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MULTIPLE POINTS OF DILATION-STABLE LÉVY PROCESSES[1]

By Narn-Rueih Shieh

National Taiwan University

Let X be a symmetric Lévy process in $\mathbb{R}^d$, $d = 2, 3$. We assume that X has independent α_j-stable components, $1 < \alpha_d \le \cdots \le \alpha_1 < 2$ (a process with stable components, by Pruitt and Taylor), or more generally that X is d-dimensionally self-similar with similarity exponents H_j, $H_j = 1/\alpha_j$ (a dilation-stable process, by Kunita). Let a given integer $k \ge 2$ be such that $k(H-1) < H$, $H = \sum_{j=1}^d H_j$. We prove that the set of k-multiple points E_k is almost surely of Hausdorff dimension

$$\dim E_k = \min\left(\frac{k-(k-1)H}{H_1}, d - \frac{k(H-1)}{H_d}\right).$$

In the stable components case, the above formula was proved by Hendricks for $d = 2$ and was suspected by him for $d = 3$.

1. Introduction and main result. Let $X_t = X_t(\omega)$, $t \in \mathbb{R}_+$, be a Lévy process, defined on a probability space $(\Omega, \mathscr{F}, P)$ and taking values in $\mathbb{R}^d$, $d \ge 2$. We assume that $X(0) = 0$. Let γ_t, $t > 0$, be a multiplicative semigroup $\gamma_{ts} = \gamma_t \circ \gamma_s$ of nonsingular linear transformations on $\mathbb{R}^d$. We say that X is *stable with respect to* γ_t if $X_t =_d \gamma_t X_1$ for all $t > 0$, where $=_d$ denotes distributional equivalence. See the recent book by Jurek and Mason (1994) for the theory of general operator-stable processes. We are interested in the case $\gamma_t = C\delta_t\theta_t C^{-1}$, where C, θ_t are orthogonal transformations and δ_t are diagonal transformations with diagonal entries $t^{H_1}, \ldots, t^{H_d}$. For the simplicity of the context of proof, we assume further in what follows that C, θ_t are identities so that $\gamma_t = \delta_t$. Then, X is *d-dimensionally self-similar with similarity exponents* $H_1, \ldots, H_d$, in the sense that for each $c > 0$, the following equivalence of the time-scaled and the space-scaled processes of $X = (X_1, \ldots, X_d)$ holds:

$$\{X(ct)\}_{t\ge 0} =_d \{(c^{H_1}X_1(t), \ldots, c^{H_d}X_d(t))\}_{t\ge 0}.$$

The consideration of the above-mentioned Lévy processes comes from a lecture of Kunita (1993), who named X a *dilation-stable process.* There are much earlier works by Pruitt and Taylor (1969a) and Hendricks (1974, 1979). They considered an important case, namely, assuming that X is also of independent components, and it was called a *process with stable components*; note that the component X_j is a (strictly) α_j-stable Lévy process, $\alpha_j = 1/H_j$. In this paper, we derive a dimension formula for the multiple points of sample paths of dilation-stable Lévy processes. For any positive integer $k \ge 2$, a point $x \in \mathbb{R}^d$

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is said to be a *k-multiple point* of the path $X(\cdot,\omega)$ if there are distinct times $0 < t_1 < \cdots < t_k$ such that $X(t_i,\omega) = x$ for $i = 1,\dots,k$. We assume that

$$\tfrac{1}{2} < H_1 \le \cdots \le H_d < 1, \tag{1.1}$$

or equivalently,

$$1 < \alpha_d \le \cdots \le \alpha_1 < 2. \tag{1.1$'$}$$

We assume in this paper that the process X is *symmetric*, in the sense that $X(t) =_d -X(t)$ for all $t > 0$. We also assume that the process X is genuinely in $\mathbb{R}^d$; that is, X is not concentrated on any proper subspace of $\mathbb{R}^d$. Let a given integer $k \ge 2$ be such that

$$k(H-1) < H, \tag{H}$$

where

$$H = H_1 + \cdots + H_d.$$

Then, $X(t)$ has k-multiple points a.s.; see Section 2. Since we have assumed that $1/2 < H_j < 1$, condition (H) is nonvoid only when $d = 2, 3$ and the situation is

$$d = 2\colon\quad 1 < H_1 + H_2 < \frac{k}{k-1},$$

$$d = 3\colon\quad \frac{3}{2} < H_1 + H_2 + H_3 < 2 \ (k = 2 \text{ only}).$$

Letting E_k denote the (random) set of k-multiple points, our purpose is to prove the following dimension formula for E_k. Let $\dim(\cdot)$ denote the Hausdorff dimension of a Borel set in $\mathbb{R}^d$. We refer to Taylor (1986) for a convenient reference on the theory of random fractals arising from the sample paths of stochastic processes, in which the detailed definitions and properties of Hausdorff and other dimension indices have been described.

THEOREM 1. *Let X be a dilation-stable symmetric Lévy process in $\mathbb{R}^d$, as described above. Under condition (H), almost surely,*

$$\dim E_k = \min\left(\frac{k-(k-1)H}{H_1}, d - \frac{k(H-1)}{H_d}\right). \tag{1.2}$$

Note that (1.2) reduces to a well-known formula proved by Taylor (1967) in the α-stable Lévy case, in which $\alpha = 1/H_j$ for all j, namely that $\dim E_k = d - k(d-\alpha)$. In the case of processes with stable components, (1.2) is consistent with an $\mathbb{R}^2$ result proved by Hendricks (1974). However, Hendricks did not prove the $\mathbb{R}^3$ result in his works (1974, 1979), although he indeed remarked on some difficulty and suspicion in (1974), page 127 (iv). Thus, our Theorem 1 completes his work on the multiple points of the processes with stable components. We should remark that, in the case $d = 3$, $\dim E_2$ in (1.2) is always the first term, since the second term is always bigger; the latter is a consequence of the fact that $2 - H < 1/2 < H_1$ (I thank Y. M. Xiao for this observation). We remark that Hendricks also made crucial usage of the independent components assumption, and it seems impossible to follow his arguments to obtain our (1.2) for more general dilation-stable processes even in the $\mathbb{R}^2$ case. Our ingredients for proving Theorem 1 are some basic scaling estimates for the potential kernels and some intersection local time techniques for general Lévy processes given in Le Gall (1987) and Le Gall, Rosen and Shieh (1989). We observe that the two possible values in (1.2) come from two kinds of coverings: by ellipsoids with semiaxis lengths $\varepsilon^{H_1}, \dots, \varepsilon^{H_d}$ and by balls with radius ε^{H_d}. For the α-stable case, the two kinds of coverings have "the same effect" since the scalings are the same in various directions; thus the two possibly different values reduce to the same, as was proved by Taylor (1967). Hendricks [(1974), page 127(i)] mentioned his idea of successively using up the two components; such an approach seems to restrict to the $\mathbb{R}^2$ and the independent components case only.

Finally, we should mention that the background for studying processes with stable components, as that remarked by Pruitt and Taylor (1969a) and Hendricks (1974), arises from Jain and Pruitt's (1969) work on the collisions of independent stable processes, while the background for studying dilation-stable processes arises from the investigation on "Lévy flows," see Kunita (1993, 1996) and Applebaum and Kunita (1994).

2. Some preliminaries and basic estimates.

First, we cite a result of Lamperti (1962), Example 1 showing that dilation-stable Lévy processes are indeed an extension of α-stable Lévy processes.

LEMMA 2.1. *If X is a d-dimensionally self-similar Lévy process in $\mathbb{R}^d$ and the similarity exponents $H_1 = \cdots = H_l$, for some l: $1 \le l \le d$, then $(X_1, \dots, X_l)$ is a (strictly) α-stable Lévy process in $\mathbb{R}^l$, $\alpha = 1/H_j$, $j = 1, \dots, l$.*

Note that we have assumed that $1/2 < H_j < 1$, so that $\alpha_j = 1/H_j$ are in the range (1, 2); thus we have the excluded subordinators, and the Cauchy and Wiener processes from our consideration. We also remark that, in view of Lemma 2.2 below and the well-known Port–Stone criterion, our process is transient; see also Choi and Sato (1995) for the transience of more general operator-stable processes. Let $P_t(dy)$ be the probability distribution of X_t. Then, it is known [Kunita (1993), Jurek and Mason (1994), Chapter 4] that $P_t(dy)$ has a bounded continuous density $p_t(y)$, $y \in \mathbb{R}^d$, and that the following scaling holds:

$$p_t(y) = \frac{p_1(\gamma_{t^{-1}} y)}{t^H} \qquad \forall\, t > 0,\ y \in \mathbb{R}^d, \tag{2.1}$$

where $H = H_1 + \cdots + H_d$. Note that $1 \le d/2 < H < d$. Observe that, by the symmetry assumption,

$$p_1(0) = \int_x \left(p_{1/2}(x)\right)^2 dx > 0.$$

Then it follows from (2.1) that $p_t(y) > 0$, $\forall\, t, y$. Note that the above positivity assertion is the "type A" condition in Taylor (1967). The following lemma can be seen in Hudson and Mason (1981) and Kunita (1993), from which it is easy to construct a dilation-stable symmetric Lévy process, with given H_j, yet not of independent components.

LEMMA 2.2. *For a dilation-stable Lévy process X, the characteristic function of X_1 is determined by*

$$E\exp(i(z, X_1)) = \exp\left[\int_0^\infty \int_S \bigl(\exp(i(z, \gamma_r x)) - 1 - i(z, \gamma_r x)1_D(\gamma_r x)\bigr)\frac{\lambda(dx)\,dr}{r^2}\right],$$

where $\lambda(dx)$ is a finite Borel measure on $S = \{x: |x| = 1\}$, and $D = \{x: |x| \le 1\}$.

Note that, in Lemma 2.2, if we take $\lambda(dx)$ to be a symmetric measure on S, then the resulting X is symmetric; while if λ is not concentrated on the coordinate axes, then X is *not* of independent components.

Following the general definition and notation in Taylor (1967), we define the potential kernel $U(y)$ with respect to the process X in Theorem 1, $0 < U(y) \le \infty$, to be

$$U(y) = \int_0^\infty p_t(y)\,dt, \qquad y \in \mathbb{R}^d,$$

which is lower-semicontinuous in y.

LEMMA 2.3. *For $0 < \rho_1 < \rho_2$,*

$$0 < \inf_{\rho_1 \le |y| \le \rho_2} U(y) \le \sup_{\rho_1 \le |y| \le \rho_2} U(y) < \infty.$$

PROOF. By (2.1), we have

$$U(y) = \int_0^1 \frac{p_1(\gamma_{t^{-1}} y)}{t^H}\,dt + \int_1^\infty \frac{p_1(\gamma_{t^{-1}} y)}{t^H}\,dt. \tag{2.2}$$

Positivity of the lower bound comes from that of $p_1(0)$. We prove finiteness of the upper bound as follows. The second term in the r.h.s of (2.2) converges uniformly in $|y| \le \rho_2$, since $|\gamma_{t^{-1}} y| \le \rho_2$ for $|y| \le \rho_2$ and $t \ge 1$ [we also note that $p_1(\cdot)$ is continuous]. As for the first term, note that $\alpha_d = 1/H_d$ is the minimum of $\alpha_j = 1/H_j$, $1 \le j \le d$. By the estimate for the stable density given in Pruitt and Taylor (1969b), considering the slowest possible decay, we have

$$p_1(\gamma_{t^{-1}} y) \le \text{Const} \cdot |\gamma_{t^{-1}} y|^{-(1+\alpha_d)} \le \text{Const} \cdot t^{H_1(1+\alpha_d)}$$

for all t: $0 < t \le 1$ and all y: $\rho_1 \le |y|$; the Const in the above display is a certain absolute constant depending only on d and ρ_1. Thus, the first term in the r.h.s. of (2.2) converges uniformly in $\rho_1 \le |y|$ whenever $H - H_1(1+\alpha_d) < 1$. The latter is true, since

$$H - H_1(1+\alpha_d) < H - 2H_1 < 2(1-H_1) < 1.$$

We note that $H < 2$ is a consequence of condition (H). □

LEMMA 2.4. *There exist $C, C' > 0$ such that*

$$\frac{C'}{(\max_j |y_j|^{1/H_j})^{H-1}} \le U(y) \le \frac{C}{(\max_j |y_j|^{1/H_j})^{H-1}} \qquad \forall\, y \ne 0.$$

PROOF. We have

$$U(y) = \frac{U(\gamma_{t^{-1}} y)}{t^{H-1}} \qquad \forall\, t,\, y. \tag{2.3}$$

In fact,

$$\begin{aligned} U(\gamma_t y) &= \int_0^\infty p_s(\gamma_t y)\, ds \\ &= \int_0^\infty \frac{p_1(\gamma_{s^{-1}} \gamma_t y)}{s^H}\, ds \\ &= \int_0^\infty \frac{p_1(\gamma_{u^{-1}} y)}{(tu)^H} t\, du \\ &= \frac{\int_0^\infty p_u(y)\, du}{t^{H-1}}. \end{aligned}$$

Now, for any $y \ne 0$, let $t_y = \max_j |y_j|^{1/H_j}$; then

$$1 \le \left|\gamma_{t_y^{-1}} y\right| \le \sqrt{d}.$$

By Lemma 2.3 and (2.3), we have

$$0 < C' \le t_y^{H-1} U(y) \le C < \infty. \qquad \square$$

For each compact $K \subset \mathbb{R}^d$, the capacity of K with respect to the process X is defined, as in Taylor (1967), to be

$$\text{Cap}(K) = \sup\left\{\mu(K) \colon \sup_x W_\mu(x) \le 1\right\},$$

where μ is a finite Borel measure supported by K and $W_\mu(x)$ is defined by

$$W_\mu(x) = \int_K U(y-x)\mu(dy).$$

The following scaling property of $\text{Cap}(K)$ is evident.

LEMMA 2.5. *We have*

$$\text{Cap}(\gamma_t K) = t^{H-1}\, \text{Cap}(K) \qquad \forall\, t,\, K.$$

Let B_r be the ball in $\mathbb{R}^d$ with center 0 and radius r, and let $\Gamma_d(r)$ be its volume.

LEMMA 2.6. *We have*

$$\frac{\Gamma_d(r)}{\int_{|y|\le 2r} U(y)\,dy} \le \mathrm{Cap}(B_r) \le \frac{\Gamma_d(4r)}{\int_{|y|\le r} U(y)\,dy}.$$

PROOF. The inequality appears in Hendricks (1979), Lemma 2. The estimate in the proof of Lemma 2.3 shows that

$$\int_{|y|\le a} U(y)\,dy$$

is finite positive for each $a > 0$. □

LEMMA 2.7. *Fix $k \ge 1$. There exists an absolute constant $C_{k,d}$ such that*

$$\int_{|y|\le 2r} U(y)^k\,dy \le C_{k,d} \int_{|y|\le r} U(y)^k\,dy$$

for all $r > 0$.

The proof follows directly from Lemma 2.4.

By the following Lemma 2.8 and the result in Evans (1987) or Le Gall, Rosen and Shieh (1989), under condition (H), the set E_k of k-multiple points of a dilation-stable Lévy process is (a.s.) nonempty.

LEMMA 2.8. *Under condition (H), for each $a > 0$,*

$$0 < \int_{|y|\le a} U^k(y)\,dy < \infty.$$

PROOF. We use an argument similar to that of the proof of Lemma 2.3. We show that both of the following integrals are finite, and we note that the first one is positive by the positivity of $p_1(0)$,

$$\int_{|y|\le a} \left[\int_0^1 p_t(y)\,dt\right]^k dy$$

and

$$\int_{|y|\le a} \left[\int_1^\infty p_t(y)\,dt\right]^k dy.$$

As before, the second one is all right; to prove the first one, by the generalized Minkowski inequality, it suffices to prove the finiteness of

$$\int_0^1 \left[\int_{|y|\le a} p_t^k(y)\,dy\right]^{1/k} dt,$$

which we estimate as equal to

$$\int_0^1 \left[\int_{|y|\le a} p_t^{k-1}(y)\,p_t(y)\,dy\right]^{1/k} dt \le \int_0^1 \left[\sup_{|y|\le a} p_t^{k-1}(y)\right]^{1/k} dt$$

since $p_t(y)$ is a density function for each t. Moreover, since

$$p_t^{k-1}(y) = \frac{p_1^{k-1}(\gamma_{t^{-1}}y)}{t^{(k-1)H}},$$

and $p_1(\cdot)$ is bounded, the above integral is finite by condition (H). $\square$

REMARK. From the proof of Lemma 2.8 and a result of Shieh (1992), it is possible to consider the k-multiple points of the path $X(\cdot, t)$ with t restricted to a certain subset of $\mathbb{R}_+$ for which the Hausdorff dimension is less than 1.

3. Proof of Theorem 1. First, we reduce our consideration from the self-intersections of a single process X to the intersections of the ranges of k independent copies $X^1, \dots, X^k$ of X, with starting points $x^1, \dots, x^k$, respectively. Such a reduction has been rigorously justified for general Lévy processes in Le Gall, Rosen and Shieh (1989), Section 3. Thus, our E_k now is the intersection $\bigcap_{j=1}^k X^j[0,\infty)$. The proof of Theorem 1 is largely based on the following "canonical" measure μ_k supported by E_k.

PROPOSITION 3.1. *There exists (a.s.) a Borel measure μ_k supported by E_k whose moments are given, for each $l = 1, 2, \dots$ and compact K, by*

$$E[\mu_k(K)]^l = \int_{K^l} dy_1 \cdots dy_l \prod_{j=1}^k \left[\sum_\sigma U(y_{\sigma(1)} - x^j) \prod_{i=2}^l U(y_{\sigma(i)} - y_{\sigma(i-1)})\right],$$

where the sum extends over all permutations of $\{1, \dots, l\}$.

From now on, we suppress the subscript k from μ_k. To construct μ, there are two (essentially the same) ways: to consider μ as the image of the measure α in Le Gall, Rosen and Shieh (1989), Theorem 3, under the mapping $(t_1, \dots, t_k) \longrightarrow X^1(t_1)$, $(t_1, \dots, t_k)$ being a k-multiple time; or to consider μ as the limiting measure of a certain normalized intersection of k sausage measures, as in Le Gall (1987), Theorem 2.1. Here we follow the latter approach, since it is more direct and useful for the present purpose. Note that Le Gall (1987) treated the spherically symmetric Lévy processes, while this quite restrictive assumption can be removed to fit our dilation-stable case. Also note that the Hawkes condition in Le Gall's paper is essentially condition (H) in our dilation-stable case. In the Appendix we show how to modify Le Gall's theorem.

The proof of "$\leq$" in (1.2). For each ε, $0<\varepsilon<1$, and $y\in\mathbb{R}^d$, we consider the ellipsoid $R_\varepsilon(y)$ with center y, axes parallel to the coordinates, and semiaxis lengths $\varepsilon^{H_1},\dots,\varepsilon^{H_d}$; we also consider the ball $B_\varepsilon(y)$ with center y and radius ε^{H_d}. We suppress $y=0$ from $R_\varepsilon(0)$ and $B_\varepsilon(0)$. Now, we observe that the ellipsoid and the ball are related by

$$R_\varepsilon(y)=\gamma_\varepsilon B_1(y),$$
$$B_\varepsilon(y)=\gamma_\varepsilon Q_\varepsilon(y),$$

where $Q_\varepsilon(y)$ is the ellipsoid with center y, axes parallel to the coordinates and semiaxis lengths $\varepsilon^{H_d-H_j}$, $j=1,\dots,d$. Note that $Q_\varepsilon(y)\subset B_1(y)$. Also note that, in the α-stable case, since $H_1=\cdots=H_d=1/\alpha$, the two objects $R_\varepsilon(y)$ and $B_\varepsilon(y)$ are the same. Let the measure functions $\phi_i(r)$, $r>0$, be defined by

$$\phi_1(r)=r^{(k-(k-1)H)/H_1},$$
$$\phi_2(r)=r^{d-(k(H-1)/H_d))}.$$

We prove that the Hausdorff ϕ_i measure of $E_k\cap K$, denoted henceforth by $\phi_i-m(E_k\cap K)$, is (a.s.) finite for both $i=1,2$, for each compact K. The "$\leq$" part of (1.2) then follows from this assertion. For the proof of ϕ_1-m, we make use of $R_\varepsilon(y)$. Let $\mathscr{B}_\varepsilon$, $0<\varepsilon<1$, be the class of abutting boxes in $\mathbb{R}^d$ with edges parallel to the coordinates, edge-lengths $\varepsilon^{H_1},\dots,\varepsilon^{H_d}$ and lower-left vertices $(\varepsilon^{H_1}k_1,\dots,\varepsilon^{H_d}k_d)$, $k_i\in\mathrm{Z}$. Let $N(\varepsilon)$ be the (random) numbers of those elements of $\mathscr{B}_\varepsilon$ intersecting $E_k\cap K$. For any Borel A, let A_ε^{nbd} denote $\bigcup_{y\in A}R_\varepsilon(y)$; we note that, for each box $A\in\mathscr{B}_\varepsilon$, vol $A=\varepsilon^H$ and diam $A=\varepsilon^{H_1}$. Geometry shows that

$$\begin{aligned} N(\varepsilon)\varepsilon^H &\leq \mathrm{Leb}\big((E_k\cap K)_{\sqrt{d}\varepsilon}\big)\\ &\leq \mathrm{Leb}\big(S^1_{\sqrt{d}\varepsilon}\cap\cdots\cap S^k_{\sqrt{d}\varepsilon}\cap K^{nbd}_{\sqrt{d}\varepsilon}\big), \end{aligned}\tag{3.1}$$

where S^j_ε denotes the sausage of X^j associated with R_ε. That is,

$$S^j_\varepsilon=\bigcup_{t\geq 0}\big(X^j(t)+R_\varepsilon\big).$$

The expectation of the last term in (3.1) is $O(\mathrm{Cap}(R_\varepsilon)^k)$; this follows from Le Gall's theorem modified in the Appendix: in $L^l(dP)$, $l=1,2,\dots,$

$$\mu(K)=\lim_{\varepsilon\to 0}\frac{1}{\mathrm{Cap}(R_\varepsilon)^k}\mathrm{Leb}\big(S^1_\varepsilon\cap\cdots\cap S^k_\varepsilon\cap K\big).$$

We have, by Lemma 2.5,

$$\mathrm{Cap}(R_\varepsilon)=\mathrm{Cap}(\gamma_\varepsilon B_1)=\varepsilon^{H-1}\,\mathrm{Cap}(B_1).$$

Therefore,

$$\liminf_{\varepsilon\to 0}E\big[N(\varepsilon)\varepsilon^{H-k(H-1)}\big]<\infty.$$

Since diam $A_\varepsilon = \varepsilon^{H_1}$ for all $A_\varepsilon \in \mathscr{B}_\varepsilon$, the above display asserts that

$$\liminf_{\varepsilon\to 0} N(\varepsilon)\phi_1(\operatorname{diam} A_\varepsilon) < \infty \quad \text{a.s.},$$

which certainly implies that $(\phi_1 - m)(E_k \cap K) < \infty$ a.s. For the proof of $\phi_2 - m$, we make use of $B_\varepsilon(y)$. Let $\mathscr{B}_\varepsilon$ now denote the class of abutting cubes constructed similarly as above, with common edge lengths ε^{H_d}. Let S_ε^j now denote the sausage of X^j associated with B_ε, and $A_\varepsilon^{nbd} = \bigcup_{y\in A} B_\varepsilon(y)$. We note now that, for $A \in \mathscr{B}_\varepsilon$, vol $A = \varepsilon^{dH_d}$ and diam $A = \sqrt{d}\varepsilon^{H_d}$. Intead of (3.1), we now have

$$N(\varepsilon)\varepsilon^{dH_d} \le \operatorname{Leb}\big(S^1_{\sqrt{d}\varepsilon} \cap \cdots \cap S^k_{\sqrt{d}\varepsilon} \cap K^{nbd}_{\sqrt{d}\varepsilon}\big). \tag{3.1$'$}$$

The expectation of the r.h.s of (3.1)′ is now $O(\operatorname{Cap}(B_\varepsilon)^k)$, while we have

$$\operatorname{Cap}(B_\varepsilon) = \operatorname{Cap}(\gamma_\varepsilon Q_\varepsilon) \le \varepsilon^{H-1}\operatorname{Cap}(B_1).$$

Thus, we have as above that

$$\liminf_{\varepsilon\to 0} N(\varepsilon)\phi_2(\operatorname{diam} A_\varepsilon) < \infty \quad \text{a.s.}$$

The proof of "≥" in (1.2). We have had the measure μ supported by E_k. Let s denote the minimum value in (1.2), that is,

$$s = \min\bigg(\frac{k-(k-1)H}{H_1}, d - \frac{k(H-1)}{H_d}\bigg).$$

Fix a compact K and we prove that if

$$0 < \tau < s,$$

then a.s.,

$$\mu(B_\varepsilon(y) \cap K) \le C\varepsilon^\tau, \tag{3.2}$$

for all the balls $B_\varepsilon(y)$ with center y and radius ε. The C depends on the path and on K, but not on $B_\varepsilon(y)$. Indeed, once (3.2) is proved, then by Frostman's lemma in Kahane [(1985), page 130], the τ-dimensional Hausdorff measure of $E_k \cap K$ is positive. Then "≥" part of (1.2) is obtained by letting a sequence $\tau \uparrow s$. We also observe that it suffices to prove (3.2) for μ-a.e. y and for a sequence $\varepsilon \downarrow 0$ which we choose to be 2^{-l}, $l = 1, 2, \dots$. Set

$$\theta = s - \tau.$$

Let $B_l(y)$ now be the ball with center y and radius 2^{-l}. Let

$$A_l = \bigg\{y \in K\colon \mu[B_l(y) \cap K] \ge 2^{\theta l}\int_{|y|\le 2^{-l}} U^k(y)\,dy\bigg\}.$$

We aim to prove that

$$\sum_{l=1}^{\infty} E\mu(A_l) < \infty \tag{3.3}$$

and that

$$\int_{|y|\le r} U^k(y)\,dy = O(r^s) \qquad \text{as } r \downarrow 0. \tag{3.4}$$

Assume that (3.3) and (3.4) have been proved. Then, by (3.3), a.s. for μ-a.e. y, $y \notin A_l$ if $l \ge l_0$, some $l_0 = l_0(y, \omega)$; thus by (3.4),

$$\begin{aligned}\mu\big[B_l(y)\cap K\big] &\le 2^{\theta l}\int_{|y|\le 2^{-l}} U^k(y)\,dy\\ &= O\big(2^{\theta l - sl}\big),\\ &= O\big(2^{-\tau l}\big),\end{aligned}$$

which is what we desire.

To prove (3.3), we observe that

$$\begin{aligned}&\int_{A_l}\mu(dy)\big[\mu(B_l(y))\big]^l\\ &\qquad \ge \mu(A_l)\cdot 2^{\theta l^2}\bigg[\int_{|y|\le 2^{-l}} U^k(y)\,dy\bigg]^l.\end{aligned}$$

Thus

$$E\mu(A_l) \le \frac{E\int_{A_l}\mu(dy)\big[\mu(B_l(y))\big]^l}{2^{\theta l^2}\big[\int_{|y|\le 2^{-l}} U^k(y)\,dy\big]^l}. \tag{3.5}$$

Applying the $(l+1)$th moment formula for μ in Proposition 3.1, we see the numerator in the r.h.s. of (3.5) is

$$O\bigg\{\big[(l+1)!\big]^k\bigg[\int_{|y|\le 2^{-(l-1)}} U(y)^k\,dy\bigg]^l\bigg\}.$$

Thus, in view of (3.5) and Lemma 2.7, we have

$$\sum_{l=1}^{\infty} E\mu(A_l) \le \text{Const}\sum_{l=1}^{\infty}\frac{[(l+1)!]^k}{2^{\theta l^2}} < \infty,$$

by Stirling's formula for $(l+1)!$.

To prove (3.4), we use the following device. We note first that

$$B_r = \gamma_\varepsilon R_{\varepsilon, r},$$

where

B_r is the ball with center 0 and radius r;

$R_{\varepsilon, r}$ is the ellipsoid with center 0, edges parallel to the coordinates, and semi-axis lengths $r\varepsilon^{-H_1}, \dots, r\varepsilon^{-H_d}$.

Thus,

$$\int_{B_r} U^k(y)\,dy = \int_{R_{\varepsilon, r}} U^k(\gamma_\varepsilon z)\varepsilon^H\,dz.$$

Case $H_d < k(H-1) < H$. In this case, we take $\varepsilon^{H_1} = r$. Then $R_{\varepsilon,r} \supset B_1$, and we may write

$$\int_{R_{\varepsilon,r}} = \int_{B_1} + \int_{R_{\varepsilon,r}-B_1}.$$

By Lemma 2.4 we have

$$\int_{R_{\varepsilon,r}-B_1} U^k(\gamma_\varepsilon z)\,dz \le \varepsilon^{-k(H-1)} \int_{R_{\varepsilon,r}-B_1} \frac{dz}{\max_j |z_j|^{k(H-1)/H_j}}.$$

The last integral in the above display is $O(1)$ as $r \downarrow 0$, since $H_j \le H_d < k(H-1)$. By Lemma 2.4 again,

$$\int_{B_1} U^k(\gamma_\varepsilon z)\,dz \le \varepsilon^{-k(H-1)} \int_{B_1} \frac{dz}{\max_j |z_j|^{k(H-1)/H_j}}.$$

The last integral in the above display is finite, since the dominating power is $k(H-1)/H_d$ which is less than d by our assumption that $k(H-1) < H$. Therefore

$$\int_{B_r} U^k(y)\,dy = O\big(r^{(H-k(H-1))/H_1}\big),$$

and we note that the power of r in the above r.h.s is the first term in s.

Case $0 < k(H-1) \le H_d$. This happens only in the case $d = 2$, since $H > 3/2$ in case $d = 3$. Now, we take $\varepsilon^{H_d} = r$, $d = 2$. Then $R_{\varepsilon,r} \subset B_1$, and we have

$$\begin{aligned}\int_{R_{\varepsilon,r}} & U^k(\gamma_\varepsilon z)\varepsilon^H\,dz \\ &= \int_{B_1} U^k(\gamma_\varepsilon T_{\varepsilon^{-1}} z)\varepsilon^H \varepsilon^{2H_2-H}\,dz,\end{aligned}$$

where T_ε: $(z_1, z_2) \to (\varepsilon^{H_1-H_2} z_1, z_2)$, which maps $R_{\varepsilon,r}$ onto B_1. Since

$$\gamma_\varepsilon T_{\varepsilon^{-1}} z = \varepsilon^{H_2} z, \qquad H_2/H_1 \ge 1,$$

we now have

$$\begin{aligned}\int_{B_r} U^k(y)\,dy &= O\bigg(\varepsilon^{2H_2} \int_{B_1} \frac{dz}{\max_j |\varepsilon^{H_2} z_j|^{k(H-1)/H_j}}\bigg) \\ &= O\big(\varepsilon^{2H_2-k(H-1)}\big) \\ &= O\big(r^{(2H_2-k(H-1))/H_2}\big),\end{aligned}$$

and we note that the power of r in the above r.h.s. is the second term in the s (with $d = 2$).

REMARKS. (i) The first part of the proof indeed gives the same upper bound estimate for the packing dimension (Dim) of E_k. Therefore we may include in Theorem 1 that $\operatorname{Dim} E_k = \dim E_k$; see Taylor (1986) for a description of the relation between the two dimensional indices.

(ii) As we have remarked in Section 1, by our assumption on the range of the H_j, each component process X_j is α_j-stable with $1 < \alpha_j < 2$. As was seen in Pruitt and Taylor (1969a) and Hendricks (1974), a different type of results should appear if we want to extend the range.

(iii) The proof of Theorem 1 is motivated by those in Le Gall (1987), Theorems 2.1 and 3.1, yet we pay special attention to the facts that a dilation-stable process cannot be spherically symmetric if the similarity exponents are different and that the potential kernel needs to be estimated properly.

(iv) As we see from the proof, the "nonisotropic" nature of the process is the driving force behind our thinking (the different roles of ellipsoids and balls). This nonisotropic nature is different, in fact stronger, than the nonsymmetric α-stable process. The latter has its nature arising from the nonuniform distribution of its difining measure on the unit spherical surface [cf. Taylor (1967), Sections 2, 3]. In the case of dilation-stable processes with different exponents, in view of Lemma 2.2, even if we start with the uniform distribution on the unit spherical surface, we can only have a symmetric, yet not spherically symmetric, Lévy process. These require that the efforts on dilation-stable processes should be more considerable than those on stable ones.

APPENDIX

Le Gall's Theorem 2.1 (1987). In his 1987 work (Theorem 2.1), Le Gall treated the intersections of k independent spherically symmetric Lévy processes with common transition densities. The potential kernel $U(x)$ in his work is then radial. That is $U(x) = U(|x|)$, and he assumed $U(r)$ to be decreasing in r. The assumption is indeed used in obtaining a majorization in his Lemma 2.3, of page 346. Such a majorization is crucial in proving his (2.b) and (2.c), page 347, which are central to obtaining his Theorem 2.1. We will show that we still have such a majorization in the present case, and thus we can exactly follow Le Gall's arguments to obtain a result corresponding to his Theorem 2.1 for our dilation-stable processes under condition (H). Note that the formulation of the assertions in LeGall's Theorem 2.1 is a general one (nothing to do with the spherical symmetry) and indeed he also pointed out (page 351) the possibility of the extension to the more general case. Now, let B_ε be either the ball $|y| \le \varepsilon^{H_d}$ or the ellipsoid $\sum_{i=1}^d (y_i/\varepsilon^{H_i})^2 \le 1$, $0 < \varepsilon < 1$. For X^i, the independent copies of a dilation-stable Lévy process X with starting point x^i, let S^i_ε be the sausage associated with B_ε and let $T^i_\varepsilon(y)$ be the hitting time of $y + B_\varepsilon$. The basic relation holds:

$$P\{y \in S^i_\varepsilon\} = P\{T^i_\varepsilon(y) < \infty\} = \int_{B_\varepsilon} U(y + x - x^i)\mu_\varepsilon(dx),$$

where μ_ε is the capacitory measure for B_ε. Let $c(\varepsilon)$ denote $\mu_\varepsilon(B_\varepsilon) = \mathrm{Cap}(B_\varepsilon)$. Then, by Lemmas 2.5 and 2.6 in the above Sections 2, $c(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$. Let $y_1, \dots, y_l$ be distinct points that do not belong to any hyperplane which is parallel to the coordinates and contains the x^i (these hyperplanes have $\mathbb{R}^d$-Lebesgue measure zero). Let $\varepsilon_1, \dots, \varepsilon_l$ be small enough so that $y_i + B_{\varepsilon_j}$ are separated by a positive distance ρ, from each other and from all the x^i. By the strong Markov property, we have

$$\begin{aligned}
&P\{T^i_{\varepsilon_1}(y_1) \le \cdots \le T^i_{\varepsilon_l}(y_l) < \infty\} \\
&\qquad = P\{T^i_{\varepsilon_1}(y_1) \le \cdots \le T^i_{\varepsilon_{l-1}}(y_{l-1}) < \infty\} E\{P^{v^i_{l-1}}\{T^i_{\varepsilon_l}(y_l) < \infty\}\},
\end{aligned}$$

where, for $j = 2, 3, \dots,$ the (random) points v^i_j are

$$v^i_j = X^i(T^i_{\varepsilon_j}(y_j)).$$

We observe that

$$\begin{aligned}
P\{T^i_{\varepsilon_1}(y_1) < \infty\} &= \int_{B_{\varepsilon_1}} U(y_1 + x - x^i)\mu_{\varepsilon_1}(dx) \\
&\le c(\varepsilon_1) f_{\varepsilon_1}(y_1 - x^i),
\end{aligned}$$

where

$$f_{\varepsilon_1}(z) = \sup_{x \in B_{\varepsilon_1}} U(z + x).$$

Moreover, since $v^i_j \in y_j + B_{\varepsilon_j}$, $j = 2, 3, \dots,$

$$\begin{aligned}
&E\{P^{v^i_{j-1}}\{T^i_{\varepsilon_j}(y_j) < \infty\}\} \\
&\qquad = E\left\{\int_{B_{\varepsilon_j}} U(y_j + x - v^i_{j-1})\mu_{\varepsilon_j}(dx)\right\} \\
&\qquad \le c(\varepsilon_j) f_{\varepsilon_{j-1}, \varepsilon_j}(y_j - y_{j-1}),
\end{aligned}$$

where

$$f_{\varepsilon_{j-1}, \varepsilon_j}(z) = \sup_{x \in B_{\varepsilon_j} + B_{\varepsilon_{j-1}}} U(z + x).$$

Therefore, we have the following recursive relation:

$$\begin{aligned}
&\{c(\varepsilon_1) \cdots c(\varepsilon_l)\}^{-1} P\{T^i_{\varepsilon_1}(y_1) \le \cdots \le T^i_{\varepsilon_l}(y_l) < \infty\} \\
&\qquad \le f_{\varepsilon_1}(y_1 - x^i) \prod_{j=2}^{l} f_{\varepsilon_{j-1}, \varepsilon_j}(y_j - y_{j-1}).
\end{aligned}$$

Under our separation condition mentioned above, the functions f_{ε_1}, $f_{\varepsilon_{j-1}, \varepsilon_j}$ are integrable in y_j, y_{j-1} up to any order, thanks to Lemmas 2.3 and 2.4 in

Section 2. The above majorization is our replacement of that given in Le Gall [(1987), page 346]. With this majorization and the same arguments as his, we have the following modification of Le Gall's result, valid for dilation-stable Lévy processes.

THEOREM A. *Let X be the dilation-stable Lévy process in Theorem 1; then the measure μ in Proposition 3.1 exists as the following limit: in $L^l(dP)$, $l = 1, 2, \dots,$*

$$\mu(K) = \lim_{\varepsilon \to 0} \frac{1}{\mathrm{Cap}(R_\varepsilon)^k} \mathrm{Leb}\big(S_\varepsilon^1 \cap \cdots \cap S_\varepsilon^k \cap K\big),$$

where K is any compact in $\mathbb{R}^d$, R_ε is either the ball $|y| \le \varepsilon^{H_d}$ or the ellipsoid $\sum_{i=1}^d (y_i/\varepsilon^{H_i})^2 \le 1$, $0 < \varepsilon < 1$ and S_ε^i is the sausage of X^i associated with R_ε.

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DEPARTMENT OF MATHEMATICS
NATIONAL TAIWAN UNIVERSITY
TAIPEI
TAIWAN 10617
E-MAIL: shiehnr@math.ntu.edu.tw