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BRANCHING PROCESSES IN LÉVY PROCESSES: LAPLACE FUNCTIONALS OF SNAKES AND SUPERPROCESSES

By Jean-François Le Gall and Yves Le Jan

Ecole Normale Supérieure and Université Paris 11

We use the exploration process introduced in a previous work to develop a new construction of superprocesses with a general branching mechanism. This construction depends on a path-valued process called the Lévy snake, which is of independent interest. Our method of proof involves a calculation of the Laplace functional of the occupation field of the Lévy snake. This calculation relies on an evaluation of the corresponding moment functionals, which requires precise information about the underlying genealogical structure.

1. Introduction.

The present work is a continuation of our previous paper [13], where we developed a representation of the genealogy of a general continuous-state branching process in terms of functionals of a spectrally positive Lévy process. Our main purpose here is to apply this representation to a snake-like construction of superprocesses with a general branching mechanism.

A key role in [13] was played by the so-called height process $H = (H_t, t \geq 0)$. This process is defined as a local time functional of a Lévy process X with no negative jumps and Laplace exponent ψ. It can be interpreted informally as describing the motion of a particle which explores the continuous genealogical tree of a continuous-state branching process by moving up and down along the branches. Although H is not Markovian in general, it is closely related to a Markov process $\rho = (\rho_t,\ t \geq 0)$ taking values in the set $M_f(\mathbb{R}_+)$ of all finite measures in $\mathbb{R}_+$, which was also introduced in [13] as a functional of the Lévy process X. The process ρ is called here the exploration process (this terminology is a little different from [13]). For every $t \geq 0$, the topological support of the measure ρ_t is the interval $[0, H_t]$. Furthermore, the transition mechanism of ρ is reminiscent of the Brownian snake in [8] and [10]: If $0 \leq t < t'$, the measure $\rho_{t'}$ is obtained by first restricting ρ_t to a smaller interval $[0, m(t, t')]$ and then concatenating the restricted measure with a (random) measure $\rho^{(t)}_{t'-t}$, independent of the past until time t and distributed as $\rho_{t'-t}$.

Superprocesses are obtained by combining the branching mechanism with a spatial motion given by a general Markov process ξ. We refer to Dynkin [5], [7] and Dawson and Perkins [3] for the general theory of superprocesses, and to [6] for the connections with partial differential equations. Pathwise constructions of superprocesses were given in [8] in the quadratic branching mechanism case and via the Brownian snake, and in [14] in the stable case

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via a projective limit. In the present work, we use the pair (ρ_t, H_t) to develop a path-valued process construction of general superprocesses, similar to the Brownian snake construction. To this end we introduce a path-valued process $(W_t,\ t \geq 0)$, such that the pair (ρ_t, W_t) is a Markov process called the Lévy snake. For every $t \geq 0$, W_t is a path of the spatial motion ξ with length H_t. Furthermore, the evolution mechanism of W_t is easily described in terms of that of ρ_t, using the previous notation: If $0 \leq t < t'$, the path $W_{t'}$ is obtained by first restricting the path W_t to the interval $[0, m(t,t')]$ and then concatenating in a Markovian way a path of the spatial motion ξ whose length is the supremum of the support of $\rho^{(t)}_{t'-t}$. Theorem 5.1 below associates with the Lévy snake a superprocess with branching mechanism ψ and spatial motion ξ, whose historical paths are precisely the paths W_t.

Let $\widehat{W}_t$ denote the terminal point of the path W_t. Our method consists in proving that the occupation field of the process $(\widehat{W}_t,\ t \geq 0)$ has the same distribution as the total occupation measure of the (ξ, ψ) superprocess. To this end, we obtain an analytic expression for the Laplace functional of the occupation field of $(\widehat{W}_t,\ t \geq 0)$. The derivation of this expression depends on certain moments computations, which are of independent interest. A key role is played by Proposition 3.2, which provides a recursive description of the genealogical structure of the exploration process at n uniformly distributed instants. In the special case where the underlying Lévy process X is Brownian motion [corresponding to the quadratic branching mechanism $\psi(u) = u^2$], this genealogical structure already was pointed out in [9].

A forthcoming companion paper [11] develops the probabilistic study of the Lévy snake under slightly more stringent assumptions on the branching mechanism ψ. In particular, the strong Markov property of the Lévy snake is obtained and local times of the height process $(H_t,\ t \geq 0)$ are constructed. This makes it possible to give a more explicit expression for the associated superprocess in terms of these local times and the process $(\widehat{W}_t,\ t \geq 0)$ (this expression was announced in [12]).

The present paper is organized as follows. In Section 2, we recall the basic facts about the height process H and the exploration process ρ, and we then introduce the Lévy snake. The key Lemmas 2.1 and 2.2, which give, respectively, the invariant measure and the potential kernel of the process ρ, are crucial to our applications. Section 3 is devoted to the proof of the important Proposition 3.2, which in a sense determines the genealogical structure associated with the exploration process. In Section 4, we identify the Laplace functional of the occupation field of $\widehat{W}$ (Theorem 4.2). We rely on moments computations which follow from Proposition 3.2. Theorem 4.2 can be viewed as our principal result since the connection with superprocesses, which is developed in Section 5, then follows by standard arguments.

2. Preliminaries.

2.1. *The exploration process.* We consider a Lévy process $X = (X_t,\ t \geq 0)$ on the real line started at $X_0 = 0$ under the probability measure P. We refer to Bertoin's recent book [1] (especially Chapter 7) for the basic properties of Lévy processes that are used below. We assume that X has no negative jumps and does not drift to $+\infty$. Then, for every $\lambda \geq 0$,

$$E[\exp -\lambda X_t] = \exp(t\psi(\lambda)),$$

with a function ψ of the form

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} \pi(dr)\,(e^{-r\lambda} - 1 + r\lambda),$$

where $\alpha \geq 0$, $\beta \geq 0$ and the Lévy measure π is a Radon measure on $(0, \infty)$ such that

$$\int_{(0,\infty)} (r \wedge r^2)\,\pi(dr) < \infty.$$

We also assume that at least one of the following conditions holds: $\beta > 0$ or $\int_{(0,1)} r\,\pi(dr) = \infty$. This excludes the case when the paths of X are of finite variation. The finite variation case is treated in detail in Section 3 of [13]: It yields analogous connections with branching processes, but the underlying branching structure becomes discrete.

Note that the process X is recurrent or drifts to $-\infty$ according as $\alpha = 0$ or $\alpha > 0$.

Without loss of generality, we may and will assume that the process X is the canonical process on the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ of right-continuous paths with left limits (cadlag) in $\mathbb{R}$. We also denote by $(\mathscr{G}_t)$ the canonical filtration on $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, completed as usual by the collection of P-negligible sets of $\mathscr{G}_\infty$.

Set $S_t = \sup_{[0,t]} X_s$ and $I_t = \inf_{[0,t]} X_s$. It is well known that both processes $S - X$ and $X - I$ are strong Markov. Under our assumptions, 0 is a regular point for both these processes. Furthermore, the process $-I_t$ is a local time at 0 for $X - I$. The associated excursion measure will be denoted by N. It is a σ-finite measure on $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ such that $X_0 = 0$, N a.e. and

$$\sigma := \inf\{s > 0,\ X_s = 0\} = \sup\{s \geq 0,\ X_s > 0\} < \infty, \qquad N \text{ a.e.}$$

We also denote by $L = (L_t,\ t \geq 0)$ the local time at 0 of $S - X$. Here we need to specify the normalization of L. Set $L^{-1}(t) = \inf\{s \geq 0,\ L_s > t\}$, with $\inf \varnothing = \infty$, and make the convention that $S_{L^{-1}(t)} = \infty$ when $L^{-1}(t) = \infty$. It is well known (see, e.g., [1], Chapter 6) that both processes $(L^{-1}(t),\ t \geq 0)$ and $(S_{L^{-1}(t)},\ t \geq 0)$ are subordinators (killed at an independent exponential time in the transient case), which are called, respectively, the ladder time process and the ladder height process of X. By [2], Proposition 9, we may and will fix the normalization of L so that, for every $\lambda > 0$,

$$\begin{aligned} E[\exp(-\lambda S_{L^{-1}(t)})] &= \exp\left(-t\frac{\psi(\lambda)}{\lambda}\right) \\ &= \exp\left(-t\left(\alpha + \beta\lambda + \int_0^\infty (1 - e^{-\lambda r})\,\pi([r,\infty))\,dr\right)\right). \end{aligned} \tag{1}$$

We leave it to the reader to check that this normalization is the same as the one used in [13] in the case $\beta = 0$.

In particular, the subordinator $S_{L^{-1}(t)}$ has a drift β and, if m denotes Lebesgue measure on $\mathbb{R}$, we have

$$m(\{S_{L^{-1}(r)};\ 0 \le r \le t,\ L^{-1}(r) < \infty\}) = \beta\,(t \wedge L_\infty).$$

Thus, when $\beta > 0$, we have

$$L_t = \beta^{-1}\, m(\{S_r,\ 0 \le r \le t\}). \tag{2}$$

We now recall the definition of the *height process* H. Fix $t > 0$ and let $\widehat{X}^{(t)}$ be the time-reversed process

$$\widehat{X}^{(t)}_s = X_t - X_{(t-s)-} \quad \text{if } 0 \le s < t,$$
$$\widehat{X}^{(t)}_t = X_t.$$

Then $(\widehat{X}^{(t)}_s,\ 0 \le s \le t) \overset{(d)}{=} (X_s,\ 0 \le s \le t)$. We set $\widehat{S}^{(t)}_s = \sup_{[0,s]} \widehat{X}^{(t)}_r$ and let $H_t = \widehat{L}^{(t)}_t$ be the local time at 0 at time t of $\widehat{S}^{(t)} - \widehat{X}^{(t)}$, normalized as explained previously. We also take $H_0 = 0$. Notice that the process H is not Markov except in certain very special cases (when π vanishes).

Let us specify the version of H that we will use. When $\beta > 0$, we can use formula (2) to get

$$H_t = \beta^{-1} m(\{I^r_t,\ 0 \le r \le t\}), \tag{3}$$

where $I^r_t = \inf_{[r,t]} X_s$. The right-hand side of (3) obviously gives a continuous version of the process $(H_t,\ t \ge 0)$. When $\beta = 0$, Proposition 4.3 of [13] provides a lower semicontinuous version of H with values in $[0,\infty]$. By Theorem 4.7 of [13], this version is continuous with values in $[0,\infty)$ if and only if

$$\int^\infty \frac{d\lambda}{\psi(\lambda)} < \infty.$$

We then turn to the *exploration process* ρ (our terminology is a little different from [13], where H itself was sometimes called the exploration process). For every $t \ge 0$, we let ρ_t be the random measure on $[0,\infty)$ defined by

$$\langle \rho_t, g\rangle = \int_{[0,t]} d_r I^r_t\, g(H_r), \tag{4}$$

where the notation $d_r I^r_t$ refers to integration with respect to the nondecreasing function $r \longrightarrow I^r_t$. Although it looks a little different, this definition is equivalent to the one given in [13] in the case $\beta = 0$: Using Proposition 4.3 of [13] (and its notation) one easily verifies that $H_r = H^r_t$, $d_r I^r_t$ a.e., for every $t \ge 0$, a.s. Notice that the total mass $\langle \rho_t, 1\rangle$ of the exploration process is the reflected Lévy process $X_t - I_t$.

The process $(\rho_t,\ t \ge 0)$ is a cadlag strong Markov process with values in the space $M_f(\mathbb{R}_+)$ of all finite measures on $\mathbb{R}_+$, equipped with the topology of weak convergence. This is proved in [13], Proposition 4.4, when $\beta = 0$. The case $\beta > 0$ is analogous and easier. The strong Markov property follows from an important identity, which we now recall. We first introduce some notation. If $\mu \in M_f(\mathbb{R}_+)$ and $a \in \mathbb{R}_+$, we let $k_a\mu$ be the unique element of $M_f(\mathbb{R}_+)$ such that, for every $r \geq 0$,

$$k_a\mu([0, r]) = \mu([0, r]) \wedge a$$

($k_a\mu$ is the measure μ "truncated at mass a"). By convention, we also take $k_a\mu = 0$ when $a < 0$. Notice that this notation is slightly different from the one in [13]. If $\mu \in M_f(\mathbb{R}+)$, we set $H(\mu) = \sup(\operatorname{supp}\mu)$, with $H(\mu) = 0$ if $\mu = 0$ by convention. If $H(\mu) < \infty$, we define the concatenation of μ with another measure $\nu \in M_f(\mathbb{R}_+)$ as the measure $[\mu, \nu]$ such that

$$\int [\mu, \nu](dr)\, \varphi(r) = \int \mu(dr)\, \varphi(r) + \int \nu(dr)\, \varphi(H(\mu) + r).$$

Then, if T be a stopping time of the filtration $(\mathscr{G}_t)$, and $X^{(T)}$ denotes the shifted process $X_t^{(T)} = X_{T+t} - X_T$, we have a.s. for every $t > 0$,

$$\rho_{T+t} = \bigl[k_{\langle \rho_T, 1\rangle + I_t^{(T)}}\rho_T, \rho_t^{(T)}\bigr], \tag{5}$$

with an obvious notation for $I_t^{(T)}$ and $\rho_t^{(T)}$. See (4.8) in [13] for a proof in the case $\beta = 0$, which is easily extended.

We will also use the following fact: a.s. for every $t \geq 0$, either $\rho_t = 0$ (which occurs if and only if $H_t = 0$) or the topological support of ρ_t is the whole interval $[0, H_t]$. See [13], Section 4, when $\beta = 0$, and note that the case $\beta > 0$ is easy because ρ_t is bounded below by β times Lebesgue measure on $[0, H_t]$. Because of this property, we may and will write $H(\rho_t)$ instead of H_t.

The definition of the processes ρ and H also makes sense under the excursion measure N. Indeed, from our construction, both ρ_t and H_t depend only on the values taken by $X - I$ on the excursion of $X - I$ that straddles t. Thus, we may define the process H and then ρ via formula (4), by exactly the same formulas that we used under P (namely, formula (3) when $\beta > 0$ and Proposition 4.3 of [13] when $\beta = 0$). The "law" of $(\rho_t,\ t \geq 0)$ under N is easily identified with the excursion measure of the Markov process ρ away from 0.

We will now state two important lemmas concerning the exploration process. These lemmas give, respectively, the invariant measure of ρ and the potential kernel of the same process killed when it hits 0. We let U be a subordinator defined under a probability measure P^0, with Laplace transform

$$\begin{aligned} E^0[\exp -\lambda U_t] &= \exp\biggl(-t\biggl(\beta\lambda + \int_0^\infty (1 - e^{-\lambda r})\, \pi([r, \infty))\, dr\biggr)\biggr) \\ &= \exp\biggl(-t\biggl(\frac{\psi(\lambda)}{\lambda} - \alpha\biggr)\biggr). \end{aligned}$$

For every $a \geq 0$, we let J_a be the random element of $M_f(\mathbb{R}_+)$ defined by $J_a(dr) = 1_{[0,a]}(r)\, dU_r$.

LEMMA 2.1. *For every nonnegative measurable function Φ on $M_f(\mathbb{R}_+)$,*

$$N\left[\int_0^\sigma ds\,\Phi(\rho_s)\right] = \int_0^\infty da\, e^{-\alpha a}\, E^0[\Phi(J_a)].$$

The measure M *defined by* $\mathrm{M}(\Phi) = \int_0^\infty da\, e^{-\alpha a}\, E^0[\Phi(J_a)]$ *is invariant for* ρ.

PROOF. It is enough to prove the first part of the lemma. Let $\theta > 0$ and let ζ be an independent exponential time with parameter θ (defined under P by enlarging the probability space). From excursion theory and the fact that ρ_s only depends on the excursion of $X - I$ that straddles s, we easily get

$$E[\Phi(\rho_\zeta)] = \frac{\theta\, N[\int_0^\sigma ds\, e^{-\theta s}\,\Phi(\rho_s)]}{N[1 - e^{-\theta\sigma}]}.$$

On the other hand, define for every $s \ge 0$ a random measure η_s on $\mathbb{R}_+$ by the formula

$$\langle \eta_s, f\rangle = \int_0^s dS_r\, f(L_s - L_r).$$

By reversing time at ζ and using the definition of ρ, we get

$$E[\Phi(\rho_\zeta)] = E[\Phi(\eta_\zeta)].$$

Note that η_s remains constant during each excursion of $S - X$ away from 0. Denote by N^* the excursion measure of $S - X$ away from 0 corresponding to the local time L. Also let (a_j, b_j), $j \in J$, be the excursion intervals of $S - X$. We have then

$$\begin{aligned} E[\Phi(\eta_\zeta)] &= \theta\, E\left[\sum_{j\in J} \Phi(\eta_{a_j})\exp(-\theta a_j)\int_0^{b_j - a_j} dr\,\exp(-\theta r)\right] \\ &= E\left[\int_0^\infty dL_s\,\exp(-\theta s)\,\Phi(\eta_s)\right] N^*[1 - \exp(-\theta\sigma)]. \end{aligned}$$

By a classical formula of fluctuation theory ([2], Proposition 9), we know that, with our normalization of L,

$$N[1 - e^{-\theta\sigma}]\, N^*[1 - e^{-\theta\sigma}] = \theta.$$

Comparing with the beginning of the proof, we get

$$N\left[\int_0^\sigma ds\, e^{-\theta s}\,\Phi(\rho_s)\right] = E\left[\int_0^\infty dL_s\, e^{-\theta s}\,\Phi(\eta_s)\right].$$

Letting θ go to 0 leads to

$$N\left[\int_0^\sigma ds\,\Phi(\rho_s)\right] = E\left[\int_0^\infty dL_s\,\Phi(\eta_s)\right] = E\left[\int_0^\infty da\, 1_{\{a < L_\infty\}}\,\Phi(\eta_{L^{-1}(a)})\right].$$

However, (1) implies that L_∞ has an exponential distribution with parameter α and that, conditionally on $\{L^{-1}(a) < \infty\}$, $\eta_{L^{-1}(a)}$ has the same distribution as J_a. The desired result follows. $\square$

For $\mu \in M_f(\mathbb{R}_+)$, we denote by P_μ the law of the process ρ started at μ. We abuse notation by also writing $\sigma = \inf\{s > 0,\ \rho_s = 0\}$ under P_μ.

LEMMA 2.2. *For any nonnegative measurable function Φ on $\mathbb{R}_+ \times M_f(\mathbb{R}_+)$,*

$$\mathsf{E}_\mu\left[\int_0^\sigma ds\, \Phi\Big(\inf_{s' \le s} H(\rho_{s'}), \rho_s\Big)\right] = \int_0^{\langle \mu, 1\rangle} dr \int \mathsf{M}(d\nu)\, \Phi(H(k_r\mu), [k_r\mu, \nu]).$$

PROOF. From [13], or by using (5), we know that P_μ is the law under P of the process $(\rho_s^\mu,\ s \ge 0)$ defined by

$$\rho_s^\mu = [k_{\langle \mu, 1\rangle + I_s}\mu, \rho_s].$$

Thus we have to evaluate

$$E\left[\int_0^{\sigma_\mu} ds\, \Phi\Big(\inf_{s' \le s} H(\rho_{s'}^\mu), \rho_s^\mu\Big)\right],$$

where $\sigma_\mu = \inf\{s,\ \rho_s^\mu = 0\} = \inf\{s,\ I_s = -\langle \mu, 1\rangle\}$. Denote by $(a_j, b_j),\ j \in J$, the excursion intervals of $X - I$ away from 0 before time σ_μ and by $e_j,\ j \in J$, the corresponding excursions. Then, $\rho_s = \rho_{s-a_j}(e_j)$ for every $s \in (a_j, b_j),\ j \in J$, P a.s. Also it is clear that $\inf_{[0,s]} H(\rho_{s'}^\mu) = H(\rho_{a_j}^\mu) = H(k_{\langle \mu, 1\rangle + I_{a_j}}\mu)$ for every $s \in (a_j, b_j),\ j \in J$. It follows that

$$\begin{aligned} &E\left[\int_0^{\sigma_\mu} ds\, \Phi\Big(\inf_{s' \le s} H(\rho_{s'}^\mu), \rho_s^\mu\Big)\right] \\ &\qquad = E\left[\sum_{j \in J} \int_0^{b_j - a_j} dr\, \Phi(H(k_{\langle \mu, 1\rangle + I_{a_j}}\mu), [k_{\langle \mu, 1\rangle + I_{a_j}}\mu, \rho_r(e_j)])\right]. \end{aligned}$$

By excursion theory, the point measure

$$\sum_{j \in J} \delta_{I_{a_j}, e_j}$$

is a Poisson point measure with intensity $1_{[-\langle \mu, 1\rangle, 0]}(u)\, du\, N(de)$. Hence,

$$E\left[\int_0^{\sigma_\mu} ds\, \Phi\Big(\inf_{s' \le s} H(\rho_{s'}^\mu), \rho_s^\mu\Big)\right] = \int_0^{\langle \mu, 1\rangle} du\, N\left[\int_0^\sigma dr\, \Phi(H(k_u\mu), [k_u\mu, \rho_r])\right]$$

and the desired result follows from Lemma 2.1. □

Without risk of confusion, we will abuse notation by writing $N(d\rho)$ for the σ-finite measure on $\mathbb{D}([0,\infty),\ M_f(\mathbb{R}_+))$ which is the excursion measure of ρ away from 0: In other words, we identify N with its image under the mapping $(\rho_t,\ t \ge 0)$. The process X is then defined under $N(d\rho)$ by the formula $X_t = \langle \rho_t, 1\rangle$.

2.2. *The Lévy snake.* We now consider a Borel right Markov process $\xi = (\xi_t,\ t \ge 0;\ \Pi_x,\ x \in E)$ with cadlag paths and values in a Polish space E. For every $a \geq 0$, we denote by $\mathbb{D}([0,a],E)$ the canonical Skorokhod space of all cadlag mappings from $[0,a]$ into E. We then set

$$\mathscr{W} = \left(\bigcup_{a\geq 0} \mathbb{D}([0,a],E)\right) \cup \mathbb{D}([0,\infty),E).$$

We set $\zeta_w = a$ if $w \in \mathbb{D}([0,a],E)$, $\zeta_w = \infty$ if $w \in \mathbb{D}([0,\infty),E)$. The subset $\{w \in \mathscr{W}, \zeta_w = 0\}$ is trivially identified with E. The set $\mathscr{W}$ is a Polish space for the distance

$$d(w,w') = |\zeta_w - \zeta_{w'}| \wedge 1 + \delta(w(\cdot \wedge \zeta_w), w'(\cdot \wedge \zeta_{w'})),$$

where δ denotes the Skorokhod distance on $\mathbb{D}([0,\infty),E)$, and $|\infty - \infty| = 0$ by convention.

Let $w \in \mathscr{W}$ and $a \in [0,\zeta_w] \cap [0,\infty)$, $b \in [a,\infty)$. We construct a probability measure $R_{a,b}(w,dw')$ on $\mathscr{W}$ by the following prescriptions:

1. $\zeta_{w'} = b$, $R_{a,b}(w,dw')$ a.s.
2. $w'(t) = w(t)$, for every $t \in [0,a]$, $R_{a,b}(w,dw')$ a.s.
3. The law of $(w'(a+t),\ 0 \leq t \leq b-a)$ under $R_{a,b}(w,dw')$ is the law of $(\xi_t,\ 0 \leq t \leq b-a)$ under $\Pi_{w(a)}$.

This definition is extended to the case $b = \infty$ by obvious modifications. When $\zeta_w = \infty$, we also take $R_{\infty,\infty}(w,dw') = \delta_w(dw')$.

Let us fix a point $x \in E$ and let $\rho \in \mathbb{D}([0,\infty),\ M_f(\mathbb{R}_+))$ be such that $\rho_0 = 0$. For every $t < t'$, set

$$m_\rho(t,t') = \inf_{t\leq r\leq t'} H(\rho_r).$$

The Kolmogorov extension theorem can be used to construct the unique probability measure Q_x^ρ on $\mathscr{W}^{\mathbb{R}_+}$ such that, if $(W_s^0,\ s \geq 0)$ denotes the canonical process on $\mathscr{W}^{\mathbb{R}_+}$, we have for every $0 \leq t_1 < t_2 < \cdots < t_n$, and every nonnegative measurable functions $f_1,\ldots,f_n$ on $\mathscr{W}$,

$$\begin{aligned} Q_x^\rho&[f_1(W_{t_1}^0)\cdots f_n(W_{t_n}^0)] \\ &= \int R_{0,H(\rho_{t_1})}(x,dw_1) \int R_{m_\rho(t_1,t_2),H(\rho_{t_2})}(w_1,dw_2) \\ &\qquad \cdots \int R_{m_\rho(t_{n-1},t_n),H(\rho_{t_n})}(w_{n-1},dw_n) f_1(w_1)\cdots f_n(w_n). \end{aligned}$$

It is easy to verify that both mappings $\rho \to H(\rho_t)$ and $\rho \to m_\rho(t,t')$ are measurable. Thus we can define a σ-finite measure on the product

$$\Omega = \mathbb{D}([0,\infty),\ M_f(\mathbb{R}_+)) \times \mathscr{W}^{\mathbb{R}_+}$$

by the formula

$$\mathbb{N}_x(d\rho\, d\omega) = N(d\rho)\, Q_x^\rho(d\omega).$$

From our construction, the process $(\rho_t, W_t^0;\ t > 0)$ is Markovian under $\mathbb{N}_x$ with a transition kernel given by

$$\mathsf{Q}_t F(\mu, w) = \mathsf{E}_\mu\biggl[\int R_{m_\rho(0,t),\, H(\rho_{t\wedge\sigma})}(w, dw')\, F(\rho_{t\wedge\sigma}, w')\biggr]$$

and an entrance law given by

$$F \to N\biggl[\int R_{0,\, H(\rho_t)}(x, dw)\, F(\rho_t, w)\biggr].$$

Also note that $\zeta_{W_t^0} = H_t\, [= H(\rho_t)]$, $\mathbb{N}_x$ a.e., for every $t \geq 0$.

LEMMA 2.3. *The process $(W_s^0,\ s > 0)$ is continuous in $\mathbb{N}_x$ measure.*

PROOF. From our construction of $\mathbb{N}_x$, the proof reduces to checking that, for every $t > 0$, $H(\rho_{t+r}) \to H(\rho_t)$ and $m_\rho(t, t+r) \to H(\rho_t)$ in N measure as $r \to 0$ (note that $N[H_t = a] = 0$ for every $a \geq 0$). The a.s. lower semicontinuity of the process H immediately implies that $m_\rho(t, t+r) \to H(\rho_t)$, N a.e. as $r \to 0$. Then it is enough to verify that for any fixed $\delta > 0$ and any $\varepsilon > 0$,

$$\lim_{r \downarrow 0} N[H_{t+r} - H_t > \varepsilon] = 0,$$
$$\lim_{r \downarrow 0} N[H_t - H_{t+r} > \varepsilon] = 0$$

uniformly in $t \in [2\delta, \infty)$. By applying the Markov property under N at time δ, we see that it suffices to prove the previous convergences when N is replaced by P_μ for some $\mu \in M_f(\mathbb{R}_+)$. The first convergence is then easy from the subadditivity property of H (we have $H_{t+r} \leq H_t + H_r^{(t)}$, where $H^{(t)}$ is the height process attached to the shifted Lévy process $X_s^{(t)} = X_{t+s} - X_s$; see [13], Lemma 4.5). As for the second one, we first observe that, for $r > 0$, we have from (5),

$$\rho_t = \bigl[k_{\langle \rho_{t-\delta}, 1\rangle + I_\delta^{(t-\delta)}} \rho_{t-\delta}, \rho_\delta^{(t-\delta)}\bigr], \qquad \rho_{t+r} = \bigl[k_{\langle \rho_t, 1\rangle + I_r^{(t)}} \rho_t, \rho_r^{(t)}\bigr], \tag{6}$$

where the law under P_μ of $\rho_\delta^{(t-\delta)}$ does not depend on t and coincides with the law of ρ_δ under P. Note that $I^{(t)}$ and $\rho_\delta^{(t-\delta)}$ are independent. Furthermore, $H(\rho_\delta^{(t-\delta)}) > 0$, P a.s., and the quantity $I_r^{(t)}$ is small in probability when $r \downarrow 0$, uniformly in t. It follows that, with a probability close to 1 when r is small, the killing operator $k_{\langle \rho_t, 1\rangle + I_r^{(t)}}$ in (6) acts only on the part of the measure ρ_t corresponding to $\rho_\delta^{(t-\delta)}$, and

$$H(\rho_{t+r}) \geq H\bigl(k_{\langle \rho_t, 1\rangle + I_r^{(t)}} \rho_t\bigr) \geq H(\rho_t) - \varepsilon.$$

This completes the proof. □

From Lemma 2.3 and standard arguments (see, e.g., [16], page 87), we get that the process $(W^0_s,\ s \ge 0)$ has a measurable modification under $\mathbb{N}_x$, which will be denoted by $(W_s,\ s \ge 0)$. Precisely, we can choose an increasing sequence (D_n) of discrete countable subsets of $\mathbb{R}_+$, with union dense in $\mathbb{R}_+$, such that the process $(W_s,\ s > 0)$ is defined by the formula

$$W_s = \begin{cases} \lim\limits_{n\to\infty} W^0_{\lambda_n(s)}, & \text{if the limit exists,} \\ x, & \text{otherwise,} \end{cases}$$

where $\lambda_n(s) = \inf\{s' > s,\ s' \in D_n\}$.

The pair $(\rho_s, W_s;\ s \ge 0)$ is called the (ξ, ψ) Lévy snake with initial point x. We will often use the notation $\widehat{W}_s = W_s(\zeta_{W_s})$ when $\zeta_{W_s} < \infty$. When $\zeta_{W_s} = \infty$, we take $\widehat{W}_s = x$ by convention. Note that $\widehat{W}_s = W_s(H_s)$, $\mathbb{N}_x$ a.e., for every $s \ge 0$.

2.3. *The jump truncation procedure.* Some of the subsequent results are more easily obtained in the special case when π is supported on $(0, A]$ for some $A < \infty$. We will now explain the procedure that allows us to reduce the general case to this special situation.

For every integer $k \ge 1$, we let $\pi^{(k)}$ denote the restriction of π to $(0, k]$, and we set

$$\psi^{(k)}(\lambda) = \left(\alpha + \int_{(k,\infty)} r\pi(dr)\right)\lambda + \beta\lambda^2 + \int \pi^{(k)}(dr)\,(e^{-r\lambda} - 1 + r\lambda).$$

Notice that $\psi^{(k)} \downarrow \psi$ as $k \uparrow \infty$. We may replace ψ by $\psi^{(k)}$ in the previous developments and introduce, in particular, the excursion measure $N^{(k)}$, the $(\xi, \psi^{(k)})$ Lévy snake and its excursion measures $\mathbb{N}^{(k)}_x$ on Ω. It turns out that the $(\xi, \psi^{(k)})$ Lévy snake can be embedded in the (ξ, ψ) Lévy snake via a suitable time change.

To explain this embedding, we introduce the stopping times $U^{(k)}_j$, $j \ge 0$, and $T^{(k)}_j$, $j \ge 1$, defined inductively as

$$\begin{aligned} U^{(k)}_0 &= 0, \\ T^{(k)}_j &= \inf\{t \ge U^{(k)}_{j-1},\ \Delta X_t > k\}, \qquad j \ge 1, \\ U^{(k)}_j &= \inf\{t \ge T^{(k)}_j,\ X_t = X_{T^{(k)}_j -}\}, \qquad j \ge 1. \end{aligned}$$

We then let $\Gamma^{(k)}$ be the random set

$$\Gamma^{(k)} = \bigcup_{j=0}^{\infty} [U^{(k)}_j, T^{(k)}_{j+1})$$

and define $\eta^{(k)}_t = \int_0^t 1_{\Gamma^{(k)}}(r)\,dr$, $\gamma^{(k)}_t = \inf\{r,\ \eta^{(k)}_r > t\}$. Then it is very easy to verify that the process $(X_{\gamma^{(k)}_t},\ t \ge 0)$ is distributed under P (resp. under N)

according to the law of the Lévy process with Laplace exponent $\psi^{(k)}$ (resp. according to the associated excursion measure). Informally, $X^{(k)}$ is obtained from X by removing the jumps of size greater than k. Similarly, our construction of the exploration process implies that the process $\rho_t^{(k)} = \rho_{\gamma_t^{(k)}}$ is distributed under N according to the excursion measure $N^{(k)}$.

We now want to consider the process $(W_{\gamma_t^{(k)}},\ t \ge 0)$ which is well defined thanks to our choice of a measurable modification for W. Let $0 < \varepsilon < A < \infty$ and let h be a bounded nonnegative measurable function on $\mathscr{W}$ such that $h(x) = 0$. We claim that, for every integer $p \ge 1$,

$$\mathbb{N}_x\left[\left(\int_\varepsilon^A h(W_{\gamma_t^{(k)}})\,dt\right)^p\right] = \mathbb{N}_x^{(k)}\left[\left(\int_\varepsilon^A h(W_t)\,dt\right)^p\right]. \tag{7}$$

Let us prove (7). Using the definition of $\mathbb{N}_x$, we have

$$\begin{aligned}
&\mathbb{N}_x\left[\left(\int_\varepsilon^A h(W_{\gamma_t^{(k)}})\,dt\right)^p\right]\\
&\quad= \mathbb{N}_x\left[\left(\int_0^\infty dr\, 1_{\Gamma^{(k)}\cap[\gamma_\varepsilon^{(k)},\,\gamma_A^{(k)}]}(r)\, h(W_r)\right)^p\right]\\
&\quad= \int_{[0,\infty)^p} dr_1\cdots dr_p \int N(d\rho)\prod_{i=1}^p 1_{\Gamma^{(k)}\cap[\gamma_\varepsilon^{(k)},\,\gamma_A^{(k)}]}(r_i)\ Q_x^\rho\left[\prod_{i=1}^p h(W_{r_i}^0)\right]\\
&\quad= \int N(d\rho)\int_{[\varepsilon,A]^p} dt_1\cdots dt_p\, Q_x^\rho\big[h(W^0_{\gamma_{t_1}^{(k)}})\cdots h(W^0_{\gamma_{t_p}^{(k)}})\big].
\end{aligned}$$

In the second equality, we used the fact that W is a modification of W^0 under $\mathbb{N}_x$ to replace each term W_{r_i} by $W_{r_i}^0$.

The form of the finite-dimensional marginals under Q_x^ρ implies that, for N a.e. choice of ρ, the law of $(W^0_{\gamma_t^{(k)}},\ t \ge 0)$ under Q_x^ρ is $Q_x^{\rho^{(k)}}$ [notice that if ρ is fixed, $(\gamma_t^{(k)}, t \ge 0)$ becomes a deterministic function]. Therefore, the previous quantities are also equal to

$$\begin{aligned}
&\int_{[\varepsilon,A]^p} dt_1\cdots dt_p \int N(d\rho)\, Q_x^{\rho^{(k)}}[h(W_{t_1}^0)\cdots h(W_{t_p}^0)]\\
&\qquad= \int_{[\varepsilon,A]^p} dt_1\cdots dt_p\, \mathbb{N}_x^{(k)}[h(W_{t_1})\cdots h(W_{t_p})]
\end{aligned}$$

and (7) follows.

Set

$$u^{(k)}(x) = \mathbb{N}_x^{(k)}\left[1-\exp\left(-\int_0^\infty h(W_t)\,dt\right)\right].$$

Since $h(x) = 0$ and $N^{(k)}[\sigma > \varepsilon] \le N[\sigma > \varepsilon] < \infty$, the right-hand side of (7) is clearly bounded by C^p for some constant C independent of p. It then follows from (7) and a monotonicity argument that

$$u^{(k)}(x) = \mathbb{N}_x\left[1-\exp\left(-\int_0^\infty h(W_{\gamma_t^{(k)}})\,dt\right)\right] = \mathbb{N}_x\left[1-\exp\left(-\int_{\Gamma^{(k)}} h(W_t)\,dt\right)\right].$$

Since the sets $\Gamma^{(k)}$ clearly increase to $[0,\infty)$ as $k\uparrow\infty$, we conclude that

$$\lim_{k\uparrow\infty}\uparrow u^{(k)}(x)=u(x):=\mathbb{N}_x\biggl[1-\exp\biggl(-\int_0^\infty h(W_t)\,dt\biggr)\biggr].$$

This monotonicity property will be used in Section 5.

3. Moments of the exploration process. Our goal is to derive recursive expressions for the moments

$$\mathbb{N}_x\biggl[\biggl(\int_0^\sigma g(W_t)\,dt\biggr)^n\biggr],$$

where $n\ge 1$ is an integer. From the definition of the Lévy snake, it will turn out that the expression of these moments involves quantities of the type

$$N\biggl[\int_{\{0<t_1<\cdots<t_n<\sigma\}} dt_1\cdots dt_n\,F(\rho_{t_1},\dots,\rho_{t_n},m_\rho(t_1,t_2),\dots,m_\rho(t_{n-1},t_n))\biggr],$$

where F is a measurable function on $M_f(\mathbb{R}_+)^n\times\mathbb{R}_+^{n-1}$. The next lemma allows us to compute such quantities. To simplify notation, we write $|\mu|=\langle\mu,1\rangle$ for $\mu\in M_f(\mathbb{R}_+)$.

Lemma 3.1. *We have*

$$N\biggl[\int_{\{0<t_1<\cdots<t_n<\sigma\}} dt_1\cdots dt_n\,F(\rho_{t_1},\dots,\rho_{t_n},m_\rho(t_1,t_2),\dots,m_\rho(t_{n-1},t_n))\biggr]$$
$$=\int Q^{(n)}(d\mu_1\cdots d\mu_n da_2\cdots da_n)\,F(\mu_1,\dots,\mu_n,H(k_{a_2}\mu_1),\dots,H(k_{a_n}\mu_n)),$$

where $Q^{(n)}$ is the measure on $M_f(\mathbb{R}_+)^n\times\mathbb{R}_+^{n-1}$ which can be defined by induction as follows. First $Q^{(1)}=\mathrm{M}$, and then, for every $n\ge1$, $Q^{(n+1)}$ is the image of

$$Q^{(n)}(d\mu_1\cdots d\mu_n da_2\cdots da_n)1_{[0,|\mu_n|]}(a)\,da\,\mathrm{M}(d\theta)$$

under the mapping $(\mu_1,\dots,\mu_n,a_2,\dots,a_n,a,\theta)\to(\mu_1,\dots,\mu_n,[k_a\mu_n,\theta],a_2,\dots,a_n,a)$.

Proof. The case $n=1$ is Lemma 2.1. The proof is then complete by induction on n using the Markov property of ρ under N and Lemma 2.2. □

We will now derive a recursive relation between the measures $Q^{(n)}$ introduced in Lemma 3.1. We first need to introduce some notation. For every $n\ge1$, we set $\Theta^{(n)}=M_f(\mathbb{R}_+)^n\times\mathbb{R}_+^{n-1}$ $[\Theta^{(1)}=M_f(\mathbb{R}_+)]$ and we take $\Theta=\bigcup_{n=1}^\infty\Theta^{(n)}$.

Let $n\ge2$ and let $(\mu_1,\dots,\mu_n,a_2,\dots,a_n)\in\Theta^{(n)}$ be such that $a_j\le|\mu_{j-1}|\wedge|\mu_j|$ and $k_{a_j}\mu_{j-1}=k_{a_j}\mu_j$ for every $j\in\{2,\dots,n\}$ (these properties hold $Q^{(n)}$ a.e.). We define several quantities depending on $(\mu_1,\dots,\mu_n,a_2,\dots,a_n)$. First, we set

$$b=\inf_{2\le j\le n}a_j,\qquad h=H(k_b\mu_1).$$

Notice that $k_b\mu_j=k_b\mu_1$ for every $j\in\{2,\dots,n\}$.

We then set $b_- = \mu_1([0,h))$, $b_+ = \mu_1([0,h])$ and observe that $b_- \le b \le b_+$. We let $j_1 < j_2 < \cdots < j_{k-1}$ be the successive integers in $\{2,\dots,n\}$ such that

$$a_{j_1} \in [b_-, b_+],\ a_{j_2} \in [b_-, a_{j_1}], \dots, a_{j_{k-1}} \in [b_-, a_{j_{k-2}}].$$

By construction, $2 \le k \le n$ and $b = a_{j_{k-1}}$. We also take $j_0 = 1$ and $j_k = n+1$ by convention.

We let ν_0 be the restriction of μ_1 (or of any μ_j) to $[0,h)$, and for every $j \in \{1,\dots,n\}$, we define $\nu_j \in M_f(\mathbb{R}_+)$ by taking $\nu_j([0,r]) = \mu_j((h,h+r])$. Figure 1 illustrates the definition of the measures ν_j. In this figure, measures are represented by vertical segments (the length of the segment corresponding to the total mass of the measure) and the horizontal lines give the successive truncation levels.

Finally, for every $l \in \{1,\dots k\}$, we define

$$\Delta_l = \big(\mu_1^{(l)},\dots,\mu_{j_l-j_{l-1}}^{(l)}, a_2^{(l)},\dots,a_{j_l-j_{l-1}}^{(l)}\big) \in \Theta^{(j_l-j_{l-1})}$$

by setting

$$\mu_i^{(l)} = \nu_{j_{l-1}+i-1}, \qquad 1 \le i \le j_l - j_{l-1},$$
$$a_i^{(l)} = a_{j_{l-1}+i-1} - a_{j_{l-1}}, \qquad 2 \le i \le j_l - j_{l-1},$$

where by convention $a_1 = b_+$.

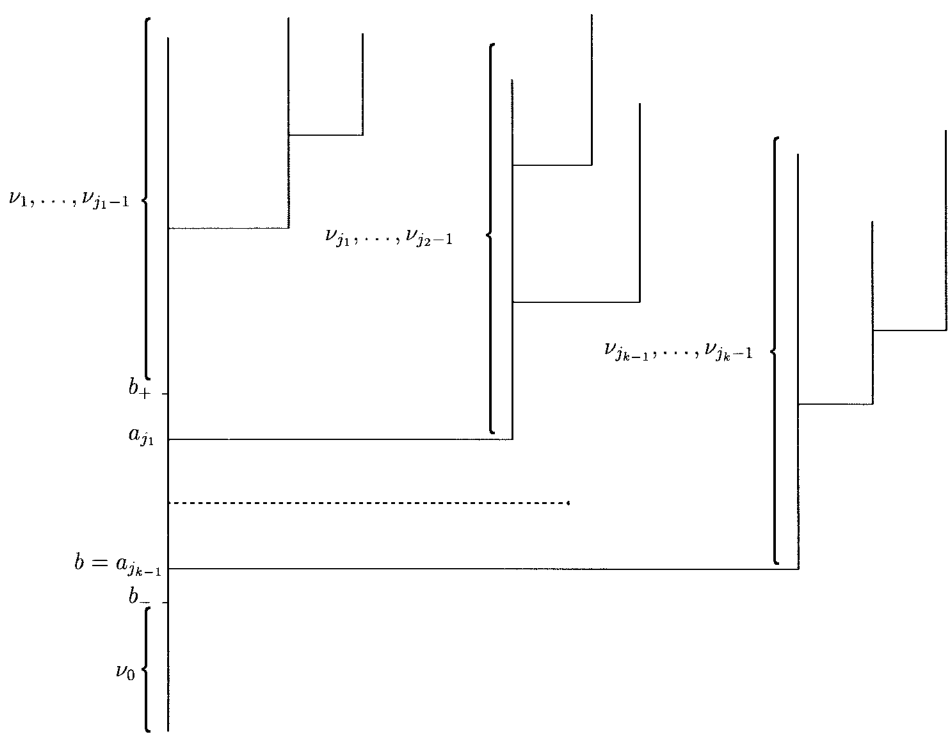

FIG. 1.

PROPOSITION 3.2. *For every integer $p \in \{2, \dots, n\}$, for every measurable subsets $A_0, \dots, A_n$ of Θ, we have*

$$Q^{(n)}[k = p,\ \nu_0 \in A_0,\ \Delta_1 \in A_1, \dots, \Delta_p \in A_p] = \left(\beta 1_{\{p=2\}} + \int \frac{y^{p-1}}{(p-1)!}\,\widetilde{\pi}(dy)\right) \mathrm{M}(A_0) \sum_{\substack{n_1+\cdots+n_p=n \\ n_i \ge 1}} Q^{(n_1)}(A_1) \cdots Q^{(n_p)}(A_p),$$

where $\widetilde{\pi}(dy) = \pi([y, \infty))\,dy$.

Proposition 3.2 will play a key role in the next section. We will derive it from a slightly more precise result, which will be proved by induction on n. We keep the previous notation and also set $\Delta b = b_+ - b_-$, $c_i = a_{j_i} - b_-$ for $1 \le i \le k-1$. Then, if B is a Borel subset of $\mathbb{R}_+^p$, we claim that

$$\begin{aligned} &Q^{(n)}[k = p,\ (\Delta b, c_1, \dots, c_{p-1}) \in B,\ \nu_0 \in A_0,\ \Delta_1 \in A_1, \dots, \Delta_p \in A_p] \\ &\quad = \left(\beta 1_{\{p=2\}} 1_B(0,0) + \int \widetilde{\pi}(dy) \int_0^y dz_1 \int_0^{z_1} dz_2 \right. \\ &\qquad\qquad \left. \cdots \int_0^{z_{p-2}} dz_{p-1}\, 1_B(y, z_1, \dots, z_{p-1}) \right) \\ &\qquad \times \mathrm{M}(A_0) \sum_{\substack{n_1+\cdots+n_p=n \\ n_i \ge 1}} Q^{(n_1)}(A_1) \cdots Q^{(n_p)}(A_p). \end{aligned} \tag{8}$$

Obviously, Proposition 3.2 follows from (8).

Before proceeding to the proof of (8), we state a useful lemma, which gives the "law" under $\mathrm{M}(d\mu)$ of the splitting of μ at a uniformly distributed mass level.

LEMMA 3.3. *If $\mu \in M_f(\mathbb{R}_+)$ and $a \in (0, |\mu|)$, define $r = r(\mu, a)$ by $r = H(k_a\mu)$ and then $\tau_r\mu, \sigma_r\mu \in M_f(\mathbb{R}_+)$ by $\tau_r\mu = \mu_{|[0,r)}$, $\sigma_r\mu([0,u]) = \mu((r, r+u])$ for every $u \ge 0$. Then*

$$\begin{aligned} &\int \mathrm{M}(d\mu) \int_0^{|\mu|} da\, F(\tau_r\mu, \sigma_r\mu, \mu(\{r\}), a - |\tau_r\mu|) \\ &\quad = \iint \mathrm{M}(d\mu_1)\mathrm{M}(d\mu_2) \\ &\qquad \times \left(\beta F(\mu_1, \mu_2, 0, 0) + \int \widetilde{\pi}(dy) \int_0^y dz\, F(\mu_1, \mu_2, y, z)\right). \end{aligned} \tag{9}$$

PROOF. Recall from Lemma 2.1 the definition of M and the notation U for a subordinator with drift β and Lévy measure $\widetilde{\pi}$. For every $a \ge 0$, set $\eta_a = \inf\{t, U_t \ge a\}$. The left-hand side of (9) can be written as

$$\int_0^\infty dt\, e^{-\alpha t} \int_0^\infty da\, E^0\Big[1_{\{a < U_t\}} F\Big(1_{[0,\eta_a)}(s)\, dU_s, \sigma_{\eta_a}(1_{[0,t]}(s)\, dU_s), \Delta U_{\eta_a}, a - U_{\eta_a -}\Big)\Big].$$

We may assume that F is factorized in the form $F(\mu_1, \mu_2, u, v) = 1_{A_1}(\mu_1)1_{A_2}(\mu_2)1_B(u, v)$. Then the strong Markov property of U at time η_a shows that the previous expression is also equal to

$$E^0\biggl[\int_0^\infty da \exp(-\alpha\eta_a)\, 1_{A_1}(1_{[0,\,\eta_a)}(s)\, dU_s)1_B(\Delta U_{\eta_a}, a - U_{\eta_a -})\biggr]\,\mathsf{M}(A_2)$$
$$= \mathsf{M}(A_2) \times \biggl(E^0\biggl[\sum_{t:\,\Delta U_t > 0} \exp(-\alpha t)1_{A_1}(1_{[0,\,t)}(s)\, dU_s)\int_0^{\Delta U_t} 1_B(\Delta U_t, z)\, dz\biggr]$$
$$+ E^0\biggl[\int_0^\infty da\, 1_{\{\Delta U_{\eta_a} = 0\}} \exp(-\alpha\eta_a)1_{A_1}(1_{[0,\,\eta_a)}(s)\, dU_s)\, 1_B(0,0)\biggr]\biggr).$$

The first term of the sum inside parentheses is equal to

$$E^0\biggl[\int_0^\infty dt\, e^{-\alpha t}\, 1_{A_1}(1_{[0,\,t)}(s)\, dU_s)\int \widetilde{\pi}(dy)\int_0^y 1_B(y, z)\, dz\biggr]$$
$$= \biggl(\int \widetilde{\pi}(dy)\int_0^y 1_B(y, z)\, dz\biggr)\,\mathsf{M}(A_1),$$

whereas the change of variable $s = \eta_a$ gives for the second term

$$E^0\biggl[\int_0^\infty dU_t\, 1_{\{\Delta U_t = 0\}} e^{-\alpha t} 1_{A_1}(1_{[0,\,t)}(s)\, dU_s)\, 1_B(0,0)\biggr] = \beta\, 1_B(0,0)\mathsf{M}(A_1).$$

Lemma 3.3 now follows. □

PROOF OF (8). We first consider the case $n = 2$. Then, necessarily, $k = 2$. Furthermore, from the construction of $Q^{(2)}$, we have with the notation of Lemma 3.3,

$$Q^{(2)}[k = 2,\ (\Delta b, c_1) \in B,\ \nu_0 \in A_0,\ \nu_1 \in A_1,\ \nu_2 \in A_2]$$
$$= \iint \mathsf{M}(d\mu)\mathsf{M}(d\mu')\int_0^{|\mu|} da\, 1_B(\mu(\{r\}), a - |\tau_r\mu|)\, 1_{A_0}(\tau_r\mu)1_{A_1}(\sigma_r\mu)1_{A_2}(\mu')$$
$$= \mathsf{M}(A_2)\int \mathsf{M}(d\mu)\int_0^{|\mu|} da\, 1_B(\mu(\{r\}), a - |\tau_r\mu|)\, 1_{A_0}(\tau_r\mu)1_{A_1}(\sigma_r\mu)$$
$$= \biggl(\beta\, 1_B(0,0) + \int \widetilde{\pi}(dy)\int_0^y dz\, 1_B(y, z)\biggr)\,\mathsf{M}(A_0)\mathsf{M}(A_1)\mathsf{M}(A_2)$$

by Lemma 3.3.

We then complete the proof by induction on n. We fix $n \geq 2$ and assume that the desired result holds up to the order n. We then prove that the formula also holds at the order $n + 1$. When there is a risk of confusion, we write $k^{(n)}$, $b^{(n)}$, $b_-^{(n)}$ and so forth for the quantities defined at the order n.

Fix $p \geq 2$ and $n_1, \ldots, n_p \geq 1$ such that $n_1 + \cdots + n_p = n + 1$. We may assume that $A_i \subset \Theta^{(n_i)}$ for every $i \in \{1, \ldots, p\}$. We first consider the case $n_p \geq 2$. On the event in consideration, we must have $k^{(n+1)} = k^{(n)} = p$ and

$$a_{n+1} > b^{(n)} = \inf_{1 \leq i \leq n} a_i = b_-^{(n)} + c_{p-1}^{(n)}.$$

Furthermore, from the equality $\mu_{n+1} = [k_{a_{n+1}}\mu_n, \theta_{n+1}]$, we have $\nu_{n+1} = [k_{a_{n+1}-b^{(n)}}\nu_n, \theta_{n+1}]$ and

$$\Delta_1^{(n+1)} = \Delta_1^{(n)}, \dots, \Delta_{p-1}^{(n+1)} = \Delta_{p-1}^{(n)}, \qquad \Delta_p^{(n+1)} = \big(\Delta_p^{(n)}, (\nu_{n+1}, a_{n+1} - b^{(n)})\big),$$

with an obvious notation. We then obtain

$$\begin{aligned} &Q^{(n+1)}[k = p,\ (\Delta b, c_1, \dots, c_{p-1}) \in B,\ \nu_0 \in A_0,\ \Delta_1 \in A_1, \dots, \Delta_p \in A_p] \\ &\quad = Q^{(n)}\bigg[k = p,\ (\Delta b, c_1, \dots, c_{p-1}) \in B,\ \nu_0 \in A_0,\ \Delta_1 \in A_1, \dots, \Delta_{p-1} \in A_{p-1}; \\ &\qquad\qquad \int_b^{|\mu_n|} da_{n+1} \int \mathsf{M}(d\theta)\, 1_{\{(\Delta_p, ([k_{a_{n+1}-b}\nu_n, \theta], a_{n+1}-b)) \in A_p\}}\bigg] \\ &\quad = \bigg(\beta 1_{\{p=2\}} 1_B(0,0) + \int \widetilde{\pi}(dy) \int_0^y dz_1 \int_0^{z_1} dz_2 \\ &\qquad\qquad \cdots \int_0^{z_{p-2}} dz_{p-1}\, 1_B(y, z_1, \dots, z_{p-1})\bigg) \\ &\qquad \times \mathsf{M}(A_0) Q^{(n_1)}(A_1) \cdots Q^{(n_{p-1})}(A_{p-1}) \int Q^{(n_p - 1)}(d\Delta) \int_0^{|\nu_{n_p - 1}|} da \\ &\qquad\qquad \times \int \mathsf{M}(d\theta) 1_{\{(\Delta, ([k_a \nu_{n_p - 1}, \theta], a)) \in A_p\}} \end{aligned}$$

using the induction hypothesis. The desired result then follows from the construction of the measures $Q^{(m)}$.

Consider then the case $n_p = 1$. We need to treat separately two subcases. If $p \ge 3$, we are in the situation where $k^{(n)} = k^{(n+1)} - 1 = p - 1$ and

$$b_-^{(n)} \le a_{n+1} \le b^{(n)} = \inf_{1 \le i \le n} a_i, \qquad c_{p-1}^{(n+1)} = a_{n+1} - b_-^{(n)}.$$

Then

$$\begin{aligned} &Q^{(n+1)}[k = p,\ (\Delta b, c_1, \dots, c_{p-1}) \in B,\ \nu_0 \in A_0,\ \Delta_1 \in A_1, \dots, \Delta_p \in A_p] \\ &\quad = Q^{(n)}\bigg[k = p - 1, \int_{b-}^{b} da\, 1_{\{(\Delta b, c_1, \dots, c_{p-2}, a - b_-) \in B\}}; \\ &\qquad\qquad \nu_0 \in A_0,\ \Delta_1 \in A_1, \dots, \Delta_{p-1} \in A_{p-1}\bigg] \mathsf{M}(A_p) \\ &\quad = \bigg(\int \widetilde{\pi}(dy) \int_0^y dz_1 \int_0^{z_1} dz_2 \cdots \int_0^{z_{p-2}} dz_{p-1}\, 1_B(y, z_1, \dots, z_{p-1})\bigg) \\ &\qquad \times \mathsf{M}(A_0) Q^{(n_1)}(A_1) \cdots Q^{(n_{p-1})}(A_{p-1}) \mathsf{M}(A_p) \end{aligned}$$

using the induction hypothesis in the last equality. This is the desired formula since $Q^{(1)} = \mathsf{M}$.

It remains to treat the case when $n_p = 1$ and $p = 2$. This corresponds to the situation where

$$0 \le a_{n+1} < b_-^{(n)} = |\nu_0^{(n)}|.$$

The proof of the desired result then reduces to checking that, writing $r = r(\nu_0, a)$ with the notation of Lemma 3.3, we have

$$
\begin{aligned}
&Q^{(n+1)}[k=2,\ (\Delta b, c_1) \in B,\ \nu_0 \in A_0,\ \Delta_1 \in A_1,\ \Delta_2 \in A_2] \\
&\quad = Q^{(n)}\biggl[\int_0^{|\nu_0|} da\, 1_B(\nu_0(\{r\}), a - \nu_0([0,r)))\, 1_{A_0}(\tau_r \nu_0) \\
&\qquad\qquad \times 1_{A_1}(\sigma_r \mu_1, \dots, \sigma_r \mu_n, a_2 - \nu_0([0,r]), \dots, a_n - \nu_0([0,r]))\biggr] \mathrm{M}(A_2) \\
&\quad = \biggl(\beta 1_B(0,0) + \int \widetilde{\pi}(dy) \int_0^y dz\, 1_B(y,z)\biggr) \mathrm{M}(A_0) Q^{(n)}(A_1) \mathrm{M}(A_2).
\end{aligned}
$$

The last equality follows from Lemma 3.3 after noticing that it is enough to consider an event A_1 "decomposed at the first node" as in the statement of the proposition. We leave details to the reader. This completes the proof of (8) and Proposition 3.2. □

4. The Laplace functional of the Lévy snake.

4.1. *Moments of the occupation measure.* In this section, we combine the results of the previous section with spatial motion to get a recursive formula for the moments of the occupation measure under $\mathbb{N}_x$. For every integer $p \geq 2$, we set

$$\gamma_p = \beta\, 1_{\{p=2\}} + \int \frac{y^{p-1}}{(p-1)!}\, \widetilde{\pi}(dy).$$

We denote by $\mathscr{B}_{b+}(\mathbb{R}_+ \times E)$ the space of all bounded nonnegative measurable functions on $\mathbb{R}_+ \times E$.

PROPOSITION 4.1. *For every integer $n \geq 1$ and every $g \in \mathscr{B}_{b+}(\mathbb{R}_+ \times E)$, define for $t \geq 0$ and $x \in E$,*

$$T^n g(t,x) = \frac{1}{n!} \mathbb{N}_x\biggl[\biggl(\int_0^\sigma ds\, g(t + H_s, \widehat{W}_s)\biggr)^n\biggr].$$

Then

$$T^1 g(t,x) = \Pi_x\biggl[\int_0^\infty dr\, e^{-\alpha r} g(t + r, \xi_r)\biggr]$$

and, if $n \geq 2$,

$$T^n g(t,x) = \sum_{p=2}^n \gamma_p \sum_{\substack{n_1 + \cdots + n_p = n \\ n_i \geq 1}} T^1(T^{n_1} g \cdots T^{n_p} g)(t,x).$$

PROOF. The case $n = 1$ is a straightforward consequence of Lemma 2.1 and the construction of the Lévy snake:

$$\begin{aligned} T^1 g(t,x) &= \int N(d\rho)\int_0^{\sigma} ds\, Q_x^{\rho}[g(t+H(\rho_s), \widehat{W}_s^0)] \\ &= \int N(d\rho)\int_0^{\sigma} ds\, \Pi_x[g(t+H(\rho_s), \xi_{H(\rho_s)})] \\ &= \int_0^{\infty} dr\, e^{-\alpha r}\, \Pi_x[g(t+r, \xi_r)]. \end{aligned}$$

The recursive formula for $T^n g$ is basically a consequence of Proposition 3.2. Let us take $t = 0$ for simplicity. Then

$$\begin{aligned} T^n g(0,x) &= \mathbb{N}_x\left[\int_{\{0<t_1<\cdots<t_n<\sigma\}} dt_1\cdots dt_n \prod_{i=1}^n g(H_{t_i}, \widehat{W}_{t_i})\right] \\ &= \int N(d\rho)\int_{\{0<t_1<\cdots<t_n<\sigma\}} dt_1\cdots dt_n \\ &\qquad\qquad \times \int R_{0,H(\rho_{t_1})}(x, dw_1)\int R_{m_\rho(t_1,t_2),\, H(\rho_{t_2})}(w_1, dw_2) \\ &\qquad \cdots \int R_{m_\rho(t_{n-1},t_n),\, H(\rho_{t_n})}(w_{n-1}, dw_n) \prod_{i=1}^n g(H(\rho_{t_i}), \widehat{w}_i) \\ &= \int Q^{(n)}(d\mu_1\cdots d\mu_n da_2\cdots da_n) \\ &\qquad \times \int R_{0,H(\mu_1)}(x, dw_1)\int R_{H(k_{a_2}\mu_1),H(\mu_2)}(w_1, dw_2) \\ &\qquad \cdots \int R_{H(k_{a_n}\mu_{n-1}),H(\mu_n)}(w_{n-1}, dw_n) \prod_{i=1}^n g(H(\mu_i), \widehat{w}_i), \end{aligned}$$

where we used Lemma 3.1.

Fix $\Delta = (\mu_1, \dots, \mu_n, a_2, \dots, a_n) \in \Theta^{(n)}$ such that $a_j \le |\mu_{j-1}| \wedge |\mu_j|$ and $k_{a_j}\mu_{j-1} = k_{a_j}\mu_j$ for every $j \in \{2, \dots, n\}$. To simplify notation, let $\Gamma_x^{(n)}(\Delta, dw_1\cdots dw_n)$ be the probability measure on $\mathscr{W}^n$ defined by

$$\begin{aligned} \Gamma_x^{(n)}(\Delta, dw_1\cdots dw_n) = R_{0,\, H(\mu_1)}(x, dw_1)\, R_{H(k_{a_2}\mu_1),\, H(\mu_2)}(w_1, dw_2) \\ \cdots R_{H(k_{a_n}\mu_{n-1}),\, H(\mu_n)}(w_{n-1}, dw_n). \end{aligned}$$

Recall the notation ν_0, k, $j_0, j_1, \dots, j_k$ and $\Delta_1, \dots, \Delta_k$ introduced in the previous section. It is intuitively clear, and can be verified by induction on n, that, under $\Gamma_x^{(n)}(\Delta, dw_1\cdots dw_n)$,

$$w_1(t) = \cdots = w_n(t) \quad \text{for } t \le H(\nu_0)$$

and, conditionally on $w_1(H(\nu_0))$, the vectors

$$\big(w_1(H(\nu_0)+\cdot), \dots, w_{j_1-1}(H(\nu_0)+\cdot)\big), \dots, \big(w_{j_{k-1}}(H(\nu_0)+\cdot), \dots, w_n(H(\nu_0)+\cdot)\big)$$

are independent with respective distributions

$$\Gamma_{w_1(H(\nu_0))}^{(j_1-j_0)}(\Delta_1, \cdot), \dots, \Gamma_{w_1(H(\nu_0))}^{(j_k-j_{k-1})}(\Delta_k, \cdot).$$

From this observation, we get

$$T^n g(0,x) = \int Q^{(n)}(d\Delta)\,\Pi_x\Bigg[\prod_{l=1}^{k}\bigg(\int \Gamma^{(j_l-j_{l-1})}_{\xi_{H(\nu_0)}}(\Delta_l, dw_1\cdots dw_{j_l-j_{l-1}}) \times \prod_{i=1}^{j_l-j_{l-1}} g(H(\nu_0)+H(\mu_i^{(l)}),\widehat{w}_i)\bigg)\Bigg]$$

$$= \sum_{p=2}^{n}\gamma_p\int_0^\infty dt\, e^{-\alpha t} \times \Pi_x\Bigg[\sum_{\substack{n_1+\cdots+n_p=n\\ n_i\ge 1}}\prod_{j=1}^{p}\bigg(\int Q^{(n_j)}(d\Delta)\int\Gamma^{(n_j)}_{\xi_t}(\Delta, dw_1\cdots dw_{n_j}) \times \prod_{i=1}^{n_j} g(t+H(\mu_i),\widehat{w}_i)\bigg)\Bigg]$$

$$= \sum_{p=2}^{n}\gamma_p\sum_{\substack{n_1+\cdots+n_p=n\\ n_i\ge 1}} T^1\bigg(\prod_{j=1}^{p} T^{n_j} g\bigg)(0,x).$$

In the second equality, we used Proposition 3.2 and the fact that the "law" of $H(\nu)$ under $\mathrm{M}(d\nu)$ is $e^{-\alpha t}\,dt$. This completes the proof. □

4.2. *Identification of the Laplace functional.* We denote by $\mathscr{H}$ the set of all functions $g \in \mathscr{B}_{b+}(\mathbb{R}_+ \times E)$ whose support is contained in $[0,T]\times E$ for some $T>0$.

THEOREM 4.2. *Let $g\in\mathscr{H}$. For every $(t,x)\in E$, set*

$$u(t,x) = \mathbb{N}_x\bigg[1-\exp-\int_0^\sigma g(t+H_s,\widehat{W}_s)\,ds\bigg].$$

Then u is the unique nonnegative solution of the integral equation

$$u(t,x)+\Pi_x\bigg[\int_0^\infty \psi(u(t+r,\xi_r))\,dr\bigg] = \Pi_x\bigg[\int_0^\infty g(t+r,\xi_r)\,dr\bigg]. \tag{10}$$

PROOF. We first assume that π is supported on $(0,A]$ for some $A<\infty$. Then ψ can be extended to an analytic function on $\mathbb{R}$ and, furthermore,

$$\psi(u) = \alpha u + \sum_{p=2}^{\infty}(-1)^p\gamma_p u^p.$$

We fix $g\in\mathscr{H}$ such that g is supported on $[0,T]\times E$ and for every $\lambda>0$ we set

$$u_\lambda(t,x) = \mathbb{N}_x\bigg[1-\exp-\lambda\int_0^\sigma g(t+H_s,\widehat{W}_s)\,ds\bigg].$$

The proof of the next lemma is postponed to the end of this section.

LEMMA 4.3. *There exists a constant $C = C(g) < \infty$ such that, for every $n \geq 1$,*

$$|T^n g(t, x)| \leq C^n \, 1_{[0,T]}(t).$$

It follows that, for $0 < \lambda < C^{-1}$,

$$\sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \mathbb{N}_x \left[\left(\int_0^{\sigma} g(t + H_s, \widehat{W}_s) \, ds \right)^n \right] < \infty.$$

By Fubini's theorem, we get for $0 < \lambda < C^{-1}$,

$$\begin{aligned} u_\lambda(t, x) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\lambda^n}{n!} \mathbb{N}_x \left[\left(\int_0^{\sigma} g(t + H_s, \widehat{W}_s) \, ds \right)^n \right] \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \lambda^n T^n g(t, x). \end{aligned} \tag{11}$$

Set $\psi^*(u) = \psi(u) - \alpha u = \sum_{p=2}^{\infty} (-1)^p \gamma_p u^p$. We can again use Fubini's theorem to evaluate

$$\begin{aligned} &\Pi_x \left[\int_0^{\infty} dr \, e^{-\alpha r} \psi^*(u_\lambda(t + r, \xi_r)) \right] \\ &\quad = \Pi_x \left[\int_0^{\infty} dr \, e^{-\alpha r} \sum_{p=2}^{\infty} (-1)^p \gamma_p \left(\sum_{n=1}^{\infty} (-1)^{n-1} \lambda^n T^n g(t + r, \xi_r) \right)^p \right] \\ &\quad = \Pi_x \left[\int_0^{\infty} dr \, e^{-\alpha r} \sum_{p=2}^{\infty} (-1)^p \gamma_p \sum_{\substack{n_1, \dots, n_p \\ n_i \geq 1}} (-1)^{\sum n_i - p} \lambda^{\sum n_i} T^{n_1} g(t + r, \xi_r) \right. \\ &\qquad\qquad \left. \cdots T^{n_p} g(t + r, \xi_r) \right] \\ &\quad = \sum_{n=2}^{\infty} (-1)^n \lambda^n \left(\sum_{p=2}^{n} \gamma_p \sum_{\substack{n_1 + \dots + n_p = n \\ n_i \geq 1}} T^1(T^{n_1} g \cdots T^{n_p} g)(t, x) \right) \\ &\quad = \sum_{n=2}^{\infty} (-1)^n \lambda^n T^n g(t, x), \end{aligned}$$

by Proposition 4.1. Comparing with (11) gives

$$\begin{aligned} u_\lambda(t, x) + \Pi_x \left[\int_0^{\infty} dr \, e^{-\alpha r} \psi^*(u_\lambda(t + r, \xi_r)) \right] &= \lambda \, T^1 g(t, x) \\ &= \lambda \, \Pi_x \left[\int_0^{\infty} dr \, e^{-\alpha r} g(t + r, \xi_r) \right]. \end{aligned}$$

This equation can also be written in the form

$$u_\lambda(t, x) - \alpha \Pi_x \left[\int_0^{\infty} dr \, e^{-\alpha r} u_\lambda(t + r, \xi_r) \right] = \Pi_x \left[\int_0^{\infty} dr \, e^{-\alpha r} h_\lambda(t + r, \xi_r) \right],$$

where $h_\lambda = \lambda g - \psi \circ u_\lambda$. Note that h_λ is bounded and supported on $[0, T] \times E$. It is then easy to see, for instance by using the resolvent equation for the space–time process (r, ξ_r), that the previous equation is equivalent to

$$u_\lambda(t, x) = \Pi_x\left[\int_0^\infty dr\, h_\lambda(t + r, \xi_r)\right],$$

which gives

$$u_\lambda(t, x) + \Pi_x\left[\int_0^\infty \psi(u_\lambda(t + r, \xi_r))\, dr\right] = \lambda\, \Pi_x\left[\int_0^\infty g(t + r, \xi_r)\, dr\right]. \tag{12}$$

This is the desired integral equation, except that we have assumed $0 < \lambda < C^{-1}$, and we want to get it with $\lambda = 1$. However, from the definition of u_λ, we easily get that, for every fixed x, the function $\lambda \to u_\lambda(t, x)$ is analytic and can be extended to an analytic function on the half-plane $\{z \in \mathbb{C},\ \mathrm{Re}\, z > 0\}$. Since ψ is also analytic, we obtain that the left-hand side of (12) is also an analytic function of λ. By analytic continuation, we conclude that (12) holds for every $\lambda > 0$.

It remains to get rid of the assumption on the support of π. To this end, we use the truncation procedure of Section 2.3. In the notation of this subsection, we introduce

$$u^{(k)}(t, x) = \mathbb{N}_x^{(k)}\left[1 - \exp - \int_0^\sigma g(t + H_s, \widehat{W}_s)\, ds\right].$$

Since $H_s = \zeta_{W_s}$, $\mathbb{N}_x^{(k)}$ a.e., we may replace $g(t+H_s, \widehat{W}_s)$ in the previous formula by a function of W_s. We can then apply Section 2.3 to obtain

$$\lim_{k\uparrow\infty} \uparrow\ u^{(k)}(t, x) = u(t, x).$$

By the first part of the proof, we know that $u^{(k)}$ solves

$$u^{(k)}(t, x) + \Pi_x\left[\int_0^\infty \psi^{(k)}(u^{(k)}(t + r, \xi_r))\, dr\right] = \Pi_x\left[\int_0^\infty g(t + r, \xi_r)\, dr\right].$$

Recall that $\psi^{(k)} \downarrow \psi$ as $k \uparrow \infty$ and also note that $u(t, x) \le \mathbb{N}_x[\int_0^\sigma g(t + H_s, \widehat{W}_s)\, ds] = T^1 g(t, x)$. By letting $k \uparrow \infty$ in the previous equation and using simple monotonicity arguments, we arrive at the desired integral equation for u. Finally, the uniqueness of the nonnegative solution of this integral equation is a straightforward consequence of Gronwall's lemma, using the fact that ψ is Lipschitz. □

PROOF OF LEMMA 4.3. For every $u \in \mathbb{R}$, set

$$\overline{\psi}(u) = \alpha\, u + \beta\, u^2 + \int \pi(dr)(e^{ru} - 1 - ru) = \alpha\, u + \sum_{p=2}^\infty \gamma_p\, u^p.$$

Fix $\varepsilon > 0$ so small that $\int_{[\varepsilon T,\infty)} (\overline{\psi}(u))^{-1}\,du > T$. Then the formula

$$\int_{\varepsilon T}^{v(t)} \frac{du}{\overline{\psi}(u)} = T - t, \qquad 0 \le t \le T,$$

defines a nonnegative function $(v(t),\ 0 \le t \le T)$, which solves the integral equation

$$v(t) = \varepsilon T + \int_t^T \overline{\psi}(v(s))\,ds. \tag{13}$$

For $0 \le t \le T$, define by induction

$$v_1(t) = \varepsilon T,$$
$$v_n(t) = \sum_{p=2}^{n} \gamma_p \sum_{\substack{n_1+\cdots+n_p=n \\ n_i \ge 1}} \int_t^T v_{n_1}(s)\cdots v_{n_p}(s)\,ds.$$

We then claim that, for every $n \ge 1$, $t \in [0, T]$,

$$v_1(t) + \cdots + v_n(t) \le v(t). \tag{14}$$

The bound (14) is trivial when $n = 1$. It then follows by induction from the integral equation (13), observing that for $n \ge 1$,

$$\int_t^T \overline{\psi}(v_1(s) + \cdots + v_n(s))\,ds \ge v_2(t) + \cdots + v_{n+1}(t).$$

The last bound is easy from the series expansion of $\overline{\psi}$ and the recursive definition of the functions v_n.

To complete the proof, choose $\delta > 0$ small enough so that $T^1(\delta g)(t, x) \le v_1(t)$ for $t \in [0, T]$. Proposition 4.1 and an immediate induction argument imply that, for every $n \ge 1$, $t \in [0, T]$,

$$T^n(\delta g)(t, x) \le v_n(t) \le v(t).$$

Since $T^n(\delta g) = \delta^n T^n g$ and v is bounded above over $[0, T]$, the desired result follows. □

5. Application to superprocesses.

5.1. *Superprocesses and their canonical measures.* In this subsection, we briefly recall a few basic facts about superprocesses, some of which are part of the folklore of the subject. The (ξ, ψ) superprocess is the Borel right Markov process with cadlag paths and values in $M_f(E)$, denoted by $(Y_t,\ t \ge 0;\ \mathbb{Q}_m,\ m \in M_f(E))$, whose transition kernel is characterized by the following identity: For $\varphi \in \mathscr{B}_{b+}(E)$, $t > 0$,

$$\mathbb{Q}_m[\exp -\langle Y_t, \varphi\rangle] = \exp -\langle m, v_t^\varphi\rangle, \tag{15}$$

where $(v_t^\varphi(y);\ t \geq 0,\ y \in E)$ is the unique nonnegative solution of the integral equation

$$v_t(y) + \Pi_y\left[\int_0^t \psi(v_{t-r}(\xi_r))\,dr\right] = \Pi_y[\varphi(\xi_t)]. \tag{16}$$

See in particular Theorem 2.1.3 in [3]. We may and will assume that Y is the canonical process on $\mathbb{D}([0,\infty),\ M_f(E))$.

Many results about superprocesses are more conveniently expressed in terms of their canonical measures. To give a brief presentation of canonical measures, fix $y \in E$ and let $t > 0$. The law of Y_t under $\mathbb{Q}_{\delta_y}$ is an infinitely divisible distribution on $M_f(E)$. By standard results about random measures (see [15], Chapter 1), there exist an element $\gamma_t \in M_f(E)$ and a σ-finite measure $S_t(y, d\mu)$ on $M_f(E)$, such that $S_t(y, \{0\}) = 0$,

$$\int(\langle\mu, 1\rangle \wedge 1)\,S_t(y, d\mu) < \infty$$

and for $\varphi \in \mathscr{B}_{b+}(E)$,

$$v_t^\varphi(y) = -\log \mathbb{Q}_{\delta_x}[\exp -\langle Y_t, \varphi\rangle] = \langle\gamma_t, \varphi\rangle + \int(1 - e^{-\langle\mu,\varphi\rangle})\,S_t(y, d\mu).$$

Let us verify that $\gamma_t = 0$ under our assumptions. Taking $\varphi = \lambda > 0$, we deduce from (16) that $v_t^\lambda(y) = v_t^\lambda$ does not depend on y and

$$\int_{v_t^\lambda}^{\lambda} \frac{dv}{\psi(v)} = t.$$

Under our assumptions on ψ, $v^{-1}\psi(v)$ converges to ∞ as $v \to \infty$ and it follows that $\lambda^{-1}v_t^\lambda$ tends to 0 as $\lambda \to \infty$, which clearly implies $\gamma_t = 0$.

Furthermore, the Markov property of superprocesses implies that, for $t' > t > 0$,

$$v_{t'}^\varphi(y) = \int(1 - \exp(-\langle\mu,\ \varphi\rangle))\,S_{t'}(y, d\mu) = \int(1 - \exp(-\langle\mu,\ v_{t'-t}^\varphi\rangle))\,S_t(y, d\mu).$$

Using this last property in particular, we can construct (cf. the arguments in Chapter 19 of [4]) a σ-finite measure $\mathscr{N}_y$ on $\mathbb{D}((0,\infty),\ M_f(E))$, which is characterized by the following properties:

1. For every $t > 0$, the "law" of Y_t under $\mathscr{N}_y(\cdot \cap \{Y_t \neq 0\})$ is $S_t(y, d\mu)$.
2. The process $(Y_t,\ t > 0)$ is Markovian under $\mathscr{N}_y$ with the transition kernels of the (ξ, ψ) superprocess.
3. $\mathscr{N}_y[Y_t = 0,\ \forall t > 0] = 0$.

The measures $\mathscr{N}_y$, $y \in E$, are called the canonical measures of the (ξ, ψ) superprocess. We can recover the measures $\mathbb{Q}_m$ from the collection $(\mathscr{N}_y,\ y \in E)$ by the following "cluster" construction. Let $m \in M_f(E)$ and let $\sum_{j\in J} \delta_{Y^j}$ be a Poisson point measure on $\mathbb{D}((0,\infty),\ M_f(E))$ with intensity $\int m(dy)\mathscr{N}_y(\cdot)$. Then $\mathbb{Q}_m$ is the law of the process $(\widetilde{Y}_t,\ t\ge 0)$ defined by

$$\widetilde{Y}_0 = m,$$
$$\widetilde{Y}_t = \sum_{j\in J} Y_t^j, \quad \text{if } t>0.$$

5.2. *The Lévy snake and superprocesses.* We fix a point $x\in E$ and consider the (ξ,ψ) Lévy snake $(\rho_s, W_s;\ s\ge 0)$ with initial point x.

THEOREM 5.1. *There exists a cadlag process $(Z_t,\ t>0)$ defined on Ω and with values in $M_f(E)$, such that:*

(i) $\mathbb{N}_x$ *a.e., for every* $g\in\mathscr{B}_{b+}(\mathbb{R}_+\times E)$,

$$\int_0^\sigma g(H_s,\widehat{W}_s)\,ds = \int_0^\infty dt\int Z_t(dy)\,g(t,y).$$

(ii) *The law of $(Z_t,\ t>0)$ under $\mathbb{N}_x$ is $\mathscr{N}_x$.*

PROOF. We first verify that, for $g\in\mathscr{H}$,

$$\mathbb{N}_x\left[1-\exp-\int_0^\sigma g(H_s,\widehat{W}_s)\,ds\right] = \mathscr{N}_x\left[1-\exp-\int_0^\infty dt\int Y_t(dy)\,g(t,y)\right]. \tag{17}$$

By standard results about superprocesses (see, e.g., [6], Theorem I.1.8) we know that the function

$$v(s,x) := -\log\mathbb{Q}_{\delta_x}\left[\exp-\int_0^\infty dt\int Y_t(dy)\,g(s+t,y)\right]$$

solves (10). The Poisson cluster construction implies that the right-hand side of (17) is equal to $v(0,x)$. The identity (17) thus follows from Theorem 4.2.

Then, for every $t>0$, $\varepsilon>0$, let Z_t^ε be the element of $M_f(E)$ defined by

$$\langle Z_t^\varepsilon, f\rangle = \frac{1}{\varepsilon}\int_0^\sigma 1_{[t,t+\varepsilon]}(H_s)\,f(\widehat{W}_s)\,ds.$$

It follows from (17) that the finite-dimensional marginals under $\mathbb{N}_x$ of the process $(Z_t^\varepsilon;\ t>0,\varepsilon>0)$ are the same as those of the process

$$\left(\frac{1}{\varepsilon}\int_t^{t+\varepsilon} Y_r\,dr;\ t>0,\varepsilon>0\right)$$

under $\mathscr{N}_x$. A little care is needed here because $\mathbb{N}_x$ and $\mathscr{N}_x$ are both infinite measures. We leave details to the reader. As a consequence, for every $t>0$, the limit

$$Z_t = \lim_{n\uparrow\infty} Z_t^{1/n}$$

exists $\mathbb{N}_x$ a.e. and the process $(Z_t,\ t > 0)$ has the same finite-dimensional marginals as $(Y_t,\ t > 0)$ under $\mathscr{N}_x$. Thus, this process also has a cadlag modification under $\mathbb{N}_x$, and we still denote by $(Z_t,\ t > 0)$ this cadlag modification. Finally, if f is bounded and continuous on E, and v is a continuous function on $\mathbb{R}_+$ with compact support contained in $(0,\infty)$, we have $\mathbb{N}_x$ a.e.,

$$\begin{aligned}\int_0^\sigma v(H_s)\, f(\widehat{W}_s)\, ds &= \lim_{n\uparrow\infty} \int_0^\sigma n\biggl(\int_{(H_s-n^{-1})^+}^{H_s} v(r)\, dr\biggr) f(\widehat{W}_s)\, ds \\ &= \lim_{n\uparrow\infty} \int_0^\infty dr\, v(r)\, \langle Z_r^{1/n}, f\rangle \\ &= \int_0^\infty dr\, v(r) \langle Z_r, f\rangle,\end{aligned}$$

where to justify the last convergence we observe that, for $0 < \delta < T < \infty$, $\sup\{\langle Z_t^\varepsilon, 1\rangle,\ 0 < \varepsilon \le 1,\ \delta \le t \le T\}$ has the same distribution under $\mathbb{N}_x$ as

$$\sup\biggl\{\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \langle Y_r, 1\rangle\, dr;\ 0 < \varepsilon \le 1,\ \delta \le t \le T\biggr\} < \infty, \qquad \mathscr{N}_x \text{ a.e.}$$

Property (i) of Theorem 5.1 now follows easily, which completes the proof. □

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DMI—ECOLE NORMALE SUPÉRIEURE
45, RUE D'ULM
75230 PARIS CEDEX 05
FRANCE
E-MAIL: legall@dmi.ens.fr

DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ PARIS-SUD
CENTRE D'ORSAY
91405 ORSAY CEDEX
FRANCE
E-MAIL: yves.lejan@math.u-psud.fr