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FUNCTIONAL ERDŐS–RENYI LAWS FOR SEMIEXPONENTIAL RANDOM VARIABLES[1]

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For an i.i.d. sequence of random variables with a semiexponential distribution, we give a functional form of the Erdős–Renyi law for partial sums. In contrast to the classical case, that is, the case where the random variables have exponential moments of all orders, the set of limit points is not a subset of the continuous functions. This reflects the bigger influence of extreme values. The proof is based on a large deviation principle for the trajectories of the corresponding random walk. The normalization in this large deviation principle differs from the usual normalization and depends on the tail of the distribution. In the same way, we prove a functional limit law for moving averages.

1. Introduction. Let $Y, Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variables with $E[Y] = 0$. Consider the partial empirical means

$$\xi_{n,m} = \frac{1}{k_n} \sum_{j=m+1}^{m+k_n} Y_j, \qquad m = 0, 1, 2, \ldots, n - k_n, \; n = 1, 2, \ldots$$

over blocks of length k_n. How fast should the block length k_n increase in order to have nontrivial fluctuations? In other words, we want to choose k_n such that, P-a.s.,

$$0 < \limsup_{n\to\infty} \sup_{0\le m\le n-k_n} \xi_{n,m} =: \alpha < \infty.$$

If $E[\exp(\lambda Y)] < \infty$ for all $\lambda \in \mathbb{R}$, it is well known, and goes back to Erdős–Renyi [5], that the block length k_n should be of order $\log n$. The constant α is then given in terms of the distribution of Y. The same question was answered earlier in [12] for moving averages of the form

$$\xi_n = \frac{1}{k_n} \sum_{j=n+1}^{n+k_n} Y_j.$$

The statements have been extended to the case where $E[\exp(\lambda Y)] < \infty$ for some $\lambda > 0$. Refinements of these results, among them exact rates of conver-

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gence, have been obtained in [3]. Under the assumption that $E[\exp(\lambda Y)] < \infty$ for some $\lambda > 0$, functional limit theorems for

$$(1)\qquad \xi_{n,m}(t) = \frac{1}{k_n} \sum_{j=m+1}^{m+[k_n t]} Y_j + \left(t - \frac{[k_n t]}{k_n}\right) Y_{m+[k_n t]+1}, \qquad 0 \le t \le 1$$

have been proved in [2] and [11]. The proof of a functional limit law can be based on a large deviation principle in function space. The most renowned example is the proof of Strassen's law of the iterated logarithm based on Schilder's theorem, given in [13]. Under the above assumption, sample path large deviation principles have been derived in [1] and [9]; see also [4].

Our goal in this paper is to give functional limit laws for $\xi_{n,m}$, where the size of k_n has to be determined, in the case where Y has a semiexponential distribution. That is, $E[Y^p] < \infty$ for all $p \ge 0$ but $E[\exp(\lambda Y)] = \infty$ for each $\lambda > 0$. This was left as an open problem in [2]; see [2], Remark 3.1. We also consider moving averages of the form

$$(2)\qquad \xi_n(t) = \frac{1}{k_n} \sum_{j=n+1}^{n+[k_n t]} Y_j + \left(t - \frac{[k_n t]}{k_n}\right) Y_{n+[k_n t]+1}, \qquad 0 \le t \le 1.$$

It turns out that the block length k_n is of bigger order than $\log n$ and depends on the distribution of Y. The set of limit points of $\{\xi_{n,m}\}$ or $\{\xi_n\}$ is not a subset of the continuous functions as in [11]. This reflects the bigger influence of extreme values on the partial sums. The proof is based on a large deviation principle for the trajectories of the corresponding random walk. This large deviation principle does not have the usual normalization; due to the influence of extreme values, the convergence of $(1/n)\sum_{j=1}^n Y_j$ to 0 is slower than exponential in n. The large deviation principle is of independent interest and complements results of [1], [9] or [8].

2. Statement of the results.

Throughout the paper, $Y, Y_1, Y_2, \dots$ will be i.i.d. with

$$E[Y] = 0 \quad \text{and} \quad E[\exp(\lambda Y)] < \infty \quad \text{for all } \lambda \le 0.$$

We will assume that

$$a_1(t)\exp(-b(t)t^r) \le P[Y \ge t] \le a_2(t)\exp(-b(t)t^r)$$

for t large enough, where $0 < r \le 1$ and a_1, a_2 and b are slowly varying functions and $b(t)/t^{1-r}$ is nonincreasing. Let $\xi_{n,m}(t)$ $(0 \le t \le 1)$ and $\xi_n(t)$ $(0 \le t \le 1)$ be defined as in (1) and (2). Let $E = \{x \in L^1[0,1]\colon x(0) = 0\}$ and let d

be the metric on E given by $d(x,y)=\int_0^1 |x(s)-y(s)|\,ds$, $x,y\in E$. Let

$$I(x)=\begin{cases}\displaystyle\sum_{t:\,x(t^+)\neq x(t^-)}(x(t^+)-x(t^-))^r, & x \text{ nondecreasing pure jump function},\\ +\infty, & \text{otherwise},\end{cases}$$

and

$$\tilde{I}(x)=\begin{cases}x(1), & x \text{ nondecreasing},\\ +\infty, & \text{otherwise}.\end{cases}$$

REMARK. Note that the level set $\{x\colon \tilde{I}(x)\le 1\}$ contains all distribution functions of probability distributions on $]0,1]$.

We will always equip E with its Borel σ-field $\mathscr{B}$. Let

$$Z_n(t)=\frac{1}{n}\sum_{j=1}^{[nt]}Y_j+\left(t-\frac{[nt]}{n}\right)Y_{[nt]+1},\qquad 0\le t\le 1. \tag{3}$$

The following theorems are our main results.

THEOREM 1. *Assume $0<r<1$. The distributions of (Z_n) satisfy a large deviation principle on (E,d) with normalization $b(n)n^r$ and with the good rate function I. This means, for every $A\in\mathscr{B}$, we have*

$$-\inf_{x\in \operatorname{int} A} I(x)\le \liminf_n \frac{1}{b(n)n^r}\log P[Z_n\in A]$$
$$\le \limsup_n \frac{1}{b(n)n^r}\log P[Z_n\in A]\le -\inf_{x\in \operatorname{cl}(A)} I(x)$$

and I is lower semicontinuous and has compact level sets.

THEOREM 2. *Assume $r=1$ and $b(t)\longrightarrow 0$ as $t\to\infty$. The distributions of (Z_n) satisfy a large deviation principle on (E,d) with normalization $b(n)n$ and with the good rate function $\tilde{I}$.*

Theorems 1 and 2 can be used to derive the following "Strassen-type" theorems.

THEOREM 3. *Assume $0<r<1$. Let $c>0$. Assume b is such that there is a sequence of positive constants (s_n) with*

$$s_n^r b(s_n(c\log n)^{1/r})\longrightarrow d \text{ as } n\to\infty, \tag{4}$$

where $0<d<\infty$. Let

$$k_n=[s_n(c\log n)^{1/r}] \tag{5}$$

and

$$K = \left\{ x \in E \colon I(x) \le \frac{1}{cd} \right\}. \tag{6}$$

Then, for some $n_o > 0$, (i) *and* (ii) *hold* P*-a.s.*

(i) *The set* $\{\xi_n,\ n \ge n_0\}$ *is relatively compact in* (E, d) *and the set of its limit points is* K.

(ii) *The set* $\{\xi_{n,m},\ m = 0, 1, 2, \dots, n - k_n,\ n \ge n_0\}$ *is relatively compact in* (E, d), *and the set of its limit points is* K.

THEOREM 4. *Assume* $r = 1$ *and* $b(t) \longrightarrow 0$ *as* $t \to \infty$. *Let* $c > 0$. *Assume* b *is such that there exists a sequence of positive constants* (s_n) *with*

$$s_n b(s_n\, c \log n) \longrightarrow d \text{ as } n \to \infty, \tag{7}$$

where $0 < d < \infty$. *Let*

$$k_n = [s_n\, c \log n] \tag{8}$$

and

$$K = \left\{ x \in E \colon \tilde{I}(x) \le \frac{1}{cd} \right\}. \tag{9}$$

Then, for some $n_0 > 0$, (i) *and* (ii) *in Theorem 3 hold* P*-a.s.*

In contrast to the classical case, that is, the case where the random variables have exponential moments of all orders, the set K of limit points is not a subset of the continuous functions. Note that if $r < 1$, K contains only pure jump functions, whereas the case $r = 1$ is the "borderline case" where functions in K can have jumps as well as continuous parts.

The following corollary is immediate.

COROLLARY 1. *Let* $F\colon E \to \mathbb{R}$ *be a continuous function. Then, in the setting of Theorem 3 or Theorem 4,*

$$\limsup_n F(\xi_n) = \sup_{x \in K} F(x), \qquad P\text{-a.s.} \tag{10}$$

and

$$\limsup_n \sup_{0 \le m \le n - k_n} F(\xi_{n,m}) = \sup_{x \in K} F(x), \qquad P\text{-a.s.} \tag{11}$$

3. Proofs.

PROOF OF THEOREM 1. The underlying one-dimensional large deviation principle is the following.

LEMMA 1. *The distributions of* $(1/n)\sum_{j=1}^{n} Y_j = Z_n(1)$ *satisfy a large deviation principle with normalization* $b(n)n^r$ *on* $\mathbb{R}$ *with the good rate function* I_1, *where*

$$I_1(t) = \begin{cases} t^r, & t \geq 0, \\ +\infty, & \textit{else}. \end{cases}$$

This goes back to [10]; a simple proof can be found in [6].

REMARK. Since we have $E[\exp(\lambda Y)] < \infty$ for all $\lambda < 0$, Cramér's theorem (see [4]) yields, for each $t < 0$,

$$\limsup_n \frac{1}{n} \log P\left[\frac{1}{n}\sum_{j=1}^{n} Y_j \leq t\right] < 0$$

and this implies

$$\limsup_n \frac{1}{b(n)n^r} \log P\left[\frac{1}{n}\sum_{j=1}^{n} Y_j \leq t\right] = -\infty.$$

As a consequence, I and $\tilde{I}$ are only finite on nondecreasing functions.

We now proceed as in the proof of Theorem 5.1.2 in [4]. In the next step, we show that the finite-dimensional marginals of (Z_n) satisfy a large deviation principle.

LEMMA 2. *Let* T *denote the collection of all ordered finite subsets of* $[0,1]$. *For any* $\tau = (t_1, \dots, t_d)$ *where* $0 = t_0 < t_1 < t_2 < \cdots < t_d \leq 1$, *the distributions of* $(Z_n(t_1), \dots, Z_n(t_d))$ *satisfy a large deviation principle with normalization* $b(n)n^r$ *on* $\mathbb{R}^d$ *with the good rate function* I_τ, *where*

$$I_\tau(x) = \begin{cases} \sum_{j=1}^{d} (x(t_j) - x(t_{j-1}))^r, & \textit{if } x(t_j) \geq x(t_{j-1}),\ j = 1, 2, \dots, d \\ +\infty, & \textit{otherwise}. \end{cases}$$

SKETCH OF PROOF. As in [4], page 153, define $\overline{Z}_n(t) = (1/n)\sum_{j=1}^{[nt]} Y_j$ $(0 \leq t \leq 1)$ and note that the distributions of (Z_n) and $(\overline{Z}_n)$ are exponentially equivalent in the normalization $b(n)n^r$. More precisely, $d(Z_n, \overline{Z}_n) = (1/2n)\sum_{j=1}^{n}(Y_j/n)$, and therefore, for each $\lambda > 0$, $P[d(Z_n, \overline{Z}_n) \geq \delta] \leq P[(1/n)\sum_{j=1}^{n} Y_j \geq 2\lambda\delta]$ for n large enough. Since

$$\limsup_n \frac{1}{b(n)n^r} \log P\left[\frac{1}{n}\sum_{j=1}^{n} Y_j \geq 2\lambda\delta\right] \leq -2^r\lambda^r\delta^r$$

due to Lemma 1, and $\lambda > 0$ was arbitrary, we see that

$$\limsup_n \frac{1}{b(n)n^r} \log P[d(Z_n, \overline{Z}_n) \geq \delta] = -\infty.$$

Observe that $Z_n^\tau = (\overline{Z}_n(t_1), \overline{Z}_n(t_2) - \overline{Z}_n(t_1), \ldots, \overline{Z}_n(t_d) - \overline{Z}_n(t_{d-1}))$ is a vector of i.i.d. random variables. Heuristically, if $x(t_1) < x(t_2) < \cdots < x(t_d)$,

$$\begin{aligned}
P[Z_n^\tau \approx x] \approx P\Bigg[&\frac{1}{t_1 n} \sum_{j=1}^{[t_1 n]} Y_j \approx \frac{x(t_1)}{t_1}, \\
&\frac{1}{(t_2 - t_1)n} \sum_{j=[t_1 n]+1}^{[t_2 n]} Y_j \approx \frac{x(t_2) - x(t_1)}{t_2 - t_1}, \ldots, \\
&\frac{1}{(t_d - t_{d-1})n} \sum_{j=[t_{d-1} n]+1}^{[t_d n]} Y_j \approx \frac{x(t_d) - x(t_{d-1})}{t_d - t_{d-1}}\Bigg] \\
\approx \exp\Bigg(&-b(t_1 n)(t_1 n)^r \left(\frac{x(t_1)}{t_1}\right)^r \\
&- b((t_2 - t_1)n)((t_2 - t_1)n)^r \left(\frac{x(t_2) - x(t_1)}{t_2 - t_1}\right)^r \\
&- \cdots - b((t_d - t_{d-1})n)((t_d - t_{d-1})n)^r \left(\frac{x(t_d) - x(t_{d-1})}{t_d - t_{d-1}}\right)^r\Bigg) \\
\approx \exp\Bigg(&-b(n)n^r \sum_{j=1}^{d} (x(t_j) - x(t_{j-1}))^r\Bigg) \\
= \exp(&-b(n)n^r I_\tau(x))
\end{aligned}$$

and the argument can be made precise using the contraction principle, as in [4], page 154. Note that if $x(t_j) < x(t_{j-1})$ and $\varepsilon < \frac{1}{2}(x(t_{j-1}) - x(t_j))$, we have

$$\lim_n \frac{1}{b(n)n^r} \log P\big[|Z_n(t_j) - x(t_j)| < \varepsilon,\ |Z_n(t_{j+1}) - x(t_{j+1})| < \varepsilon\big] = -\infty$$

due to the remark following Lemma 1. □

Let X denote the space of all functions x: $[0, 1] \to \mathbb{R}$, $x(0) = 0$, equipped with the topology of pointwise convergence.

LEMMA 3. *The distributions of* (Z_n) *satisfy a large deviation principle with normalization* $b(n)n^r$ *on* X *with the good rate function* I_T, *where*

$$I_T(x) = \sup_{\tau \in T} I_\tau(x). \tag{12}$$

Lemma 3 follows from Lemma 2 if we apply the large deviation theorem for projective limits of Dawson and Gärtner; see [4], Theorem 4.6.1.

LEMMA 4. *For all* $x \in X$, $I_T(x) = I(x)$.

PROOF. Clearly, $I_T(x) = I(x)$ if x is a jump function. Further, both rate functions are infinite if there is $s < t$ such that $x(s) > x(t)$. In the next step, we show that $I_T(x) = \infty$ if x is continuous on some interval $[s, t]$ with $0 \leq s < t \leq 1$, and $x(s) \neq x(t)$. Let $t_j^{(n)} = s + ((j-1)/n)(t-s)$, $j = 1, 2, \ldots, n+1$ and $\tau_n = (t_1^{(n)}, \ldots, t_{n+1}^{(n)})$. Then we have

$$\begin{aligned} x(t) - x(s) &= \sum_{j=1}^{n} x\bigl(t_{j+1}^{(n)}\bigr) - x\bigl(t_j^{(n)}\bigr) \\ &\leq \max_{1\leq j\leq n} \Bigl|x\bigl(t_{j+1}^{(n)}\bigr) - x\bigl(t_j^{(n)}\bigr)\Bigr|^{1-r} \sum_{j=1}^{n} \Bigl(x\bigl(t_{j+1}^{(n)}\bigr) - x\bigl(t_j^{(n)}\bigr)\Bigr)^{r} \end{aligned}$$

and since $\max_{1\leq j\leq n} |x(t_{j+1}^{(n)}) - x(t_j^{(n)})|^{1-r} \to 0$ for $n \to \infty$, this implies that $I_{\tau_n}(x) \to \infty$ for $n \to \infty$. Let x be nondecreasing, x not a pure jump function. Then there is a nondecreasing pure jump function x_1 and a nondecreasing continuous function x_2 such that $x = x_1 + x_2$. Due to the last step, we can find a sequence (τ_n) such that $I_{\tau_n}(x_2) \to \infty$ for $n \to \infty$, and x is continuous in all points of τ_n, for each n. Then we have $I_{\tau_n}(x) \geq I_{\tau_n}(x - x_1) = I_{\tau_n}(x_2)$, and this implies $I_{\tau_n}(x) \to \infty$ for $n \to \infty$. □

Let

$$K_L = \bigl\{x \in E\colon \operatorname{var}_{[0,1]}(x) \leq L^{2/r}\bigr\}. \tag{13}$$

LEMMA 5. *For each* L, K_L *is compact in* (E, d). *For* L *large enough,*

$$\limsup_n \frac{1}{b(n)n^r} \log P[Z_n \in K_L^c] \leq -L. \tag{14}$$

In particular, the distributions of (Z_n) *are exponentially tight in the normalization* $b(n)n^r$.

PROOF. Note that $K_L \subseteq \{x \in E\colon \sup_{0\leq s\leq 1} |x(s)| \leq L^{2/r}\}$. Let $(x_n) \subseteq K_L$. Then each x_n can be identified with a signed measure. Let $x_n = x_n^+ - x_n^-$ be the Jordan–Hahn decomposition of x_n. Due to Prohorov's theorem, we can find a subsequence (n_j) such that $(x_{n_j}^+)$, $(x_{n_j}^-)$ converge weakly to x^+, x^-. Then $x = (x^+ - x^-) \in K_L$. This implies that $d(x_{n_j}, x) \to 0$, since $(x_{n_j}^+ - x_{n_j}^-)(s) \to (x^+ - x^-)(s)$ for λ-a.a.s.

We have

$$P[Z_n \in K_L^c] \leq P\Biggl[\frac{1}{n}\sum_{j=1}^{n} |Y_j| \geq L^{2/r}\Biggr].$$

Let $L_0 = E[|Y|]$. Then $|Y_1| - L_0, |Y_2| - L_0, \ldots$ satisfy the assumptions of Lemma 1, and we have

$$\limsup_n \frac{1}{b(n)n^r} \log P\left[\frac{1}{n}\sum_{i=1}^{n}|Y_i| \geq L^{2/r}\right] \leq -(L^{2/r} - L_0)^r \leq -L,$$

where the last inequality holds for L large enough. □

Now we are able to prove the upper bound in Theorem 1.

LEMMA 6. *For each $A \in \mathscr{B}$,*

$$\limsup_n \frac{1}{b(n)n^r} \log P[Z_n \in A] \leq -\inf_{x\in \mathrm{cl}(A)} I(x).$$

PROOF. We have

$$P[Z_n \in A] \leq P[Z_n \in \mathrm{cl}(A) \cap K_L] + P[Z_n \in K_L^c].$$

Since $\mathrm{cl}(A) \cap K_L$ is closed w.r.t. the topology of pointwise convergence, the claim follows from Lemma 3 and Lemma 5. □

In the next step we prove the lower bound in Theorem 1.

LEMMA 7. *For each $A \in \mathscr{B}$,*

$$\liminf_n \frac{1}{b(n)n^r} \log P[Z_n \in A] \geq -\inf_{x\in \mathrm{int}(A)} I(x).$$

PROOF. Let $U_\delta(x) = \{y: d(x, y) < \delta\}$. It is enough to show that, for x with $I(x) < \infty$ and $A = U_\delta(x)$,

$$\liminf_n \frac{1}{b(n)n^r} \log P[Z_n \in A] \geq -I(x).$$

Assume for simplicity that x has one jump of height h at t, where $0 < t < 1$. Then, for some $\varepsilon = \varepsilon(\delta)$ small enough,

$$\begin{aligned} P[Z_n \in U_\delta(x)] \geq P\big[&Z_n(t-\varepsilon) \leq \varepsilon,\ h < Z_n(t+\varepsilon) - Z_n(t-\varepsilon) \\ &< h + \varepsilon,\ Z_n(1) - Z_n(t+\varepsilon) \leq \varepsilon\big] \end{aligned}$$

and

$$\liminf_n \frac{1}{b(n)n^r} \log P[Z_n \in U_\delta(x)] \geq -h^r$$

due to Lemma 2. The same argument carries through for any nondecreasing jump function x. □

It remains to show that I has the desired properties.

LEMMA 8. *I is lower semicontinuous and has compact level sets.*

PROOF. To see that I has relatively compact level sets, note that for each $c > 0$, there is $L = L(c)$ such that $\{x: I(x) \leq c\} \subseteq K_L$, and it was shown in Lemma 5 that K_L is compact, for each L.

Assume $d(x_n, x) \to 0$. We have to show that $I(x) \leq \liminf_n I(x_n)$. Without loss of generality, we can assume $I(x_n) < \infty$, for each n. We can then identify x_n, x with distribution functions where $x_n(s) \longrightarrow x(s)$ as $n \to \infty$ if x is continuous in s.

(i) Assume x is continuous on $[0, 1]$. Then $x_n(s) \to x(s)$ for all s, and $I_\tau(x_n) \to I_\tau(x)$ for all τ. Hence

$$\begin{aligned} I(x) &= \sup_\tau I_\tau(x) \\ &= \sup_\tau \liminf_n I_\tau(x_n) \\ &\leq \liminf_n \sup_\tau I_\tau(x_n) \\ &= \liminf_n I(x_n). \end{aligned}$$

In fact, $\liminf_n I(x_n) = \infty$ since we know that $I(x) = \infty$ in this case.

(ii) Assume x is discrete, with countably many jumps. There is a $\tau = (t_1, t_2 \ldots)$ such that x and each x_n are continuous in each t_i. Then $I(x) = I_\tau(x)$ and $I(x_n) = I_\tau(x_n)$, for all n. We conclude $I(x_n) \to I(x)$.

The general case is easy from (i) and (ii). □

The theorem now follows from Lemmas 6, 7 and 8. □

PROOF OF THEOREM 3. (i) Let $\delta > 0$.

(a) Let $K_\delta = \{y: d(y, K) < \delta\}$ where $d(y, K) = \inf\{d(y, z): z \in K\}$. Let $K_\delta^c = E \setminus K_\delta$ denote the complement of K_δ. Note that, since K_δ^c is closed and I is lower semicontinuous and has compact level sets, there is $\varepsilon > 0$ such that $\inf\{I(x): x \in K_\delta^c\} > (1/cd) + \varepsilon$. We will prove that, with probability 1, $\xi_n \in K_\delta^c$ only for finitely many n. We apply Theorem 1 to get

$$P[\xi_n \in K_\delta^c] \leq \exp\left(-b(k_n)k_n^r \left(\frac{1}{cd} + \frac{\varepsilon}{2}\right)\right) \tag{15}$$

for n large enough.

Since $(k_n^r b(k_n)/c \log n) \longrightarrow d$ as $n \to \infty$ due to (4) and (5), we have

$$\begin{aligned} P[\xi_n \in K_\delta^c] &\leq \exp\left(-cd \log n \left(\frac{1}{cd} + \frac{\varepsilon}{3}\right)\right) \\ &= \frac{1}{n^\gamma} \end{aligned} \tag{16}$$

for n large enough, where $\gamma = 1 + cd\,(\varepsilon/3) > 1$.

The claim now follows by applying the Borel–Cantelli lemma.

(b) Let $x \in K$. We will show that with probability 1, (ξ_n) has a subsequence (ξ_{n_j}) converging to x. We can assume w.l.o.g. that $I(x) < 1/cd$. Let $U_\delta(x) = \{y: d(x, y) < \delta\}$. Then there is $\varepsilon > 0$ such that $\inf\{I(y): y \in U_\delta(x)\} \le I(x) < (1/cd) - \varepsilon$. Again, Theorem 1 yields

$$
\begin{aligned}
P[\xi_n \in U_\delta(x)] &\ge \exp\left(-b(k_n)\, k_n^r \left(\frac{1}{cd} - \frac{\varepsilon}{2}\right)\right) \\
&\ge \exp\left(-cd \log n \left(\frac{1}{cd} - \frac{\varepsilon}{3}\right)\right) \qquad (17)\\
&= \frac{1}{n^\gamma}
\end{aligned}
$$

for n large enough, with $\gamma = (1 - (\varepsilon/3)\, cd) < 1$, where we used (4) and (5) in the second inequality.

Let $\lambda > 1$. Consider the subsequence ξ_{n_j} with $n_j = [j^\lambda]$. Choose λ small enough such that $\lambda\gamma < 1$. Then

$$\sum_j P[\xi_{n_j} \in U_\delta(x)] \ge \sum_j \frac{1}{[j^\lambda]^\gamma} \ge \sum_j \left(\frac{1}{j^{\lambda\gamma}}\right) = \infty.$$

Since ξ_{n_j}, $j = 1, 2, \ldots$ are independent for j large enough, the claim follows by applying the Borel–Cantelli lemma. □

(ii)

(a) We show that, with probability 1, there are only finitely many n such that $\xi_{n,m} \in K_\delta^c$ for some $m \in \{0, 1, 2, \ldots, n - k_n\}$. Choose $\lambda > 1$. Then, $\lambda^{\gamma-1} > 1$ for $\gamma = 1 + \varepsilon/3$. Going back to the proof of (i), we have

$$
\begin{aligned}
&P\left[\xi_{n,m} \in K_\delta^c \quad \text{for some } m \in \{0, 1, \ldots, n - k_n\}\right] \\
&\qquad \le nP[\xi_{n,1} \in K_\delta^c] \\
&\qquad \le n\, \frac{1}{n^\gamma},
\end{aligned}
$$

where we used (16) in the last inequality. Let $n_j = [\lambda^j]$. Then we have

$$
\begin{aligned}
&\sum_j P\left[\xi_{n_j,m} \in K_\delta^c \quad \text{for some } m \in \{0, 1, \ldots, n_j - k_{n_j}\}\right] \\
&\qquad \le \sum_j \frac{1}{[\lambda^j]^{\gamma-1}} < \infty.
\end{aligned}
$$

We conclude

$$\limsup_{j\to\infty} \sup_{0 \le m \le n_j - k_{n_j}} d(\xi_{n_j,m}, K) = 0. \qquad (18)$$

To pass from the subsequences (n_j) to the sequence n, we use the following analytical lemma, which plays the role of Lemma 1.20 in [13].

LEMMA 9. *Let K and $(\xi_{n,m})$ be defined as in Theorem 3. Assume that for each $\lambda > 1$ and $n_j = [\lambda^j]$,*

$$\limsup_{j\to\infty} \sup_{0\le m\le n_j - k_{n_j}} d(\xi_{n_j,m}, K) = 0.$$

Then we have

$$\limsup_{n\to\infty} \sup_{0\le m\le n - k_n} d(\xi_{n,m}, K) = 0.$$

PROOF. Choose δ, λ with $0 < \delta < 2/3$, $1 < \lambda < (1+\delta/2)\wedge(1+(\delta/2)(cd)^{1/r})$ and j_0 such that $d(\xi_{n_j,m}, K) \le \delta/4$ for $j \ge j_0$, $m \in \{0, 1, \ldots n_j - k_{n_j}\}$. Choose $L \in \mathbb{N}$ such that $k_{[\lambda n]+1}/k_n \le \lambda$ for $n \ge L$. Let $n(\delta) = L \vee j_0$. For $n \ge n(\delta)$, there is j such that $\lambda^j \le n \le \lambda^{j+1}$. Let $N = [\lambda^{j+1}]$. Then $n \le N \le \lambda n$. Now, observe that

$$\xi_{n,m}(t) = \frac{k_N}{k_n}\, \xi_{N,m}\left(\frac{k_n}{k_N}\, t\right), \qquad 0 \le t \le 1. \tag{19}$$

Let $\overline{x} \in K$ such that

$$d(\xi_{N,m}, \overline{x}) = d(\xi_{N,m}, K) \le \frac{\delta}{4}.$$

Then $\overline{x}(k_n/k_N\,\cdot) \in K$, and

$$\begin{aligned}
d(\xi_{n,m}, K) &\le \int_0^1 \left| \frac{k_N}{k_n}\, \xi_{N,m}\left(\frac{k_n}{k_N}\, t\right) - \overline{x}\left(\frac{k_n}{k_N}\, t\right)\right| dt \\
&\le \int_0^1 \left| \frac{k_N}{k_n}\, \xi_{N,m}\left(\frac{k_n}{k_N}\, t\right) - \frac{k_N}{k_n}\,\overline{x}\left(\frac{k_n}{k_N}\, t\right)\right| dt \\
&\quad + \left(\frac{k_N}{k_n} - 1\right) \int_0^1 \overline{x}\left(\frac{k_n}{k_N}\, t\right) dt \\
&\le \left(\frac{k_N}{k_n}\right)^2 d(\xi_{N,m}, \overline{x}) + \left(\frac{k_N}{k_n} - 1\right)\frac{1}{(cd)^{1/r}} \\
&\le \lambda^2\,\frac{\delta}{4} + (\lambda - 1)\,\frac{1}{(cd)^{1/r}} < \delta,
\end{aligned}$$

where we used

$$\frac{k_N}{k_n} \le \frac{k_{[\lambda n]+1}}{k_n} \le \lambda.$$

(b) As in the proof of (i), one can show that, with probability 1, for each $x \in K$ there is an independent subsequence (ξ_{n_j,m_j}) converging to x. □

Let Z_n be defined as in (3). The proof of Theorem 2 follows the same lines as the proof of Theorem 1. The underlying one-dimensional large deviation principle is the following.

LEMMA 10. *In the setting of Theorem 2, the distributions of* $(1/n)\sum_{i=1}^{n} Y_i = Z_n(1)$ *satisfy a large deviation principle with normalization* $b(n)n$ *on* $\mathbb{R}$ *with the good rate function* $\tilde{I}_1$, *where*

$$\tilde{I}_1(t) = \begin{cases} t, & t \geq 0, \\ +\infty, & t < 0. \end{cases}$$

We refer to [10] or [6] for a proof.

Theorem 4 is then proved from Theorem 2 exactly as Theorem 3 was proved from Theorem 1.

4. Examples. Of course, we have a lot of choices to specify F in Corollary 1.

Letting $F(x) = x(1)$, we get, in the setting of Theorem 3 or Theorem 4,

$$\limsup_n \frac{1}{k_n} \sum_{j=n+1}^{n+k_n} Y_j = \frac{1}{(cd)^{1/r}}, \qquad P\text{-a.s.} \tag{20}$$

$$\liminf_n \frac{1}{k_n} \sum_{j=n+1}^{n+k_n} Y_j = 0, \qquad P\text{-a.s.} \tag{21}$$

$$\limsup_n \sup_{0 \leq m \leq n-k_n} \frac{1}{k_n} \sum_{j=m+1}^{m+k_n} Y_j = \frac{1}{(cd)^{1/r}}, \qquad P\text{-a.s.} \tag{22}$$

$$\liminf_n \sup_{0 \leq m \leq n-k_n} \frac{1}{k_n} \sum_{j=m+1}^{m+k_n} Y_j = 0, \qquad P\text{-a.s.} \tag{23}$$

Letting $F(x) = \int_0^1 x(s)\,ds$, we get

$$\begin{aligned} \limsup_n \bigg(\Big(1 - \frac{1}{2k_n}\Big) Y_{n+1} + \Big(1 - \frac{3}{2k_n}\Big) Y_{n+2} \\ + \Big(1 - \frac{5}{2k_n}\Big) Y_{n+3} + \cdots + \frac{3}{2k_n} Y_{n+k_n-1} + \frac{1}{2k_n} Y_{n+k_n} \bigg) \\ = \frac{1}{(cd)^{1/r}}, \qquad P\text{-a.s.}, \end{aligned} \tag{24}$$

$$\begin{aligned} \liminf_n \bigg(\Big(1 - \frac{1}{2k_n}\Big) Y_{n+1} + \Big(1 - \frac{3}{2k_n}\Big) Y_{n+2} \\ + \Big(1 - \frac{5}{2k_n}\Big) Y_{n+3} + \cdots + \frac{3}{2k_n} Y_{n+k_n-1} + \frac{1}{2k_n} Y_{n+k_n} \bigg) \\ = 0, \qquad P\text{-a.s.} \end{aligned} \tag{25}$$

$$
\begin{aligned}
&\limsup_n \sup_{0\le m\le n-k_n}\Bigg(\left(1-\frac{1}{2k_n}\right)Y_{m+1}+\left(1-\frac{3}{2k_n}\right)Y_{m+2}\\
&\qquad +\left(1-\frac{5}{2k_n}\right)Y_{m+3}+\cdots+\frac{3}{2k_n}Y_{m+k_n-1}+\frac{1}{2k_n}Y_{m+k_n}\Bigg)\\
&\quad=\frac{1}{(cd)^{1/r}},\qquad P\text{-a.s.},
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
&\liminf_n \sup_{0\le m\le n-k_n}\Bigg(\left(1-\frac{1}{2k_n}\right)Y_{m+1}+\left(1-\frac{3}{2k_n}\right)Y_{m+2}\\
&\qquad +\left(1-\frac{5}{2k_n}\right)Y_{m+3}+\cdots+\frac{3}{2k_n}Y_{m+k_n-1}+\frac{1}{2k_n}Y_{m+k_n}\Bigg)\\
&\quad=0,\quad P\text{-a.s.}
\end{aligned}
\tag{27}
$$

Let us now specify the distribution of Y.

EXAMPLE 1. *Assume $P[Y\ge t]=a(t)\exp(-b_1t^r)$ for t large enough, where a is slowly varying and $0<r<1$. In other words, we specified $b(t)\equiv b_1$. Condition (4) is satisfied with $s_n\equiv 1$ and $d=b_1$. In particular, (20), (21), (22) and (23) become*

$$\limsup_n \frac{1}{(c\log n)^{1/r}}\sum_{j=n+1}^{n+[(c\log n)^{1/r}]}Y_j=\frac{1}{(b_1c)^{1/r}},\qquad P\text{-a.s.}$$

$$\liminf_n \frac{1}{(c\log n)^{1/r}}\sum_{j=n+1}^{n+[(c\log n)^{1/r}]}Y_j=0,\qquad P\text{-a.s.}$$

$$\limsup_n \sup_{m=0,1,2,\dots,n-[(c\log n)^{1/r}]}\frac{1}{(c\log n)^{1/r}}\sum_{j=m+1}^{m+[(c\log n)^{1/r}]}Y_j=\frac{1}{(b_1c)^{1/r}},\qquad P\text{-a.s.}$$

$$\liminf_n \sup_{m=0,1,2,\dots,n-[(c\log n)^{1/r}]}\frac{1}{(c\log n)^{1/r}}\sum_{j=m+1}^{m+[(c\log n)^{1/r}]}Y_j=0,\qquad P\text{-a.s.}$$

Similar results were obtained in [7].

EXAMPLE 2. *Let $b(t)=b_1\log t+b_2\log\log t$, $0<r<1$. Condition (4) is satisfied with*

$$s_n=\frac{1}{(\log\log n)^{1/r}}\quad\text{and}\quad d=\frac{b_1}{r}.$$

In particular, (20)–(27) hold with $k_n=[(c\log n/\log\log n)^{1/r}]$ and $d=b_1/r$.

EXAMPLE 3. *Let $b(t)=1/(b_1\log t+b_2\log\log t)$, $0<r\le 1$. Condition (4) is satisfied with $s_n=(\log\log n)^{1/r}$ and $d=r/b_1$. In particular, (20)–(27) hold with $k_n=[(c\log n\log\log n)^{1/r}]$ and $d=r/b_1$.*

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