(\varphi,\mu),\ S_{0T}(\varphi,\mu)\le s_0,\ \varphi_0=x\}$. Note that $\Phi_x(s_0)$ is compact according to Lemma 3.1.3 in [2]. We first show that

LEMMA 2.10. *For all $\delta>0$, $\gamma>0$, $s_0>0$, there exists $\varepsilon_0>0$ such that*

$$P[\|X^{\varepsilon}-\varphi\|<\delta,\ \|U-\mu\|<\delta]\ge\exp\left\{-\frac1\varepsilon\left(S_{0T}(\varphi,\mu)+\gamma\right)\right\}$$

for all $\varepsilon<\varepsilon_0$, $x\in\mathbb{R}^d$, $n\in[\![1,k]\!]$, $\varphi\in\Phi_x(s_0)$.

This is a simple consequence of Proposition 2.8 and Theorem 2.7. Let us prove the upper bound. Denote

$$\|(\varphi,\mu),\Phi(s)\| = \inf_{(\phi,\nu)\in\Phi(s)} \max(\|\phi-\varphi\|,\|\nu-\mu\|).$$

LEMMA 2.11. *For all* $\delta>0$, $\gamma>0$, $s_0>0$, *there exists* $\varepsilon_0>0$ *such that*

$$P\big[\|(X^\varepsilon,U),\Phi_x(s)\|\ge\delta\big]\le\exp\biggl(-\frac{s-\gamma}{\varepsilon}\biggr)$$

for all $\varepsilon<\varepsilon_0$, $x\in\mathbb{R}^d$, $n\in[\![1,k]\!]$, $s\le s_0$.

PROOF. For all $s\in[0,\infty[$, $\{S_{0T}^{WU}\le s\}$ is compact. Therefore, b and σ being bounded, $\Phi_x(s_0)$ is compact. Denote $F(s,\delta)=\{(\varphi,\mu)\colon \|(\varphi,\mu),\ \Phi(s)\|\ge\delta\}$.

Let $\gamma>0$, $(\psi,\mu)\in\{S_{0T}^{WU}\le s-\gamma\}$. If $\varphi=F_x(\psi,\mu)$, then $S_{0T}(\varphi,\mu)\le s-\gamma$. Hence, $(\varphi,\mu)\notin F(s,\delta)$ and there exists $\rho_{\psi\mu}>0$ such that

$$B\big((\varphi,\mu),\rho_{\psi\mu}\big)\cap F(s,\delta)=\varnothing,$$

where $B((\varphi,\mu),\rho)$ is the open ball the radius of which is ρ for uniform topology. Moreover, according to Theorem 2.7, there exist $\alpha_{\psi\mu}>0$ and $\varepsilon_{\psi\mu}>0$ such that, if $\varepsilon\le\varepsilon_{\psi\mu}$,

$$P\Big[\|X^\varepsilon-\varphi\|\ge\rho_{\psi\mu},\ \|\sqrt{\varepsilon}\,(W^{\varepsilon,l})_{l\in[\![1,k]\!]}-\psi\|<\alpha_{\psi\mu},\ \|U-\mu\|<\alpha_{\psi\mu}\Big]\le\exp\biggl(-\frac{R}{\varepsilon}\biggr).$$

But $\{S_{0T}^{WU}\le s-\gamma\}$ is compact. Therefore, there exist $N\in\mathbb{N}$ and

$$(\psi_i,\mu_i)_{1\le i\le N}\in\big(\{S_{0T}^{WU}\le s-\gamma\}\big)^N$$

such that

$$\big\{S_{0T}^{WU}\le s-\gamma\big\}\subset\bigcup_{i=1}^N B\big((\psi_i,\mu_i),\alpha_i\big)=O,$$

where $\alpha_i=\alpha_{\psi_i\mu_i}$. In the same way, denote ρ_i instead of $\rho_{\psi_i\mu_i}$. Notice that

$$\Big\{\big((\sqrt{\varepsilon}\,W^{\varepsilon,l})_{l\in[\![1,k]\!]},U\big)\in O\Big\}\cap\{(X^\varepsilon,U)\in F(s,\delta)\}\subset\bigcup_{i=1}^N C_i,$$

where $C_i=C(\psi_i,\mu_i)$. Then

$$P\Big[\big((\sqrt{\varepsilon}\,W^{\varepsilon,l})_{l\in[\![1,k]\!]},U\big)\in O,\ (X^\varepsilon,U)\in F(s,\delta)\Big]\le N\exp\biggl(-\frac{R}{\varepsilon}\biggr).$$

We are now close to the conclusion:

$$\begin{aligned}
P[(X^\varepsilon,U)\in F(s,\delta)] &\le P\Big[\big((\sqrt{\varepsilon}\,W^{\varepsilon,l})_{l\in[\![1,k]\!]},U\big)\in O,\ (X^\varepsilon,U)\in F(s,\delta)\Big]\\
&\quad+P\Big[\big((\sqrt{\varepsilon}\,W^{\varepsilon,l})_{l\in[\![1,k]\!]},U\big)\notin O\Big]\\
&\le N\exp\biggl(-\frac{R}{\varepsilon}\biggr)+\exp\biggl(-\frac{s-\gamma/2}{\varepsilon}\biggr)\\
&\le\exp\biggl(-\frac{s-\gamma}{\varepsilon}\biggr).
\end{aligned}$$

□

Notice that the uniformity of the large deviation principle comes from the uniformity of the estimates in Theorem 2.7 and Proposition 2.8.

2.2.1. *A few remarks.* When, for all $l \in [\![1, k]\!]$, $\sigma_l \cdot \sigma_l^*$ is definite positive, as soon as φ is absolutely continuous, for all $\mu \in C_+$ the set of ψ such that $F_x(\psi, \mu) = \varphi$ is nonempty. It is not always true when $\sigma_l \cdot \sigma_l^*$ may degenerate because $F_x(\psi, \mu) = \varphi$ may have no solution. As for a single equation, the question is: for given $x, y \in \mathbb{R}^d$ and $t > 0$, is there (φ, μ) such that $S_{0t}(\varphi, \mu) < \infty$ and $\varphi_0 = x$, $\varphi_t = y$? It is true as soon as $\Sigma = (\sigma(\cdot, 1), \dots, \sigma(\cdot, k)) \in \mathbb{R}^{d \times kd}$ satisfies the strong Hörmander condition (cf. Definition 2.12). Then we can choose $\dot{\mu}_s(l) = 1/k$ for all $(s, l) \in [0, t] \times [\![1, k]\!]$ and the question becomes the classical one in linear control theory. Let us recall the definition of the strong Hörmander assumption.

DEFINITION 2.12. We denote by $\Sigma(x, i)$ the columns of matrices $\Sigma(x)$ and we assume that $\Sigma \in C^\infty(\mathbb{R}^d, \mathbb{R}^{d \times kd})$. Denote by $A(n, x)$ the set of Lie brackets of $(\Sigma_i)_{1 \le i \le kd}$ of order lower than n at the point $x \in \mathbb{R}^d$.

We say that Σ satisfies the strong Hörmander condition if, for all $x \in \mathbb{R}^d$, there exists $n_x \in \mathbb{N}$ such that $A(n_x, x)$ generates $\mathbb{R}^d$.

We introduce the pseudo Riemannian metric associated with S_{0t}.

DEFINITION 2.13. For all $(t, x, y) \in]0, \infty[\times \mathbb{R}^d \times \mathbb{R}^d$,

$$\rho^2(t, x, y) = \inf_{\varphi, \mu} \{S_{0t}(\varphi, \mu), \varphi_0 = x, \varphi_t = y\}.$$

As noted in the Appendix, $\rho(t, \cdot, \cdot)$ is not continuous in general, even if Σ satisfies the strong Hörmander condition. We need some stronger assumptions to get the continuity of ρ:

(H1) $b(x, l) = b(x, 1) \equiv b(x)$ for all $(x, l) \in \mathbb{R}^d \times [\![1, k]\!]$ and Σ satisfies the strong Hörmander condition;

(H2) $\Sigma(x)\Sigma^*(x) \ge 1/K \cdot I_d$ for all $x \in \mathbb{R}^d$.

PROPOSITION 2.14. *On* (H1) *or* (H2), *for all* $t \in]0, \infty[$, $\rho^2(t, \cdot, \cdot)$ *is continuous on* $\mathbb{R}^d \times \mathbb{R}^d$.

PROOF. For the proof and some remarks on ρ, we refer the reader to the Appendix.

3. About degenerate systems of KPP equations. As we said in the Introduction, we want to study systems of scaled KPP equations:

$$\begin{aligned} \frac{\partial u_l^\varepsilon}{\partial t}(t, x) &= L_l^\varepsilon u_l^\varepsilon(t, x) + \frac{1}{\varepsilon} \bar{f}(x, l, u^\varepsilon(t, x)), \\ &\qquad\qquad t > 0,\ x \in \mathbb{R}^d, l \in [\![1, k]\!], \\ u_l(0, x) &= g_l(x), \qquad x \in \mathbb{R}^d, \end{aligned} \tag{6}$$

where

$$L_l^\varepsilon = \frac{\varepsilon}{2}\sum_{i,j=1}^{d} a^{ij}(x,l)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i_{\varepsilon,l}(x)\frac{\partial}{\partial x^i},$$

$a_l = \sigma_l \cdot \sigma_l^*$ and $\bar{f}(x,l,u)$ behaves like $c(x,l)u_l(1-u_l)$ plus a coupling term.

Wavefront propagation for such systems is studied in [16] on one hand and [5] on the other hand on uniform ellipticity assumptions. Moreover, according to Pardoux, Pradeilles and Rao [24], BSDEs and parabolic systems are connected. We will show that the ideas developed in [25] to study hypoelliptic KPP equations still apply despite some new technical difficulties, mainly when we want to study the convergence to the stable equilibrium state. We give an example where it occurs and we compute the convergence rate.

Before recalling the basic results on BSDEs and giving the proof of the main theorem, we set the assumptions of this section. But the degeneracy of parabolic operators is given in [16].

3.1. *Assumptions and main theorem.* Our assumptions are of two types: $(b_\varepsilon)_\varepsilon$ and σ satisfy Assumptions 2.1 and are such that $\rho(t,\cdot,\cdot)$ is continuous; that is, (H1) or (H2) is true. As we said previously, we make on g and $\bar{f}$ Freidlin–Lee assumptions [16]. We recall them below and introduce some notation.

ASSUMPTIONS 3.1. (a) For all $l \in [\![1,k]\!]$, $g_l \in C_b(\mathbb{R}^d,\mathbb{R})$ and

$$\sup_{x\in\mathbb{R}^d}\sup_{l\in[\![1,k]\!]} g(x,l) = \bar{g} < \infty,$$

$$G_0 = \{x \in \mathbb{R}^d : \exists l \in [\![1,k]\!],\ g(x,l) > 0\}.$$

Owing to comparison theorems, this assumption can be weakened: g: $\mathbb{R}^d \times \mathbb{N} \to [0,\infty[$ is bounded and $\overline{\mathring{G}_0} = \overline{G_0}$;

(b) $\bar{f}$: $\mathbb{R}^d \times \mathbb{N} \times \mathbb{R}^k \to \mathbb{R}$;

(c) if $n = l \bmod[k]$, $l \in [\![1,k]\!]$, $g(\cdot,n) = g_l \in C(\mathbb{R}^d,\mathbb{R})$, $\bar{f}(\cdot,n,\cdot) = \bar{f}_l \in C(\mathbb{R}^d \times \mathbb{R}^k,\mathbb{R})$;

(d) moreover $\bar{f}$ satisfies the following conditions:

(i) for all $y \in \mathbb{R}$, $l \in [\![1,k]\!]$: $\bar{f}_l(\cdot,y)$ is bounded;

(ii) for all $x \in \mathbb{R}^d$, $y, y' \in \mathbb{R}^d$, $l \in [\![1,k]\!]$,

$$|\bar{f}_l(x,y) - \bar{f}_l(x,y')| \le K|y-y'|.$$

Let us now introduce generalized KPP assumptions: for all $x \in \mathbb{R}^d$, $l \in [\![1,k]\!]$:

(a) $\bar{f}_l(x,0) = 0$, $\bar{f}_l(x,u) > 0$ if, for all $i \in [\![1,k]\!]$, $u_i \ge 0$, $u_l = 0$ and $\sum_{i=1}^k u_i > 0$;

(b) there exists $\Lambda > 0$ such that, for all $x \in \mathbb{R}^d$, if $u > \Lambda$ then, for all $l \in [\![1,k]\!]$, $\bar{f}_l(x,(u)^k) < 0$;

(c) for all $x \in \mathbb{R}^d$, $l, i \in [\![1, k]\!]$ we define

$$c(x, l, i) \equiv \frac{\partial \bar{f}_l}{\partial u_{l+i}}(x, 0), \qquad c_{li}(x) = \frac{\partial \bar{f}_l}{\partial u_i}(x, 0), \qquad C(x) = \big(c_{li}(x)\big)_{1 \le i, j \le k}$$

and we assume that the matrix C satisfies Assumptions 2.1 and is uniformly bounded;

(d) for all $x \in \mathbb{R}^d$, $l \in [\![1, k]\!]$, $\bar{f}_l(x, u) \le \sum_{i=1}^k c_{li}(x) u_i$;

(e) for all $\gamma > 0$, there exists $B(\gamma) > 0$ such that

$$\forall (x, l, u) \in \mathbb{R}^d \times [\![1, k]\!] \times [0, B(\gamma)]^k, \qquad \sum_{i=1}^k (c_{li}(x) - \gamma) u_i \le \bar{f}_l(x, u).$$

DEFINITION 3.1. For all $(t, x) \in]0, \infty[\times \mathbb{R}^d$,

$$\begin{aligned} V^*(t, x) &= \inf_{\tau \in \Theta_t} \sup_{\varphi, \mu} \{R_{0\tau}(\varphi, \mu), \varphi_0 = x, \varphi_t \in G_0\} \\ &= \inf_{\tau \in \Theta_t} \sup_{\varphi} \{R_{0\tau}(\varphi),\ \varphi_0 = x, \varphi_t \in G_0\}, \end{aligned}$$

where

$$R_{0s}(\varphi, \mu) = \int_0^s \sum_{l=1}^k c(\varphi_r, l) \dot{\mu}_r(l)\, dr - S_{0s}(\varphi, \mu)$$

$$R_{0s}(\varphi) = \int_0^s \zeta(\varphi_r, \dot{\varphi}_r)\, dr$$

and Θ_t is the set of stopping times τ no greater than t defined by: there exists O an open subset of $[0, t] \times \mathbb{R}^d$ such that

$$\tau(\varphi) = \min\{s \in [0, t]: (t - s, \varphi_s) \in O\}.$$

V^* is nonpositive and we denote $M = \{V^* = 0\}$ and $E = \{V^* < 0\}$ the subsets of $\mathbb{R}^+ \times \mathbb{R}^d$. Here is our main result:

THEOREM 3.2. *$\varepsilon \ln u_l^\varepsilon$ converges uniformly on compact subsets to V^* for all $l \in [\![1, k]\!]$. Moreover*:

1. $\sup_{1 \le l \le k} u_l^\varepsilon(t, x)$ *converges uniformly to* 0 *on compact subsets of* E;
2. *there exists* $h > 0$ *such that*

$$\liminf_{\varepsilon \downarrow 0} \inf_{1 \le l \le k} u_l^\varepsilon(t, x) \ge h$$

uniformly on compact subsets of $\mathring{M}$.

Before giving the proof, we notice that our assumptions include the usual systems of KPP equations. Indeed, if $\bar{f}$ is defined by

$$\bar{f}_l(x, u) = c(x, l)(u_l \wedge (\bar{g} \vee 1))(1 - (u_l \vee 0)) + \sum_{i=1}^{k-1} c(x, l, i)(u_{l+i} - u_l),$$

it is easy to check that the unique solution of the system is the solution of the classical system and one can show that the solutions of the BSDE associated with KPP systems must be bounded and nonnegative.

3.2. *The proof*. We first recall some notation and definitions and the link between BSDEs and our problem. The probability space $(\Omega, F, (F_t)_{t\geq 0}, P)$ has been defined in the previous section.

DEFINITION 3.3. Let u be a viscosity solution of (1) if u is continuous and if it is both a sub- and a supersolution of (1).

Let u be a subsolution (resp. supersolution) if, for all $(l, t_0, x_0) \in [\![1, k]\!] \times]0, +\infty[\times \mathbb{R}^d$, for all real-valued $\varphi \in C^{1,2}(]0, +\infty[\times \mathbb{R}^d)$ such that $\varphi - u_l \geq (\varphi - u_l)(t_0, x_0)$ (resp. $\leq$),

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) - L_l \varphi(t_0, x_0) - f_l(x_0, u(t_0, x_0), \nabla\varphi(t_0, x_0)\sigma(x_0, l)) \leq 0 (\text{resp.} \geq)$$

As far as we know, there is no definition of the viscosity solution for a system such that f_l depends on ∇u_m with $m \neq l$.

DEFINITION 3.4. We set

$$L_T = \prod_{l=1}^{k-1} \prod_{n\geq 1} c(T_n(l), l) \mathbb{1}_{\{T_n(l)\leq T\}} \exp\left\{ \int_0^T (1 - c(s, l))\, ds \right\},$$

where $T_n(l)$ is a jump time of $N(l)$ lower than T and we assume that $E[L_T] = 1$. We say that $\tilde{P}$ defined by $d\tilde{P}/dP = L_T$ is *the probability* under which $N(l)$ has intensity $c(\cdot, l)$ for all $l \in [\![1, k]\!]$.

If $u = (u_l)_{1\leq l\leq k} \in \mathbb{R}^k$, we denote $h^l(i) = u_{l+i} - u_l$ for all $i \in [\![1, k-1]\!]$ with $u_{j+k} = u_j$ and

$$h^l = \big(h^l(i)\big)_{1\leq i\leq k-1},$$

$$\tilde{f}(t, l, y, h, z) = f(t, l, y, h, z) + \sum_{i=1}^{k-1} \lambda_t(i) h(i),$$

$$\bar{f}(t, l, u, z) = \tilde{f}\big(t, l, u_l, h^l, z\big),$$

where $\lambda(i)/\varepsilon$ is the intensity of $N(i)$ under $P^{\lambda,\varepsilon}$, $(X^{\varepsilon,x,l}, \nu^l)$ is the diffusion–transmutation process defined in (3) and $(Y^{\varepsilon,t,x,l}, H^{\varepsilon,t,x,l}, Z^{\varepsilon,t,x,l})$ is the unique solution of the BSDE associated with (6):

$$\begin{aligned} Y_s^{\varepsilon,t,x,l} &= g\big(X_t^{\varepsilon,x,l}, \nu_t^l\big) + \frac{1}{\varepsilon}\int_s^t f\big(X_r^{\varepsilon,x,l}, \nu_r^l, Y_r^{\varepsilon,t,x,l}, H_r^{\varepsilon,t,x,l}\big)\, dr \\ &\quad - \int_t^T Z_r^{\varepsilon,t,x,l}\, dW_r - \int_t^T \sum_{i=1}^{k-1} H_r^{\varepsilon,t,x,l}(i)\, d\overline{N}_r(i). \end{aligned} \tag{7}$$

The existence and uniqueness do not depend on $\tilde{P}$ equivalent to P as well as the following theorem according to [24].

THEOREM 3.5. *If u^ε is the unique viscosity solution of* (6), *then*

$$\forall (t, x, l) \in \big[0, \infty\big[\times \mathbb{R}^d \times [\![1, k]\!], \qquad u_l^\varepsilon(t, x) = Y_0^{\varepsilon, t, x, l}.$$

We can use this formula and the comparison theorem for BSDEs to show a first property.

LEMMA 3.6. *For all $\varepsilon > 0$, $t > 0$, $x \in \mathbb{R}^d$ and $l \in [\![1, k]\!]$,*

$$0 < Y_0^{\varepsilon, t, x, l} \le \Lambda \vee \bar{g}.$$

PROOF. $0 \le Y_0^{\varepsilon, t, x, l} \le \Lambda \vee \bar{g}$ is quite simple. In order to show that the lower bound is strict, we introduce $\gamma > 0$ and $\bar{f}^\gamma$ defined on $\mathbb{R}^d \times [\![1, k]\!] \times [0, \gamma]^k$ by

$$\bar{f}_l^\gamma(x, u) \equiv \frac{k}{2K} u_l \left(1 - \frac{u_l}{\gamma}\right) + \sum_{i \ne l} \frac{1}{2K}(u_i - u_l),$$

and $0 \le g^\gamma \le \gamma$. Set $(Y^\gamma, Z^\gamma, H^\gamma)$ the unique solution of the BSDE associated with (6), where $\bar{f}$ and g are replaced by $\bar{f}^\gamma$ and g^γ. According to the comparison theorem, $Y^\gamma \le Y^{\varepsilon, x, t}$ and $0 \le Y^\gamma \le \gamma$ if γ is small enough. Then we just have to notice that, under the probability under which all $N(l)$ have the intensity $\underline{c}/2\varepsilon$, Y^γ is a supermartingale. Hence,

$$Y_0^\gamma \ge E g^\gamma(X_t, \nu_t) > 0$$

since Σ satisfies the strong Hörmander condition. □

Actually, the last inequality is a consequence of the well-known Feynman–Kac formula using that $Y_s^{\varepsilon, t, x, l} = u_{\nu_s^l}^\varepsilon(t - s, X_s^{\varepsilon, t, x, l})$: we refer to [24] for viscosity solutions.

THEOREM 3.7. *Let c: $\Omega \times [0, T] \to \mathbb{R}$, F_t-predictable, be such that*

$$E\int_0^T c^2(s)\left(1 + \exp\left[\int_0^s 2c(r)\, dr\right]\right) ds < \infty. \tag{8}$$

and let τ be a Markov time lower than T. Then

$$\begin{aligned} Y_0 &= E Y_\tau \exp\left\{\int_0^\tau c(s)\, ds\right\} \\ &\quad + E\int_0^\tau \exp\left\{\int_0^s c(r)\, dr\right\}\big[f(s, Y_s, H_s, Z_s) - c(s) Y_s\big]\, ds. \end{aligned} \tag{9}$$

PROOF. We refer to the equivalent result in [25]. □

Denote

$$v_l^\varepsilon(t, x) = \varepsilon \ln u_l^\varepsilon(t, x),$$

where v^ε is a viscosity solution of

$$\frac{\partial v_l^\varepsilon}{\partial t}(t,x) = L_l^\varepsilon v^\varepsilon(t,x) + \frac{1}{2}\|\sigma_l^*(x)\nabla v^\varepsilon(t,x)\|^2 + \frac{\bar f_l(x,u^\varepsilon(t,x))}{u_l^\varepsilon(t,x)}, \qquad 0<t,\ x\in\mathbb{R}^d,$$

$$\lim_{t\downarrow 0} v_l^\varepsilon(t,x) = -\infty \quad \text{if } g_l(x)=0,$$

$$v^\varepsilon(0,x) = \varepsilon \ln g_l(x) \quad \text{if } g_l(x)>0.$$

Let us now introduce some notation:

(a) $\rho^2(t,x,G_0) = \inf_{\varphi,\mu}\{S_{0t}(\varphi,\mu)\colon \varphi_0 = x,\ \varphi_t\in G_0\}$;
(b) $L(x,p)$ is the Hamiltonian of

$$A(x,p) = C(x) + \mathrm{Diag}\Big(\tfrac{1}{2}|\sigma_1^*(x)p|^2 + \big(b_1(x),p\big),\dots,\tfrac{1}{2}|\sigma_k^*(x)p|^2 + \big(b_k(x),p\big)\Big);$$

(c) for all $(t,x)\in\,]0,\infty[\,\times\mathbb{R}^d$,

$$u^*(t,x) = \lim_{\eta\to 0}\sup\big\{v_l^\varepsilon(s,y);\ \varepsilon\le\eta,\ (s,y)\in B\big((t,x),\eta\big),\ l\in[\![1,k]\!]\big\},$$

$$v^*(t,x) = \lim_{\eta\to 0}\inf\big\{v_l^\varepsilon(s,y);\ \varepsilon\le\eta,\ (s,y)\in B\big((t,x),\eta\big),\ l\notin[\![1,k]\!]\big\},$$

$$\rho^2(t,x,G_0) = \inf_{\varphi,\mu}\big\{S_{0t}(\varphi,\mu),\ \varphi_0 = x,\ \varphi_t\in G_0\big\}.$$

REMARK 3.8. Using the fact that $0\le \bar f_l(x,u)\le K\sum_{i=1}^k c_{li}(x)u_i$, large deviations and the continuity of ρ, we easily show

$$-\rho^2(t,x,G_0)\le v^*(t,x)\le u^*(t,x)\le \min\big(kKt-\rho^2(t,x,G_0),0\big).$$

LEMMA 3.9. *u^* and v^* are sub and super viscosity solutions of*

$$\max\left(w,\frac{\partial w}{\partial t}(t,x) - L(x,\nabla w)\right) = 0, \qquad 0<t,\ x\in\mathbb{R}^d,$$

$$\lim_{t\downarrow 0} w(t,x) = -\infty,\ x\notin\overline{G_0}, \tag{10}$$

$$w(0,x) = 0,\ x\in G_0.$$

PROOF. The proof we give is in [5] with constant coefficients.

We first consider u^*. Let $(t_0,x_0)\in\,]0,\infty[\,\times\mathbb{R}^d$. Let $\phi\ge u^*$ be a smooth function such that $\phi(t_0,x_0) = u^*(t_0,x_0)$ and (t_0,x_0) is a strict local minimum of $\phi - u^*$. Let $(\psi_1,\dots,\psi_k)\in(]0,\infty[)^k$ be an eigenvector of $A(x,\nabla\phi(t_0,x_0))$ for the eigenvalue $L(x,\nabla\phi(t_0,x_0))$. Then

$$u^*(t_0,x_0) = \lim_{\eta\to 0}\sup_{l\in[\![1,k]\!]}\big\{v_l^\varepsilon(s,y) - \varepsilon\ln\psi_l;\ \varepsilon\le\eta,\ (s,y)\in B\big((t_0,x_0),\eta\big)\big\}.$$

There exist $m \in [\![1, k]\!]$ and a sequence $(\varepsilon_n, t_n, x_n)$ which tends to $(0, t_0, x_0)$ such that

$$v_m^{\varepsilon_n}(t_n, x_n) - \varepsilon_n \ln \psi_m = \max_{l \in [\![1,k]\!]} \bigl(v_l^{\varepsilon_n}(t_n, x_n) - \varepsilon_n \ln \psi_l\bigr) \to u^*(t_0, x_0)$$

and (t_n, x_n) locally minimizes $\phi - v_m^\varepsilon + \varepsilon_n \ln \psi_m$. Hence (we omit n),

$$\frac{\partial \phi}{\partial t}(t, x) \leq L_m^\varepsilon \phi(t, x) + \frac{1}{2}\|\sigma_m^*(x)\nabla\phi(t, x)\|^2 + \frac{\bar f_m(x, u^\varepsilon(t, x))}{u_m^\varepsilon(t, x)}.$$

Actually,

$$\frac{\bar f_m(x, u^\varepsilon(t, x))}{u_m^\varepsilon(t, x)} \leq \sum_{l=1}^k c_{ml}(x)\frac{u_l^\varepsilon}{u_m^\varepsilon}(t, x) \leq \sum_{l=1}^k c_{ml}(x)\frac{\psi_l}{\psi_m},$$

since $u_l^\varepsilon(t, x)/\psi_l \leq u_m^\varepsilon(t, x)/\psi_m$. Passing to the limit leads to

$$\begin{aligned}\frac{\partial \phi}{\partial t}(t_0, x_0)\psi_m \leq{}& \left(\frac{1}{2}\|\sigma_m^*(x_0)\nabla\phi(t_0, x_0)\|^2 + \bigl(b(x_0), \nabla\phi(t_0, x_0)\bigr)\right)\psi_m \\ &+ \sum_{l=1}^k c_{ml}(x_n)\psi_l.\end{aligned}$$

Since ψ is an eigenvector of $A(x, \nabla\phi(t_0, x_0))$ for $L(x, \nabla\phi(t_0, x_0))$ and $\psi_m > 0$, the proof is complete.

Let us now consider v^*. Let $(t_0, x_0) \in]0, \infty[\times \mathbb{R}^d$ such that $v^*(t_0, x_0) < 0$. Let $\phi \leq v^*$ be a smooth function such that $\phi(t_0, x_0) = v^*(t_0, x_0)$ and (t_0, x_0) is a strict local maximum of $\phi - v^*$. Let $(\psi_1, \ldots, \psi_k) \in (]0, \infty[)^k$ be an eigenvector of $A(x, \nabla\phi(t_0, x_0))$ for $L(x_0, \nabla\phi(t_0, x_0))$. Then

$$v^*(t_0, x_0) = \lim_{\eta \to 0} \inf_{l \in [\![1,k]\!]} \bigl\{v_l^\varepsilon(s, y) - \varepsilon \ln \psi_l;\ \varepsilon \leq \eta,\ (s, y) \in B\bigl((t_0, x_0), \eta\bigr)\bigr\}.$$

There exist $m \in [\![1, k]\!]$ and a sequence $(\varepsilon_n, t_n, x_n)$ which tends to $(0, t_0, x_0)$ such that

$$v_m^{\varepsilon_n}(t_n, x_n) - \varepsilon_n \ln \psi_m = \min_{l \in [\![1,k]\!]} \bigl(v_l^{\varepsilon_n}(t_n, x_n) - \varepsilon_n \ln \psi_l\bigr) \to v^*(t, x)$$

and $\phi - v_m^\varepsilon + \varepsilon_n \ln \psi_m$ has a local maximum in (t_n, x_n). Therefore (we omit n),

$$\frac{\partial \phi}{\partial t}(t, x) - L_m^\varepsilon \phi(t, x) - \frac{1}{2}\|\sigma_m^*(x)\nabla\phi(t, x)\|^2 \geq \frac{\bar f_m(x, u^\varepsilon(t, x))}{u_m^\varepsilon(t, x)}.$$

Hence, for all $l \in [\![1, k]\!]$, $u_l^\varepsilon(t, x)$ tends to 0. Otherwise we would have a contradiction with the fact that $\bar f_m(x, u^\varepsilon(t, x))/u_m^\varepsilon(t, x)$ is bounded. Then, for n large enough,

$$M \geq \frac{\bar f_m(x, u^\varepsilon(t, x))}{u_m^\varepsilon(t, x)} \geq \frac{1}{2K}\sum_{l=1}^k \frac{u_l^\varepsilon}{u_m^\varepsilon}(t, x) \geq 0.$$

This leads to

$$\frac{\bar{f}_m(x, u^\varepsilon(t, x))}{u_m^\varepsilon(t, x)} = \sum_{l=1}^{k} c_{ml}(x) \frac{u_l^\varepsilon}{u_m^\varepsilon}(t, x) + o(1),$$

since $\|u^\varepsilon(t, x)\| \le 2KMu_m^\varepsilon(t, x)$ for n large enough. We conclude as for u^*. □

Now, we establish a uniqueness theorem for this equation.

THEOREM 3.10. *Let u and v be sub and super viscosity solutions of* (10). *If, for all* $(t, x) \in (0, \infty) \times \mathbb{R}^d$,

$$\begin{aligned} -\rho^2(t, x, G_0) &\le v(t, x) \le 0, \\ u(t, x) &\le \min\big(kKt - \rho^2(t, x, G_0), 0\big), \end{aligned} \tag{11}$$

then $v \ge u$.

PROOF. Let $T > 1$, $\mu \in (0, 1)$ and $R_0 > 0$ be such that

$$kKT - (1 - \mu)(R_0)^2 = 0.$$

Then

$$\sup_{]0, T] \times R^d} (u(t, x) - \mu v(t, x)) \le \max\Big(\sup_B (u(t, x) - \mu v(t, x)), 0\Big),$$

where $B = \{(t, y),\ t \in (0, T),\ \rho(t, y, G_0) < R_0\}$ is an open subset: ρ is continuous according to Proposition 2.14. Let $\delta \in (0, 1)$ and $\gamma > 0$ be given. We set

$$u'(t, x) = u(t, x)e^{-t} \quad \text{and} \quad v'(t, x) = \mu(v(t, x) + \gamma t)e^{-t}.$$

Here u' is a solution of

$$\max\Big(w, \frac{\partial w}{\partial t}(t, x) + w(t, x) - L(t, x, \nabla w)\Big) \le 0$$

and v' is a solution

$$\max\Big(w, \frac{\partial w}{\partial t}(t, x) + w(t, x) - L^\mu(t, x, \nabla w)\Big) \ge \mu\gamma\delta e^{-T},$$

where $L(t, x, p)$ and $L^\mu(t, x, p)$ are the eigenvalues of

$$C(x)e^{-t} + \mathrm{Diag}\Big(\tfrac{1}{2}|\sigma_1^*(x)p|^2 e^t + (b_1(x), p), \dots, \tfrac{1}{2}|\sigma_k^*(x)p|^2 e^t + (b_k(x), p)\Big)$$

and

$$\begin{aligned} \mu C(x)e^{-t} + \mathrm{Diag}\Big(&\frac{1}{2\mu}|\sigma_1^*(x)p|^2 e^t + (b_1(x), p), \dots, \\ &\frac{1}{2\mu}|\sigma_k^*(x)p|^2 e^t + (b_k(x), p)\Big). \end{aligned}$$

We assume that, for all $\mu \in]0, 1[$, there exists $(t_0, x_0) \in \Omega' = B \cap \{t \in]\delta, T[\}$ such that $u'(t_0, x_0) - v'(t_0, x_0) > 0$ and denote by $M > 0$ the supremum of $u' - v'$ on Ω'. Then, since v' is bounded on Ω' and $u' \leq 0$, for $\varepsilon > 0$ and $\alpha > 0$ small enough,

$$u'(t, x) - v'(s, y) - \frac{|t - s|^2 + |x - y|^2}{2\varepsilon^2} - \frac{\alpha}{2}\big(|x|^2 + |y|^2\big)$$

has a maximum $M_{\varepsilon, \alpha} > 0$ in $\Omega \times \Omega$ at $((t_{\varepsilon, \alpha}, x_{\varepsilon, \alpha}), (s_{\varepsilon, \alpha}, y_{\varepsilon, \alpha}))$. Then $v'(s, y) < 0$ (we omit ε, α), since $M_{\varepsilon, \alpha} > 0$. If we set $p = (x - y)/\varepsilon^2$, then

$$u'(t, x) - v'(s, y) - L(t, x, p + \alpha x) + L^{\mu}(s, y, p - \alpha y) \leq -\mu\gamma\delta e^{-T}.$$

Hence (cf. the proof of Theorem 4.3 [25])

$$\operatorname{Diag}\begin{pmatrix} \dfrac{1}{2\mu}|\sigma_l^*(y)(p - \alpha y)|^2 e^s + \big(b_l(x), p - \alpha y\big) \\ -\dfrac{1}{2}|\sigma_l^*(x)(p + \alpha x)|^2 e^s - \big(b_l(x), p + \alpha x\big) \end{pmatrix} \geq o_{\varepsilon, \alpha}(1) I_k.$$

Denote by $\xi(t, x)$ the positive eigenvector of $C(t, x)$ for $L(t, x, 0)$ such that $|\xi(t, x)| = 1$. At least for a subsequence $(\varepsilon_m, \alpha_m)$ which tends to $(0, 0)$ and such that $\alpha_m = o(\varepsilon_m)$, $(C(t, x), \xi(t, x), L(t, x, 0))$ converges to (C, ξ, L), where $C > 0$, L is its Hamiltonian and $C\xi = L\xi$. According to the Perron–Frobenius theorem, $\xi > 0$. Moreover, using that ([11], e.g.),

$$L(s, y, 0) = \min_{\xi > 0} \max_{1 \leq i \leq k} \frac{(C(s, y)\xi)_i}{\xi_i} = \max_{\xi > 0} \min_{1 \leq i \leq k} \frac{(C(s, y)\xi)_i}{\xi_i},$$

one can prove:

LEMMA 3.11. *There exists $K' \in (0, \infty)$ such that, for all $(\varepsilon_m, \alpha_m)$,*

$$|L(s, y, 0) - L(t, x, 0)| \leq K'(|x - y| + |t - s|).$$

Moreover, $L^{\mu}(s, y, 0) = \mu e^{-s} L(0, y, 0)$ and $L(0, y, 0) \in [0, kK]$. Then

$$\begin{aligned} L^{\mu}(s, y, 0) - L(t, x, 0) &\geq k(\mu - 1)kK - K'(|x - y| + |t - s|) \\ &\geq -\mu\gamma\delta e^{-T} + \frac{\gamma\delta}{2}e^{-T} + o_{\varepsilon, \alpha}(1). \end{aligned}$$

If we choose

$$\mu = \frac{k^2 K e^{-\delta} + \gamma\delta e^{-T}/2}{k^2 K e^{-\delta} + \gamma\delta e^{-T}},$$

then

$$M_{\varepsilon, \alpha} + o_{\varepsilon, \alpha}(1) \leq -\frac{\gamma\delta}{2}e^{-T},$$

according to Lemma 4.3 of [3], which is a contradiction with $M > 0$ according to the same result. Then

$$\sup_{[\delta,T]\times R^d} (u'(t,x) - v'(t,x)) \le kK \max(\delta e^{-\delta}, Te^{-T}).$$

But v is locally bounded and μ tends to 1 when γ tends to 0. Then, for all $R > 0$,

$$\lim_{\gamma\to 0} \sup_{[\delta,T]\times B_R} (u'(t,x) - v'(t,x)) = \sup_{[\delta,T]\times B_R} (u(t,x) - v(t,x))e^{-t}.$$

Hence,

$$\sup_{[\delta,T]\times R^d} (u(t,x) - v(t,x))e^{-t} \le kK \max(\delta e^{-\delta}, Te^{-T}).$$

But, for all $0 < \delta' \le \delta < T \le T' < \infty$,

$$\sup_{[\delta,T]\times R^d} (u(t,x) - v(t,x))e^{-t} \le \sup_{[\delta',T']\times R^d} (u(t,x) - v(t,x))e^{-t},$$

which completes the proof. □

COROLLARY 3.12. *$\varepsilon \ln u_l^\varepsilon$ converges uniformly to v^* on compact subsets $]0,\infty[\times\mathbb{R}^d$ for all $l \in [\![1,k]\!]$.*

COROLLARY 3.13. *If u and v are two viscosity solutions of* (10) *and satisfy* (11), *then $u \equiv v$.*

Now, we can prove the asymptotic results on u^ε and $\varepsilon \ln u^\varepsilon$.

THEOREM 3.14. *$\sup_{1\le l\le k} u_l^\varepsilon$ converges uniformly to 0 on compact subsets of E.*

PROOF. This is a consequence of Corollary 3.12: by continuity, v^* is uniformly negative on compact subsets of E. □

THEOREM 3.15. *For all $l \in [\![1,k]\!]$, $\varepsilon \ln u_l^\varepsilon$ converges uniformly to V^* on compact subsets of $(0,\infty)\times\mathbb{R}^d$.*

PROOF. According to Corollary 3.13, we just have to show the result for strongly coupled KPP equations, that is, when, for all $(x,l,u) \in \mathbb{R}^d \times [\![1,k]\!] \times ([0,\infty))^k$,

$$\bar{f}_l(x,u) = c(x,l)u_l(1-u_l) + \sum_{i=1}^{k-1} c(x,l,i)(u_{l+i} - u_l).$$

Let K be a compact subset of $[t_0, T] \times \mathbb{R}^d$ where $0 < t_0 \le T < \infty$. Let $(t, x) \in K$, $\tau \in \Theta_t$. We omit indices. We work under the probability under which the intensity of $N(i)$ is $c(X, \nu, i)/\varepsilon$:

$$\begin{aligned} Y_0 &\le E Y_\tau \exp\left\{ \frac{1}{\varepsilon} \int_0^\tau c(X_r, \nu_r)\, dr \right\} \\ &\le (\Lambda \vee \bar{g}) E \mathbb{1}_{\tau < t} \exp\left\{ \frac{1}{\varepsilon} \int_0^\tau c(X_r, \nu_r)\, dr \right\} \\ &\quad + (\Lambda \vee \bar{g}) E \mathbb{1}_{X_t \in G_0} \mathbb{1}_{\tau = t} \exp\left\{ \frac{1}{\varepsilon} \int_0^t c(X_r, \nu_r)\, dr \right\}. \end{aligned}$$

Denote $B = \{\tau < t\} \cup \{\tau = t,\ X_t \in G_0\}$. According to Proposition A.6,

$$\begin{aligned} &\limsup_{\varepsilon \downarrow 0} \varepsilon \ln E \mathbb{1}_B \exp\left\{ \frac{1}{\varepsilon} \int_0^\tau c(X_r, \nu_r)\, dr \right\} \\ &\qquad \le \sup_{\varphi, \mu} \{ R_{0\tau}(\varphi, \mu),\ \varphi_0 = x,\ \varphi_t \in G_0 \}. \end{aligned}$$

Hence, $v^*(t, x) \le V^*(t, x)$, which completes the proof on M. Let us show the lower bound on E. Let $(t, x) \in E$, $\alpha > 0$. We set

$$\tau^\alpha = \inf\{ s \le t,\ v^*(t - s, \varphi_s) > -\alpha \}.$$

Let φ be such that $R_{0\tau}(\varphi) \ge V^*(t, x) - \alpha$, $\varphi_0 = x$ and $\varphi_t \in G_0$. Notice that $\tau(\varphi) < t$. Let us choose $\beta \in]0, t - \tau(\varphi)[$, where τ is upper semicontinuous. Hence, there exists $\delta > 0$ such that

$$\sup_{s \in [0, t]} |\psi_s - \varphi_s| < \delta \Rightarrow \tau(\psi) < \tau(\varphi) + \beta.$$

Then, according to the definition of Markov times,

$$\|\psi - \varphi\|_\beta \equiv \sup_{s \in [0, \tau(\varphi) + \beta]} |\psi_s - \varphi_s| < \delta \Rightarrow \tau(\psi) < \tau(\varphi) + \beta.$$

Moreover, we choose δ such that

$$\|\psi - \varphi\|_\beta < \delta \Rightarrow \forall\ s \in [0, \tau(\varphi) + \beta],\ |v^*(t - s, \psi_s) - v^*(t - s, \varphi_s)| \le \frac{\alpha}{2}.$$

For a given $\gamma > 0$, if ε is small enough, Corollary 3.12 and Theorem 3.14 lead to

$$\begin{aligned} Y_0 &\ge E Y_{\tau(\varphi) + \beta} \mathbb{1}_{\|X - \varphi\|_\beta < \delta} \exp\left\{ \frac{1}{\varepsilon} \int_0^{\tau(\varphi) + \beta} c(X_r, Y_r)\, dr \right\} \\ &\ge E \mathbb{1}_{\|X - \varphi\|_\beta < \delta} \exp\left\{ \frac{1}{\varepsilon} \left(\int_0^{\tau(\varphi) + \beta} c(X_r)\, dr - 2\alpha - \gamma \right) \right\}. \end{aligned}$$

According to Varadhan's theorem,

$$\begin{aligned} v^*(t, x) \ge{} & \sup_\psi \left\{ \int_0^{\tau(\varphi) + \beta} c(\psi_r)\, dr - S_{0\tau(\varphi) + \beta}(\psi),\ \psi_0 = x,\ \|\psi - \varphi\|_\beta < \delta \right\} \\ & - 2\alpha - \gamma. \end{aligned}$$

If γ and β tend to 0, then, for all $\alpha > 0$, $v^*(t,x) \geq R_{0\tau(\varphi)}(\varphi) - 2\alpha \geq V^*(t,x) - 3\alpha$. $\square$

THEOREM 3.16. *There exists $h > 0$ such that*

$$\liminf_{\varepsilon \downarrow 0} \inf_{1 \leq l \leq k} Y_0^{\varepsilon,t,x,l} \geq h$$

uniformly on compact subsets of $\mathring{M}$.

PROOF. According to the assumptions, we can write $\bar{f}$ as follows:

$$\begin{aligned}\bar{f}_l(x,u) &= \sum_{i=1}^{k} c(x,l,i,u)u_{l+i} \\ &= c(x,l,u)u_l + \sum_{i=1}^{k-1} c(x,l,i,u)(u_{l+i} - u_l),\end{aligned}$$

where all $c(\cdot, l, i, \cdot)$ are bounded continuous functions. There exists $h > 0$ such that, for all $x \in \mathbb{R}^d$, $u \in ([0,h])^k$, $l, i \in [\![1,k]\!]$,

$$c(x,l,i,u) \geq \frac{1}{2K}.$$

Let $\eta > 0$ and $\bar{f}' \leq \bar{f}$ be a function which satisfies Assumption 3.1 and such that, for all $x \in \mathbb{R}^d$, $u \in ([0,h])^k$, $l \in [\![1,k]\!]$,

$$\bar{f}'(x,l,u) = (c(x,l) - k\eta)u_l\left(1 - \frac{u_l}{h}\right) + \sum_{i=1}^{k-1}(c_{li}(x) - \eta)(u_i - u_l).$$

R^η is defined like R but $C(x)$ is replaced by $C(x) - \eta\mathbf{1}$ and we denote $M^\eta = \{V^{*\eta} = 0\}$. If $(t,x) \in \mathring{M}$, there exists $\eta > 0$ such that $(t,x) \in \mathring{M}^\eta$ (proof of in [25]). This inclusion shows that we can choose $\eta > 0$ for a compact subset of $\mathring{M}$. Denote by $g' = g \wedge h$ and (Y', H', Z') the solution of the BSDE associated with the system where $\bar{f}$ and g are replaced by $\bar{f}'$ and g'. Applying comparison theorems leads to $0 \leq Y' \leq h$. Moreover, under the probability under which the intensity of $N(i)$ is $(c(X,\nu,i) - \eta)/\varepsilon$, Y' is a supermartingale and for all Markov time τ,

$$Y_0' = EY_\tau' \exp\left\{\frac{1}{\varepsilon}\int_0^\tau (c(X_r,\nu_r) - k\eta)\left(1 - \frac{Y_r'}{h}\right) dr\right\}.$$

Using the arguments of the proof of Theorem 3.17 or Theorem 4.7 in [25], we show that Y' converges uniformly to h on compact subsets of M^η. The inequality $Y' \leq Y$ completes the proof. $\square$

3.3. *One example of stabilization.* We now give an example where u^ε converges to the stable equilibrium on M without Lyapounov assumption. For the sake of simplicity, $\bar{g} \leq 1$ and we assume that the stable equilibrium point is **1**. The L_l^ε operators are given in the previous section but the nonlinearities have the following form:

$$\bar{f}_l(x,u) = c(x,l,l,u)u_l + \sum_{i=1}^{k-1} c(x,l,i,u)u_{l+i}$$
$$= c(x,l,u)u_l + \sum_{i=1}^{k-1} c(x,l,i,u)(u_{l+i} - u_l),$$

where, for all $(l,i) \in [\![1,k]\!] \times [\![1,k-1]\!]$ and all $(x,u) \in \mathbb{R}^d \times ([0,\infty))^k$:

(a) $1/K \leq c(x,l,i,u) \leq K$;

(b) if $u_l = 1$, then $c(x,l,u) \equiv 0$, and if $u_l \in [0,1[$, then $c(x,l,u) > 0$;

(c) there exists $\alpha \in C(\mathbb{R},\mathbb{R})$ uniformly Lipschitz continuous such that, for all $(x,l,u) \in \mathbb{R}^d \times [\![1,k]\!] \times \mathbb{R}^k$, $u_l \in [0,1[$,

$$\tilde{c}(x,l,u_l,h^l) = c(x,l,u) \geq \alpha(u_l) > 0.$$

Hence, $0 \leq Y^\varepsilon \leq 1$ and, under the probability under which the intensity of $N(i)$ is $c(X,\nu,i,Y^\varepsilon,H^\varepsilon)/\varepsilon$, Y^ε is a supermartingale. Moreover, as we already mentioned in the previous section,

$$Y_0 = EY_\tau \exp\left\{\frac{1}{\varepsilon}\int_0^\tau c(X_r,\nu_r,Y_r,H_r)\,dr\right\}.$$

THEOREM 3.17. $\lim_{\varepsilon\downarrow 0} Y_0^{\varepsilon,t,x,l} = 1$ *uniformly on compact subsets of* $\mathring{M}$.

PROOF. The proof is similar to the proof of Theorem 4.7 in [25]. Let K be a compact subset of $\mathring{M}$ and K' a compact neighborhood of K in $\mathring{M}$. Furthermore, $\tau(\cdot,\cdot)$ is defined by

$$\tau(t,\varphi) = \inf\{s \leq t : (t-s,\varphi_s) \notin K'\}$$

and $\eta > 0$. Using that v^ε converges uniformly to 0 on K' and that $\alpha(Y_r) \geq 0$ $dP \times dr$ a.s., we prove that $E\int_0^{\tau(t,X)} \alpha(Y_r)\,dr$ converges uniformly to 0 on K.

Since every coefficient is bounded by K and K' is a compact neighborhood of K, there exists $\delta > 0$ such that, for all $(t,x) \in K$, the distance between (t,x) and the complementary of K' is greater than $2K\delta$. Therefore, $P((t-s, X_s^{\varepsilon,x})_{0\leq s\leq\delta} \subset K')$ converges uniformly to 1 on K.

Let us assume that $Y_0^{\varepsilon,t,x,l}$ does not converge uniformly to 1 on K. Let $h > 0$ be such that

$$\forall\, \varepsilon_0 > 0,\ \exists\, \varepsilon \in (0,\varepsilon_0), \qquad (t^\varepsilon, x^\varepsilon, l^\varepsilon) \in K \times [\![1,k]\!], \qquad Y_0^{\varepsilon,t^\varepsilon,x^\varepsilon} \leq 1-h.$$

Denote $\tau^\varepsilon = \tau(t^\varepsilon, X^{\varepsilon,x^\varepsilon})$. The two previous remarks imply

$$\lim_{\varepsilon\downarrow 0} P(\tau^\varepsilon \geq \delta) = 1 \quad \text{and} \quad \lim_{\varepsilon\downarrow 0} E\int_0^{\tau^\varepsilon} \alpha(Y_r)\,dr = 0.$$

Set $\beta^{\varepsilon} = \inf\{s, Y_s^{\varepsilon, t^{\varepsilon}, x^{\varepsilon}} \geq 1 - h/2\}$ and $\gamma^{\varepsilon} = \beta^{\varepsilon} \wedge \tau^{\varepsilon}$.

$$E\int_0^{\tau^{\varepsilon}} \alpha(Y_r)\, dr \geq E\gamma^{\varepsilon} \min_{0 \leq y \leq 1-h/2} \alpha(y) \geq 0.$$

This leads to

$$\lim_{\varepsilon \downarrow 0} E\gamma^{\varepsilon} \cdot \min_{0 \leq y \leq 1-h/2} \alpha(y) = 0.$$

Hence, $\lim_{\varepsilon \downarrow 0} E\gamma^{\varepsilon} = 0$. But $\lim_{\varepsilon \downarrow 0} P(\tau^{\varepsilon} \geq \delta) = 1$, which implies that $P(\gamma^{\varepsilon} = \beta^{\varepsilon}, \beta^{\varepsilon} < t)$ converges to 1. Therefore, $P(Y_{\gamma^{\varepsilon}}^{\varepsilon, t^{\varepsilon}, x^{\varepsilon}, l^{\varepsilon}} = 1 - h/2)$ also tends to 1, and $\lim_{\varepsilon \downarrow 0} EY_{\gamma^{\varepsilon}}^{\varepsilon, t^{\varepsilon}, x^{\varepsilon}} = 1 - h/2$, according to the Lebesgue theorem. We have a contradiction with the fact that Y is a supermartingale. □

Now, we briefly study the speed of convergence to $\mathbf{1}$ if $c(x, l, u)u_l = d(x, l, \mathbf{1} - u)(u_l - 1)$ with $d(\cdot, l, \cdot)$ continuous and bounded. Set $d(x, l, 0) \equiv d(x, l) \leq 0$ and

$$\tau_M(t, \varphi) = \inf\left\{s \leq t: (t - s, \varphi_s) \notin \mathring{M}\right\},$$

$$Q_{0s}(\varphi, \mu) = \int_0^s \sum_{l=1}^k d(\varphi_r, l)\dot{\mu}_r(l)\, dr - S'_{0s}(\varphi, \mu),$$

$$J(t, x) = \sup_{\varphi, \mu}\left\{Q_{0\tau_M}(\varphi, \mu): \varphi_0 = x, \ \varphi_t \in H_0\right\},$$

where $H_0 = \{x \in \mathbb{R}^d: \exists\ l \in [\![1, k]\!], 1 - g(x, l) > 0\}$, and, for all $l \neq i$, $d_{li}(x) = c_{li}(x, \mathbf{1})$ replaces $c_{li}(x, 0)$ in the definition of S'. Let $D(x) = (d_{li}(x))_{1 \leq l, i \leq k}$ and let $M(x, p)$ be the Hamiltonian of

$$D(x) + \operatorname{Diag}\left(\tfrac{1}{2}\|\sigma_l^*(x)\nabla w(t, x)\|^2 + \left(b_l(x), \nabla w(t, x)\right)\right).$$

As for a single equation, the following result is true:

THEOREM 3.18. *$\varepsilon \ln(1 - Y_0^{\varepsilon, t, x, l})$ converges uniformly to $J(t, x)$ on compact subsets of $\mathring{M}$ and J is a viscosity solution of*

$$\begin{aligned} &\frac{\partial w}{\partial t}(t, x) - M(x, \nabla w(t, x)) = 0, && (t, x) \in \mathring{M}, \\ &w(t, x) = 0, && (t, x) \in \overline{E},\ t > 0, \\ &\lim_{t \downarrow 0} w(t, x) = -\infty, && x \notin \overline{H_0}, \\ &w(0, x) = 0, && x \in H_0, \end{aligned} \tag{12}$$

PROOF. We refer to the proof of Theorem 4.10 in [25]. Here $a^\varepsilon = 1 - u^\varepsilon$ is a solution of

$$\begin{aligned}\frac{\partial a_l^\varepsilon}{\partial t}(t,x) &= L_l^\varepsilon a_l^\varepsilon(t,x)\\ &\qquad + \frac{1}{\varepsilon}\left(d(x,l,a^\varepsilon)a_l^\varepsilon + \sum_{i=1}^{k-1} c(x,l,i,u^\varepsilon)(a_{l+i}^\varepsilon - a_l^\varepsilon)\right),\\ &\qquad\qquad t>0,\ x\in\mathbb{R}^d,\\ a_l^\varepsilon(0,x) &= 1-g_l(x), \qquad x\in\mathbb{R}^d.\end{aligned}$$

The technique used to study the speed of convergence to 0 still applies since one can prove a uniqueness theorem for (12) in a good class of solutions. Hence, $\varepsilon \ln(1 - Y_0^{\varepsilon,t,x,l})$ converges uniformly to $J(t,x)$ and J is a solution of (12) under our strong assumptions.

According to the uniqueness theorem, we can choose $c(x,l,i,u) = c(x,l,i,\mathbf{1})$ to compute J. If under the probability P the intensity of $N(i)$ is $c(X,\nu,i,\mathbf{1})/\varepsilon$, then

$$1 - Y_0 = E(1-Y_\tau)\exp\left\{\frac{1}{\varepsilon}\int_0^\tau d(X_r,\nu_r,1-Y_r,H_r)\,dr\right\}.$$

Since $d(x,l,a) \le 0$ if $a\in[0,1]^k$, the computation given in Section 4.4 of [25] is still valid. □

4. The gradient-dependent case.

Now, we want to study the case where $\bar f$ depends on ∇u^ε. More precisely, we consider the following systems:

$$\begin{aligned}\frac{\partial u_l^\varepsilon}{\partial t}(t,x) &= L_l^\varepsilon u_l^\varepsilon(t,x) + \frac{1}{\varepsilon}\bar f(x,l,u^\varepsilon(t,x),m(\varepsilon)\nabla u_l^\varepsilon\sigma_l(t,x)),\\ &\qquad\qquad t>0,\ x\in\mathbb{R}^d,\ l\in[\![1,k]\!],\\ u_l(0,x) &= g_l(x), \qquad x\in\mathbb{R}^d,\ l\in[\![1,k]\!],\end{aligned}\tag{13}$$

where $\limsup \varepsilon \ln m(\varepsilon) \le 0$ when ε tends to 0. The basic ideas are like those used in [25] to study a single KPP equation which is nonlinear in ∇u^ε. However, the following study is more difficult since Y may not be a supermartingale, even if we assume g very small. We first set our new assumptions and then we show that Z_0 is controlled by Y_0 if $\bar f$ satisfies strong assumptions. Moreover, we need that *every* operator is uniformly elliptic. Then, by comparison, we prove the result in the general case. Denote $K\in(1,\infty)$ such that:

(a) for all $l\in[\![1,k]\!]$, $(b_{\varepsilon,l})_{0<\varepsilon}\subset C^1(\mathbb{R}^d,\mathbb{R}^d)$, $\sigma_l\in C^1(\mathbb{R}^d,\mathbb{R}^{d\times d})$ satisfy Assumptions 2.1 and

$$\frac{1}{K}|x'| \le |\sigma_l(x)\cdot x'| \le K|x'|;$$

(b) for all $l \in [\![1,k]\!]$, $\bar f_l \in C^1(\mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^d, \mathbb{R})$ and $g_l \in C^1(\mathbb{R}^d, [0,\infty[)$ and $\bar f$ and g satisfy Assumptions 3.1 uniformly with respect to the last variable if we denote, for all $x, z \in \mathbb{R}^d$, $l, i \in [\![1,k]\!]$,

$$c(x,l,i,z) = \frac{\partial \bar f_l}{\partial u_{l+i}}(x,0,z), \qquad c_{li}(x) = \frac{\partial \bar f_l}{\partial u_i}(x,0,0);$$

(c) for all $\gamma > 0$, there exists $B(\gamma) > 0$ such that, for all $(x,l,u,z) \in \mathbb{R}^d \times [\![1,k]\!] \times [0, B(\gamma)]^k \times [0, B(\gamma)]^d$,

$$\sum_{i=1}^k (c_{li}(x) - \gamma) u_i \le \bar f_l(x,u,z);$$

(d) for all $x \in \mathbb{R}^d$, $l \in [\![1,k]\!]$, $u, u' \in \mathbb{R}^k$, $z, z' \in \mathbb{R}^d$,

$$\left|\frac{\partial \bar f_l}{\partial x}(x,u,z)\right| \le K(1+|x|^p),$$

$$\left|\frac{\partial \bar f_l}{\partial u}(x,u,z)\right| + \left|\frac{\partial \bar f_l}{\partial z}(x,u,z)\right| \le K;$$

(e) for all $l, i \in [\![1,k]\!]$, $x, x', z, z' \in \mathbb{R}^d$,

$$|c(x,l,i,z) - c(x',l,i,z')| \le K(|x-x'| + |z-z'|).$$

The process $(X^{\varepsilon,x,l}, \nu^l)$ is the solution of (3) and $(Y^{\varepsilon,t,x,l}, H^{\varepsilon,t,x,l}, Z^{\varepsilon,t,x,l})$ is the unique solution of

$$\begin{aligned} Y_s^{\varepsilon,t,x,l} &= g\big(X_t^{\varepsilon,x,l}, \nu_t^l\big) + \frac{1}{\varepsilon}\int_s^t f\big(X_r^{\varepsilon,x,l}, \nu_r^l, Y_r^{\varepsilon,t,x,l}, H_r^{\varepsilon,t,x,l}, Z_r^{\varepsilon,t,x,l}\big)\,dr \\ &\quad - \frac{1}{p(\varepsilon)}\int_s^t Z_r^{\varepsilon,t,x,l}\,dW_r - \int_s^t \sum_{i=1}^{k-1} H_r^{\varepsilon,t,x,l}(i)\,d\overline{N}_r(i), \end{aligned} \tag{14}$$

where $p(\varepsilon) = m(\varepsilon)/\sqrt{\varepsilon}$. Then, according to [24], $u_l^\varepsilon(t,x) \equiv Y_0^{\varepsilon,t,x,l}$.

4.1. *Control of* $|\nabla u_l^\varepsilon|$ *by* u_l^ε. According to Theorem 3.1 in [24],

$$Z_s^{\varepsilon,t,x,l} = m(\varepsilon)\,\partial Y_s^{\varepsilon,t,x,l}\big(\partial X_s^{\varepsilon,x,l}\big)^{-1}\sigma\big(X_s^{\varepsilon,x,l}, \nu_s^l\big),$$

where $\partial X^{\varepsilon,x,l}$ is the unique solution of the linear equation obtained by formal derivation with respect to x of the SDE of (3); $\partial Y^{\varepsilon,t,x,l}$ is defined in the same way by derivation of (14). However, in order to be able to choose a probability under which Y is a supermartingale, we assume that, for all $u \in [0,\alpha]^k$:

$$\bar f_\alpha(x,l,u,z) = c(x,l,z)\,u_l\left(1 - \frac{u_l}{\alpha}\right) + \sum_{i=1}^{k-1} c(x,l,i,z)(u_{i+l} - u_l),$$

$c(\cdot,l,\cdot) \ge 1/K$ and $c(\cdot,l,i,\cdot) \ge 1/K$ are smooth, bounded with bounded derivatives and $\bar g \le \alpha$. Denote by $\tilde P^\varepsilon$ the probability under which the intensity of $N(i)$ is $c(X_r, \nu_r, i, Z_r)/\varepsilon$ and $\tilde{E}^\varepsilon$ the associate expectation. We write $\bar{f}$ instead of $\bar{f}_\alpha$ in this section. A direct consequence of the Itô formula is:

LEMMA 4.1. *For all ε, t, x, l, $Y^{\varepsilon,t,x,l}$ is a positive supermartingale and is bounded by α under $\tilde{P}^\varepsilon$. Moreover,*

$$\begin{aligned}\tilde{E}^\varepsilon \frac{1}{\varepsilon}\int_0^t |Z_s^{\varepsilon,t,x,l}|^2\,ds + \tilde{E}^\varepsilon \frac{1}{\varepsilon}\int_0^t \sum_{i=1}^{k-1} |H_s^{\varepsilon,t,x,l}(i)|^2 c\big(X_s^{\varepsilon,x,l}, \nu_s^l, i, Z_s^{\varepsilon,t,x,l}\big)\,ds&\\ \le 2\alpha Y_0^{\varepsilon,t,x,l}.&\end{aligned}$$

Denote $\overline{m}(\varepsilon) = m^2(\varepsilon) \vee m(\varepsilon) \vee \varepsilon^{3/2}$.

PROPOSITION 4.2. *Let K be a compact subset of $[0,\infty[\times\mathbb{R}^d$. There exists $M > 0$ such that, for all $(t,x) \in K$, $\varepsilon \in (0,1)$,*

$$|\partial Y_0^{\varepsilon,t,x}| \le \frac{M\cdot\overline{m}(\varepsilon)}{\varepsilon^{3/2}}\big(P(X_t^{\varepsilon,x}\in G_0)\big)^{1/4} + \frac{M\cdot\overline{m}(\varepsilon)}{\varepsilon^3}\big(Y_0^{\varepsilon,t,x}\big)^{1/4}. \tag{15}$$

PROOF. Let us recall the BSDE of which $(\partial Y, \partial H, \partial Z)$ is a solution as well as some notation:

$$\begin{aligned}\bar{f}'_x(r) &= \frac{\partial\bar{f}}{\partial x}(X_r, \nu_r, Y_r, H_r, Z_r),\\ \bar{f}'_y(r) &= \frac{\partial\bar{f}}{\partial y}(X_r, \nu_r, Y_r, H_r, Z_r) = c(X_r, \nu_r, Z_r)\left(1 - \frac{2Y_r}{\alpha}\right),\\ \bar{f}'_h(r,i) &= \frac{\partial\bar{f}}{\partial h}(X_r, \nu_r, Y_r, H_r, Z_r) = c(X_r, \nu_r, i, Z_r),\\ \bar{f}'_z(r) &= \frac{\partial\bar{f}}{\partial z}(X_r, \nu_r, Y_r, H_r, Z_r),\\ \overline{N}(i) &= N(i) - \frac{1}{\varepsilon}\int_0^\cdot c(X_r, \nu_r, i, Z_r)\,dr.\end{aligned}$$

In the same way, we define $c_x(r), c_z(r), c_x(r,i), c_z(r,i)$:

$$\begin{aligned}\partial Y_s = \nabla g(X_t, \nu_t)\,\partial X_t - \frac{1}{\sqrt{\varepsilon}}\int_s^t \partial Z_r\,dWr - \int_s^t \sum_{i=1}^{k-1} \partial H_r(i)\,dN_r(i)&\\ + \frac{1}{\varepsilon}\int_s^t \bigg[\bar{f}'_x(r)\,\partial X_r + \bar{f}'_y(r)\,\partial Y_r + \bar{f}'_z(r)\,\partial Z_r&\\ + \sum_{i=1}^{k-1} \bar{f}'_h(r,i)\,\partial H_r(i)\bigg]\,dr.&\end{aligned}$$

For $s = t$, $\partial Y_t = \nabla g(X_t, \nu_t)\,\partial X_t$, and for $s = 0$,

$$\partial Y_0 = \tilde{E}\,\partial Y_t + \frac{1}{\varepsilon}\tilde{E}\int_0^t \Big[\bar{f}'_x(r)\,\partial X_r + \bar{f}'_y(r)\,\partial Y_r + \bar{f}'_z(r)\,\partial Z_r \Big]\,dr,$$

which leads to

$$\begin{aligned} |\partial Y_0| &\le \tilde{E}|\partial Y_t| \\ &\quad + \frac{1}{\varepsilon}\tilde{E}\int_0^t \Big[|\bar{f}'_x(r)|\cdot|\partial X_r| + |\bar{f}'_y(r)|\cdot|\partial Y_r| + |\bar{f}'_z(r)|\cdot|\partial Z_r| \Big]\,dr. \end{aligned} \tag{16}$$

We are going to bound each term. Let M be a real positive number that may change from one line to another but that remains independent of ε.

LEMMA 4.3. *There exists $M > 0$ such that, for $n \in \{1, 2\}$, $\forall (t, x) \in K$, $0 < \varepsilon < 1$,*

$$\tilde{E}|g'(X_t, \nu_t)\cdot \partial X_t|^n \le M\sqrt{P(X_t \in G_0)}$$

$$\tilde{E}\frac{1}{\varepsilon}\int_0^t |\bar{f}'_x(r)|\cdot|\partial X_r|\,dr \le \frac{M}{\varepsilon}\sqrt{Y_0} \quad \forall (t, x) \in K,\ 0 < \varepsilon < 1,$$

$$\tilde{E}\frac{1}{\varepsilon}\int_0^t |\bar{f}'_y(r)|\cdot|\partial Y_r|\,dr \le \frac{M}{\varepsilon^{3/2}}\sqrt{Y_0} \quad \forall (t, x) \in K,\ 0 < \varepsilon < 1,$$

$$\tilde{E}\frac{1}{\varepsilon}\int_0^t |\bar{f}'_z(r)|\cdot|\partial Z_r|\,dr \le \frac{M\cdot p(\varepsilon)}{\varepsilon}\left(P(X_t \in G_0)^{1/4} + \frac{p(\varepsilon)\vee 1}{\varepsilon}(Y_0)^{1/4}\right).$$

PROOF. We will only give the proof of the last upper bound which includes the different technics used to show the others. Notice that $|\bar{f}'_z(r)| \le M$. Then

$$\begin{aligned} &\tilde{E}\frac{1}{\varepsilon}\int_0^t |\bar{f}'_z(r)|\cdot|\partial Z_r|\,dr \\ &\qquad \le \frac{M\cdot p(\varepsilon)}{\varepsilon}\left(\tilde{E}\frac{1}{p(\varepsilon)^2}\int_0^t |\partial Z_r|^2\,dr\right)^{1/2}. \end{aligned}$$

The Itô formula leads to

$$\begin{aligned} \tilde{E}\frac{1}{p(\varepsilon)^2}\int_0^t |\partial Z_r|^2\,dr &\le \tilde{E}|\partial Y_t|^2 \\ &\quad + \frac{2}{\varepsilon}\tilde{E}\int_0^t \Big(\bar{f}'_x(r)\,\partial X_r + \bar{f}'_y(r)\,\partial Y_r + \bar{f}'_z(r)\,\partial Z_r, \partial Y_r \Big)\,dr. \end{aligned}$$

Set $q(\varepsilon) = p(\varepsilon) \vee 1$. Using that $2(a, b) \le |a|^2 + |b|^2$ and that $\bar{f}'_y$ and $\bar{f}'_z$ are bounded, we get

$$\begin{aligned} \tilde{E}\frac{1}{2\,p(\varepsilon)^2}\int_0^t |\partial Z_r|^2\,dr &\le \tilde{E}|\partial Y_t|^2 + \frac{M}{\varepsilon}\tilde{E}\int_0^t |\bar{f}'_x(r)\,\partial X_r|^2\,dr \\ &\quad + \frac{M\cdot q(\varepsilon)^2}{\varepsilon}\tilde{E}\int_0^t |\partial Y_r|^2\,dr. \end{aligned}$$

After a short computation (cf. Lemma 5.6 in [25]), we also get

$$\tilde{E}\int_0^t |\partial Y_r|^2\, dr \le \frac{M'}{\varepsilon}\left(\tilde{E}\left(\frac{1}{p(\varepsilon)^2}\int_0^t |Z_r|^2\, dr\right)^2\right)^{1/2} \le \frac{M}{\varepsilon}\sqrt{Y_0}\,.$$

Hence,

$$\tilde{E}\frac{1}{p(\varepsilon)^2}\int_0^t |\partial Z_r|^2\, dr \le \frac{M}{\varepsilon}\big(P(X_t \in G_0)\big)^{1/2} + \frac{M\cdot q(\varepsilon)^2}{\varepsilon^2}\sqrt{Y_0}\,.$$

Therefore,

$$\tilde{E}\frac{1}{\varepsilon}\int_0^t |\bar{f}'_z(r)|\cdot|\partial Z_r|\, dr \le \frac{M\cdot p(\varepsilon)}{\varepsilon}\left(P(X_t \in G_0)^{1/4} + \frac{q(\varepsilon)}{\varepsilon}(Y_0)^{1/4}\right). \qquad \square$$

Proposition 4.2 follows from the four results given previously. □

COROLLARY 4.4. *Let K be a compact subset of $\mathbb{R}_+^* \times \mathbb{R}^d$. There exists $M > 0$ such that, for all $(t, x) \in K$ and all $\varepsilon \in]0, 1[$,*

$$|Z_0^{\varepsilon,t,x}| < \frac{M\cdot m(\varepsilon)\overline{m}(\varepsilon)}{\varepsilon^{3/2}}\big(P(X_t^{\varepsilon,x} \in G_0)\big)^{1/4} + \frac{M\cdot m(\varepsilon)\cdot\overline{m}(\varepsilon)}{\varepsilon^3}\big(Y_0^{\varepsilon,t,x}\big)^{1/4}.$$

4.2. *Asymptotic behavior of u^ε and $\varepsilon \ln Y^\varepsilon$.* We follow the same approach used in the previous section. We show that $\varepsilon \ln u^\varepsilon$ converges uniformly to V^* on compact subsets of $(0,\infty)\times\mathbb{R}^d\times[\![1,k]\!]$. Then we study the asymptotic behavior of u^ε on $E = \{V^* < 0\}$ and $M = \{V^* = 0\}$. Let us recall the definition of V^* given in Definition 3.1:

$$V^*(t,x) = \inf_{\tau\in\Theta_t}\sup_{\varphi,\mu}\{R_{0\tau}(\varphi,\mu),\ \varphi_0 = x,\ \varphi_t \in G_0\},$$

where

$$R_{0a}(\varphi,\mu) = \int_0^a \sum_{l=1}^k c(\varphi_s, l)\dot{\mu}_s(l)\, ds - S_{0a}(\varphi,\mu),$$

and S is the action of $(X^\varepsilon, U^\varepsilon)$. We only write the points where the proofs are different from the previous section.

LEMMA 4.5. *u^ε is a viscosity solution of*

$$\begin{aligned}\frac{\partial u_l^\varepsilon}{\partial t}(t,x) &= L_l^\varepsilon u_l^\varepsilon(t,x)\\ &\qquad + \frac{1}{\varepsilon}\bar{f}\big(x, l, u^\varepsilon(t,x), Z_0^{\varepsilon,t,x,l}\big), \qquad t>0,\ x\in\mathbb{R}^d,\ l\in[\![1,k]\!],\\ u_l(0,x) &= g_l(x), \qquad x\in\mathbb{R}^d.\end{aligned}$$

PROOF. This is a simple generalization of the proof of Theorem 4.1 in [24]. We also refer to Theorem 5.8 in [25]. □

REMARK 4.6. According to comparison theorems,

$$-\rho^2(t, x, G_0) \le v^*(t, x) \le u^*(t, x) \le \min\big(kKt - \rho^2(t, x, G_0), 0\big).$$

THEOREM 4.7. *For all $l \in [\![1, k]\!]$, $\varepsilon \ln u_l^\varepsilon(t, x)$ converges uniformly on compact subsets to V^*.*

PROOF. According to the proof of Theorem 3.15, u^* is a subsolution of (10). If there exists $\alpha > 0$ such that $\bar{f} = \bar{f}_\alpha$, then, using that $u_l^\varepsilon(t, x) \le \exp\{-k/\varepsilon\}$ implies the convergence of $Z_0^{\varepsilon, t, x, l}$ to 0, we show that v^* is a supersolution of (10). Therefore, $u^* = v^* = V^*$.

In the general case, if $v^*(t, x) < 0$, for a subsequence (ε', t', x') which tends to $(0, t, x)$ and such that $v^{\varepsilon'}(t', x')$ tends to $v^*(t, x)$, if ε' is small enough, there exist $\alpha > 0$ and $\eta > 0$ such that, for all $(l, u) \in [\![1, k]\!] \times [0, \alpha]^k$,

$$\begin{aligned}
&\bar{f}\big(x', l, u, Z_0^{\varepsilon, t', x', l}\big)\\
&\qquad \ge \sum_{i=1}^{k} \big(c_{li}(x') - \eta\big) u_i\\
&\qquad \ge \big(c(x', l) - k\eta\big) u_l \left(1 - \frac{u_l}{\alpha}\right) + \sum_{i \ne l} \big(c_{li}(x') - \eta\big)(u_i - u_l).
\end{aligned}$$

Then, $v^* \ge V^{*\eta}$. But, if $L^\eta(x, p)$ is the Hamiltonian associated with $c(x, l, i) - \eta$, L^η converges to L when η tends to 0. Moreover, if

$$V_*(t, x) = \lim_{\eta \to 0} \inf\big\{V^{*\gamma}(s, y);\ \gamma \le \eta,\ (s, y) \in B\big((t, x), \eta\big)\big\},$$

then V_* is a supersolution of (10) according to [3]. Therefore,

$$u^* \le V_* \le v^*.$$

Hence, we have the uniform convergence on compact subsets and the limit is a viscosity solution of (10). By Corollary 3.13 the proof is complete. □

THEOREM 4.8. *By our assumptions*:

1. $\lim_{\varepsilon \downarrow 0} \sup_{1 \le l \le k} Y_0^{\varepsilon, t, x, l} = 0$ *uniformly on compact subsets of* E;
2. *there exists* $h > 0$ *such that*

$$\liminf_{\varepsilon \downarrow 0} \inf_{1 \le l \le k} Y_0^{\varepsilon, t, x, l} \ge h$$

uniformly on compact subsets of $\mathring{M}$.

PROOF. The first statement is straightforward since

$$\lim_{\varepsilon \downarrow 0} \varepsilon \ln Y_0^{\varepsilon, t, x} = V^*(t, x)$$

uniformly on compact subsets. The proof of the second statement is the same as that of the proof of Theorem 3.16. □

APPENDIX

The two following propositions refer to the so-called "small loop principle" [19, 20] of which Theorem 4.1 is a straightforward consequence, according to [25]. Let us first recall the definition of (H1) and (H2):

(H1) $b(x,l) = b(x,1) \equiv b(x)$ for all $(x,l) \in \mathbb{R}^d \times [\![1,k]\!]$ and Σ satisfies the strong Hörmander condition;
(H2) $\Sigma(x)\Sigma^*(x) \geq 1/K \cdot I_d$ for all $x \in \mathbb{R}^d$.

THEOREM A.1. *If Σ satisfies the strong Hörmander condition and if φ is such that $S_{0t}(\varphi,\mu^1) < \infty$ where $\dot{\mu}^1 \equiv 1/k$, $\varphi_0 = x$, $\varphi_1 = y$ and $\varphi = F_x(h,\mu^1)$, then, for all $\varepsilon > 0$, there exists $\gamma > 0$ such that*

$$|y-z| < \gamma \Rightarrow \exists\, h' \in H^2(0,1)/\begin{cases} \displaystyle\int_0^1 |\dot{h}'_s - \dot{h}_s|^2\, ds \leq \varepsilon, \\ F_x(h',\mu^1)_1 = z. \end{cases}$$

More precisely, we will use:

PROPOSITION A.2. *We assume that* (H1) *or* (H2) *is true and we choose (φ,μ) such that $S_{01}(\varphi,\mu) < \infty$, $\varphi_0 = x$, $\varphi_1 = y$ and $\varphi = F_x(h,\mu)$. Then, for all $\varepsilon > 0$, there exists $\gamma > 0$ such that if $|y-z| < \gamma$ there exists $h' \in (I)^k$ such that*

$$\int_0^1 \Big[|\dot{h}'_s - \dot{h}_s|^2 + |\eta(\varphi_s,0,\dot{\mu}_s) - \eta(F_x(h',\mu')_s,0,\dot{\mu}'_s)|\Big]\, ds \leq \varepsilon,$$
$$F_x(h',\mu')_1 = z.$$

PROOF. We assume that (H1) is true. For a given $\theta > 0$ we define (h^θ,μ^θ) as follows:

$$\begin{aligned}
\dot{\mu}^\theta_s(l) &= \dot{\mu}_s(l) && \text{if } s \in [0,1-\theta[, \\
\dot{\mu}^\theta_s(l) &= 1/k && \text{if } s \in [1-\theta,1], \\
\dot{h}^\theta_s(l) &= \dot{h}_s(l) && \text{if } s \in [0,\mu^\theta_{1-\theta}(l)[, \\
\dot{h}^\theta_{\mu^\theta_s(l)}(l) &= k\dot{h}_{\mu_s}(l)\dot{\mu}_s(l) && \text{if } s \in [1-\theta,1], \\
\dot{h}^\theta_s(l) &= 0 && \text{if } s \in]\mu^\theta_1(l),1].
\end{aligned}$$

It is easy to see that $F_x(h^\theta,\mu^\theta) = \varphi$. Moreover,

$$\lim_{\theta\to 0} \int_0^1 \Big[|\dot{h}^\theta_s - \dot{h}_s|^2 + |\eta(\varphi_s,0,\dot{\mu}_s) - \eta(\varphi_s,0,\dot{\mu}^\theta_s)|\Big]\, ds = 0.$$

Hence, for a given $\varepsilon > 0$, there exists $\theta > 0$ such that

$$\int_0^1 \Big[|\dot{h}_s^\theta - \dot{h}_s|^2 + |\eta(\varphi_s, 0, \dot{\mu}_s) - \eta(\varphi_s, 0, \dot{\mu}_s^\theta)| \Big] \, ds \le \frac{\varepsilon}{4}.$$

Then, we apply Theorem A.1 to φ between $1 - \theta$ and 1 with $\varepsilon' = \varepsilon/4$.

Now, we assume that (H2) is true. Set $\varepsilon' = \varepsilon/2kK(2 + 8K + K^2)(1 + kK)$. Let $\delta \in]0, \varepsilon'[$ be such that $S_{1-\delta 1}^{WU}(h, \mu) \le \varepsilon'$, and denote $z_\delta = \varphi_{1-\delta}$. Then $|\varphi_1 - z_\delta|^2/\delta \le 4K\varepsilon'$. Moreover, if $z_1, z_2 \in \mathbb{R}^d$, set

$$\dot{\psi}_{s/k} = k\Sigma^*(z_s)\big(\Sigma(z_s)\Sigma^*(z_s)\big)^{-1}\bigg(\frac{z_2 - z_1}{\delta} - \bar{b}(z_s, \dot{\mu}^1)\bigg),$$

where $z_s = ((\delta - s)z_1 + sz_2)/\delta$, $z_2 = F_{z_1}(\psi, \mu^1)_\delta$ and

$$\frac{1}{2}\int_0^\delta |\dot{\psi}_s|^2 \, ds \le kK\bigg(\frac{|z_1 - z_2|^2}{\delta} + K^2\delta\bigg).$$

Hence, if $|z - y|^2 \le \varepsilon'\delta$, there exists $\psi \in H^1(0, \delta)$ such that $z = F_{z_\delta}((\psi)^k, \mu^1)_\delta$ and

$$\frac{1}{2}\int_0^\delta |\dot{\psi}_s|^2 \, ds \le kK(2 + 8K + K^2)\varepsilon' \le \frac{\varepsilon}{2},$$

where $(\psi)^k = (\psi, \dots, \psi) \in I^k$. Therefore, we just have to set

$$\big(\dot{h}'_s, \dot{\mu}'_s\big) = \begin{cases} \big(\dot{h}_s, \dot{\mu}_s\big), & \text{if } s \in [0, 1 - \delta], \\ \big(\dot{\psi}_{s'}, \dot{\mu}^1_{s'}\big), & \text{if } s = s' + t - \delta \in [1 - \delta, 1] \end{cases}$$

to satisfy the second condition. Moreover,

$$\begin{aligned} \int_0^1 \Big[|\dot{h}'_s - \dot{h}_s|^2 + |\eta(\varphi_s, 0, \dot{\mu}_s) - \eta(F_x(h', \mu')_s, 0, \dot{\mu}'_s)| \Big] \, ds \\ \le \varepsilon'(1 + kK) + \frac{\varepsilon}{2} \le \varepsilon, \end{aligned}$$

which completes the proof. $\square$

These results allow us to prove Proposition 2.14 which we rewrite.

Proposition A.3. *On* (H1) *or* (H2), *for all* $t \in]0, \infty[$, $\rho^2(t, \cdot, \cdot)$ *is continuous on* $\mathbb{R}^d \times \mathbb{R}^d$.

Proof. Proposition A.2 applied to (b, Σ) and to $(-b, -\Sigma)$ imply that, for all $\varepsilon > 0$, there exists $\gamma > 0$ such that

$$|x - x'| + |y - y'| < \gamma \Rightarrow \rho^2(t, x', y') < \rho^2(t, x, y) + \varepsilon$$

or, in other words, if $\lim(x_n, y_n) = (x, y)$,

$$\limsup_{n \to \infty} \rho^2(t, x_n, y_n) \le \rho^2(t, x, y).$$

On the other hand,

$$\liminf_{n\to\infty} \rho^2(t, x_n, y_n) \geq \rho^2(t, x, y).$$

Indeed, let (φ_n, μ_n) be such that $x_n = \varphi_0^n$, $y_n = \varphi_t^n$ and $S_{0t}(\varphi_n, \mu_n) < \rho^2(t, x_n, y_n) + 1/n$; $\liminf S_{0t}(\varphi_n, \mu_n) \leq \rho^2(t, x, y) < \infty$. Then, there exists a subsequence $(\varphi_{n'}, \mu_{n'})$ such that

$$\lim S_{0t}(\varphi_{n'}, \mu_{n'}) = \liminf S_{0t}(\varphi_n, \mu_n) < \infty.$$

Therefore, $\sup_{n'} S_{0t}(\varphi_{n'}, \mu_{n'}) < \infty$. Hence, $(\varphi_{n'}, \mu_{n'})$ is relatively compact because $(x_{n'})$ is bounded, too. One can extract $(\varphi_{n''}, \mu_{n''})$ which converges to (φ, μ) such that $\varphi_0 = x$, $\varphi_t = y$ and $S_{0t}(\varphi, \mu) \leq \liminf S_{0t}(\varphi_n, \mu_n)$. Hence,

$$\rho^2(t, x, y) \leq S_{0t}(\varphi, \mu) \leq \liminf \rho^2(t, x_n, y_n).$$

The proof is complete. $\square$

If we only have the strong Hörmander condition on Σ, we may lose the continuity of ρ. Indeed, let us consider the following case: for all $x = (x_1, x_2) \in \mathbb{R}^2$,

$$\sigma(x, 1) = \begin{pmatrix} 1 & 0 \\ 0 & x_1^3 \end{pmatrix}, \qquad b(x, 1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and $\sigma(x, 2) \equiv 0$, $b(x, 2) \equiv 0$. Moreover, we assume $c(x, 1, 2) \equiv c(x, 2, 1) \equiv c > 0$. In this case, if $\beta \in D$, $\eta(x, 0, \beta) = c(\sqrt{\beta_1} - \sqrt{\beta_2})^2 \leq c$.

THEOREM A.4. *There exists $c_0 > 0$ such that the mapping $x \mapsto \rho(1, (0, 0), x)$ from $\mathbb{R}^2$ to $\mathbb{R}$ is not continuous in $(0, 0)$.*

PROOF. It is evident that if $\mu_s^0(2) = s$ and $\varphi_s^0 = 0$, $F_{(0,0)}(\varphi^0, \mu^0)_1 = (0, 0)$. Assume that $\mu_1(1) > 0$. Set $\psi = F^1_{(0,0)}(h)$ the unique solution of

$$\dot{\psi}_s = b(\psi_s, 1) + \sigma(\psi_s, 1) \cdot \dot{h}_s,$$
$$\psi_0 = (0, 0).$$

If $\varphi = F_{(0,0)}(h, \mu)$, $\varphi = \psi \circ \mu(1)$. Hence,

$$\inf_{\varphi} \{S_{01}(\varphi, \mu),\ \varphi_0 = \varphi_1 = (0, 0)\} = \rho_1^2(\mu_1(1), (0, 0), (0, 0)),$$

where ρ_1 is the sub-Riemannian metric associated with $(b(\cdot, 1), \sigma(\cdot, 1))$. According to Ben Arous and Léandre [6],

$$\lim_{t\to 0,\, t>0} \rho_1^2(t, (0, 0), (0, 0)) = \infty.$$

Hence,

$$\inf_{\varphi,\mu} \{S_{01}(\varphi, \mu),\ \varphi_0 = \varphi_1 = (0, 0),\ \mu_1(1) > 0\} = a_1 > 0.$$

Then, if $c < a_1$, $\rho^2(1,(0,0),(0,0)) = c$ and there exists $\varepsilon > 0$ such that if $(h,\mu) \in I^k \times C_+$ and $\varphi_1 = (0,0)$ where $\varphi = F_{(0,0)}(h,\mu)$ and the associate Malliavin matrix is definite positive, then $S_{01}(\varphi,\mu) \geq c + \varepsilon = S_{01}(\varphi^0,\mu^0) + \varepsilon$. The "small loop principle" is no longer true.

Moreover, if $y > 0$, $\rho^2(1,(0,0),(0,-y)) = \inf\{\rho_1^2(t,(0,0),(0,-y)),\ 0 < t \leq 1\}$. But, if $F^1_{(0,0)}(h)_t = (0,-y)$, then $F^1_{(0,0)}(h)_{t+y} = (0,0)$ with, for all $s \in]t, t+y]$, $\dot{h}_s = 0$. Therefore,

$$\rho_1^2(t,(0,0),(0,-y)) \geq \rho_1^2(t+y,(0,0),(0,0)).$$

But, according to the preceding results, $a \equiv \inf\{\rho_1^2(t,(0,0),(0,0)),\ 0 < t \leq 2\} > 0$. Hence, if $c < a \leq a_1$, for all $y \in]0,1]$, $\rho^2(1,(0,0),(0,-y) \geq a > \rho^2(1,(0,0),(0,0))$. $\square$

We recall that $C(x) = (c(x,i,j))_{1\leq i,j\leq k}$ is Lipschitz continuous and its coefficients belong to $[1/K, K]$ and that

$$c(x,l) = \sum_{i=1}^k c(x,l,i).$$

LEMMA A.5. *The mapping from* $C([0,t],\mathbb{R}^d) \times C_+$ *to* $\mathbb{R}$ *defined by*

$$(\varphi,\mu) \mapsto \int_0^t \sum_{l=1}^k c(\varphi_s,l)\dot{\mu}_s(l)\,ds$$

is continuous for the uniform topology.

PROOF. Let $\varphi, \varphi' \in C([0,t],\mathbb{R}^d)$ and $\mu, \mu' \in C_+$. If $n \in \mathbb{N}$,

$$\varphi_s^n = \varphi_{t_i} \quad \text{if } s \in [t_i, t_{i+1}) = \left[\frac{it}{n}, \frac{(i+1)t}{n}\right).$$

Let $\alpha > 0$ and n be such that $\|\varphi - \varphi^n\| \leq \alpha$. A short computation leads to

$$\left|\int_0^t \sum_{l=1}^k [c(\varphi_s,l)\dot{\mu}_s(l) - c(\varphi'_s,l)\dot{\mu}'_s(l)]\,ds\right| \leq Kt\|\varphi - \varphi'\| + 2\alpha Kt + 2nK\|\mu - \mu'\|,$$

which completes the proof. $\square$

Set $\tau(\varphi) = \inf\{s \leq t: (t-s, \dot{\varphi}_s) \in O\}$ and $B = \{\tau < t\} \cup \{\tau = t,\ X_t \in G_0\}$.

PROPOSITION A.6. *On* (H1) *or* (H2),

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \ln E\,\mathbb{1}_B \exp\left\{\frac{1}{\varepsilon}\int_0^\tau c(X_r,\nu_r)\,dr\right\} \leq \sup_{\varphi,\mu}\{R_{0\tau}(\varphi,\mu),\ \varphi_0 = x,\ \varphi_t \in G_0\},$$

PROOF. Denote

$$\overline{B} = \{\varphi\colon\ \varphi_0 = x,\ (t-\tau,\varphi_\tau) \in \overline{O}\} \cup \{\varphi\colon\ \varphi_0 = x,\ \tau = t,\ \varphi_t \in \overline{G_0}\},$$
$$C_1 = \{\varphi\colon\ \varphi_0 = x,\ (t-\tau,\ \varphi_\tau) \in \overline{O}\},$$
$$C_2 = \{\varphi\colon\ \varphi_0 = x,\ \tau = t,\ \varphi_t \in \overline{G_0},\ (0,\varphi_t) \notin \overline{O}\}.$$

According to Varadhan's theorem,

$$\limsup_{\varepsilon \downarrow 0} \varepsilon \ln E\,\mathbb{I}_B \exp\left\{\frac{1}{\varepsilon}\int_0^\tau c(X_r,\nu_r)\,dr\right\} \le \sup_{\varphi,\mu}\{R_{0t}(\varphi,\mu),\ \varphi \in \overline{B}\}.$$

But Proposition A.2 implies the two following lemmas:

LEMMA A.7. *Let* $(\varphi,\mu) \in C_1 \times C_+$ *be such that* $\tau(\varphi) = t$ *and* $S_{0t}(\varphi) < \infty$. *On* (H1) *or* (H2), *for all* $\varepsilon > 0$, *there exists* (φ',μ') *such that* $\varphi'_0 = x$, $\tau(\varphi') < t$ *and*

$$R_{0\tau}(\varphi',\mu') \ge R_{0\tau}(\varphi,\mu) - \varepsilon.$$

LEMMA A.8. *Let* $(\varphi,\mu) \in C_2 \times C_+$ *be such that* $S_{0t}(\varphi) < \infty$. *On* (H1) *or* (H2), *for all* $\varepsilon > 0$, *there exists* $(\varphi',\mu') \in C_2 \times C_+$ *such that* $\varphi'_t \in G_0$, $\tau(\varphi) = t$,

$$R_{0t}(\varphi',\mu') \ge R_{0t}(\varphi',\mu') - \varepsilon.$$

Hence,

$$\sup_{\varphi,\mu}\{R_{0t}(\varphi,\mu),\ \varphi \in C_1\} = \sup_{\varphi,\mu}\{R_{0\tau}(\varphi,\mu),\ \varphi_0 = x,\ \varphi_t \in G_0,\ \tau < t\}$$

and

$$\sup_{\varphi,\mu}\{R_{0t}(\varphi,\mu),\ \varphi \in C_2\} = \sup_{\varphi,\mu}\{R_{0\tau}(\varphi,\mu),\ \varphi_0 = x,\ \varphi_t \in G_0,\ \tau = t\}.$$

It follows that

$$\sup_{\varphi,\mu}\{R_{0t}(\varphi,\mu),\ \varphi \in \overline{B}\} = \sup_{\varphi,\mu}\{R_{0\tau}(\varphi,\mu),\ \varphi_0 = x,\ \varphi_t \in G_0\}.$$

For more details on the proofs of the two last lemmas, we refer to [25]. □

Acknowledgments. The author would like to thank the referees for their very competent reading and their suggestions. This work was done during Ph.D. thesis at CMI, Université de Provence, Marseille, and while visiting the Institute for Systems Research at the University of Maryland at College Park.

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