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OCCUPATION TIME LARGE DEVIATIONS FOR CRITICAL BRANCHING BROWNIAN MOTION, SUPER-BROWNIAN MOTION AND RELATED PROCESSES

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We derive a large deviation principle for the occupation time functional, acting on functions with zero Lebesgue integral, for both super-Brownian motion and critical branching Brownian motion in three dimensions. Our technique, based on a moment formula of Dynkin, allows us to compute the exact rate functions, which differ for the two processes. Obtaining the exact rate function for the super-Brownian motion solves a conjecture of Lee and Remillard. We also show the corresponding CLT and obtain similar results for the superprocesses and critical branching process built over the symmetric stable process of index β in R^d, with $d < 2\beta < 2+d$.

1. Introduction. Consider a critical branching Brownian motion in R^3: particles are initially distributed in R^3 according to a Poisson random field with uniform density $\tau > 0$. Letting ξ_t^τ denote the countable set of sites in R^3 occupied at time t, the particle at each $x \in \xi_t^\tau$ undergoes a Brownian motion until it either splits into two particles or disappears with exponential rate τ, independently of the other particles. If h is any function in $L^1(R^3)$, we define

$$N_t^\tau(h) = \sum_{x \in \xi_t^\tau} h(x).$$

In particular, if $A \subseteq R^3$ is a bounded measurable set, then $N_t^\tau(A) = N_t^\tau(1_A)$ is just the number of particles in A at time t. In case $\tau = 1$, we simply write N_t for N_t^1. As $\tau \to \infty$, the law of $\{(1/\tau)N_t^\tau,\ t \geq 0\}$ converges weakly to the law of the Dawson–Watanabe *super-Brownian motion* $\{\mu_t,\ t \geq 0\}$ in R^3 starting from the Lebesgue measure μ (cf. [4]). For $h \in L^1(R^3)$, we write, as above, $\mu_t(h) = \int_{R^3} h(x)\,\mu_t(dx)$.

Let $T > 0$ and define

$$\bar{N}_T \equiv \frac{1}{T}\int_0^T N_s\,ds \quad \text{and} \quad \bar{\mu}_T \equiv \frac{1}{T}\int_0^T \mu_s\,ds,$$

the *occupation time functionals* of N and μ. Note that

$$\bar{\mu}_T(h) = \frac{1}{T}\int_{R^3} h(x)\,L_T^x\,\mu(dx),$$

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where $\{L_T^x,\ x \in R^3,\ T \geq 0\}$ is *the local time* of the super-Brownian motion in R^3 (cf. [8], [9]).

For each $h \in L^1(R^3)$, both $\bar{N}_T(h)$ and $\bar{\mu}_T(h)$ converge almost surely as $T \to \infty$ to the space average $\langle h, \mu\rangle \equiv \int_{R^3} h(x)\,\mu(dx)$. This convergence has been the object of several papers; compare [2] for critical branching Brownian motion and [10] and [7] for super-Brownian motion. In particular, in case $\langle h, \mu\rangle > 0$, Iscoe and Lee [10] proved a large deviation principle at critical speed $T^{1/2}$ for both $\bar{N}_T(h)$ and $\bar{\mu}_T(h)$, with identical rate function Λ_h^*. Here Λ_h^* is expressed in terms of the Legendre transform of $\Lambda_h(\theta) = \Lambda(\langle h, \mu\rangle\theta)$, where

$$\Lambda(\theta) = \log E_\mu[\exp(\theta L_1^0)].$$

Note that Λ_h depends only on $\langle h, \mu\rangle$. In particular, $\Lambda_h \equiv 0$ in case $\langle h, \mu\rangle = 0$. This yields an infinite rate $\Lambda_h^*(x) = \infty$ for $x \neq 0$ and suggests a different scaling in this case. Take for example $h(x) = 1_A(x) - 1_B(x)$, where A and B are disjoint bounded sets with same Lebesgue measure $\mu(A) = \mu(B)$. Then $N_t(h) = N_t(A) - N_t(B)$, the difference between the number of particles present at time t in the sets A and B, reduces the fluctuations of the time average $\bar{N}_T(h)$. In fact, Lee and Remillard have shown in a recent paper [11] that $P(T^{1/4}\bar{\mu}_T(h) > b)$ is of the order $\exp(-O(T^{1/2}))$ and they conjecture the corresponding exact large deviation principle.

The object of this paper is to prove such a large deviation principle for both $T^{1/4}\bar{N}_T(h)$ and $T^{1/4}\bar{\mu}_T(h)$, when $\langle h, \mu\rangle = 0$ and to identify the corresponding rate functions. Unlike the previous situation when $\langle h, \mu\rangle > 0$, the rate functions for $T^{1/4}\bar{N}_T(h)$ and $T^{1/4}\bar{\mu}_T(h)$ turn out to be different.

More precisely, let $\mathscr{C}_0$ be the set of measurable $h\colon R^3 \to R$ such that

$$\langle h, \mu\rangle = 0 \quad \text{and} \quad \int_{R^3} (1 + |x|)|h(x)|\,dx < \infty. \tag{1.1}$$

Next, set

$$\sigma(h) = -\frac{1}{2\pi}\int\int |x - y|\,h(x)\,h(y)\,dx\,dy \tag{1.2}$$

and

$$\rho(h) = \frac{1}{2\pi}\int\int |x - y|^{-1}\,h(x)\,h(y)\,dx\,dy. \tag{1.3}$$

Introduce the rate functions $\Lambda_h^{\mu,*},\ \Lambda_h^{N,*}\colon R \to [0, \infty)$:

$$\Lambda_h^{\mu,*}(x) = \sup_{\theta \in R}\{x\theta - \Lambda(\sigma(h)\theta^2)\},$$

$$\Lambda_h^{N,*}(x) = \sup_{\theta \in R}\{x\theta - \Lambda((\sigma(h) + \rho(h))\theta^2)\}.$$

Our main result is the following large deviation principle.

THEOREM 1. *If $h \in \mathscr{C}_0$, then for each $x \geq 0$,*

$$\lim_{T\to\infty} \frac{1}{T^{1/2}} \log P(T^{1/4}\bar{\mu}_T(h) \geq x) = -\Lambda_h^{\mu,*}(x), \tag{1.4}$$

$$\lim_{T\to\infty} \frac{1}{T^{1/2}} \log P(T^{1/4}\bar{N}_T(h) \geq x) = -\Lambda_h^{N,*}(x). \tag{1.5}$$

Note that the first equality is precisely Conjecture 1 of [11].

The crucial step in the proof of Theorem 1 is the convergence of the moment generating functions

$$\lim_{T\to\infty} \frac{1}{T^{1/2}} \log E_\mu\big[\exp\big(T^{3/4}\theta\bar{\mu}_T(h)\big)\big] = \Lambda(\sigma(h)\theta^2), \tag{1.6}$$

for $|\theta| < \theta_{c,h}$ and

$$\lim_{T\to\infty} \frac{1}{T^{1/2}} \log E_\mu\big[\exp\big(T^{3/4}\theta\bar{N}_T(h)\big)\big] = \Lambda((\sigma(h)+\rho(h))\theta^2), \tag{1.7}$$

for $|\theta| < \theta_{cb,h}$, where $\theta_{c,h}$ and $\theta_{cb,h}$ are defined by

$$\theta^* = \sigma(h)\theta_{c,h}^2 \quad \text{and} \quad \theta^* = (\sigma(h)+\rho(h))\theta_{cb,h}^2,$$

and $\theta^* = \sup\{\theta \in R\colon \Lambda(\theta) < \infty\}$. Once (1.6) and (1.7) are proved, the large deviation results follow from the Ellis–Gärtner theorem.

Unlike [11], where the asymptotics of the left-hand side of (1.6) are investigated with non-linear PDE techniques, our proof is based on Dynkin's moment formula [8]. This approach provides a graphical method for organizing the series expansion of (1.6), which enables us to prove convergence up to the critical value $\theta_{c,h}$, a condition which is necessary for the derivation of the full large deviation principle.

As a byproduct of (1.6), we also get a large deviation principle for

$$\left\{x\colon R^3 \to \frac{1}{T^{3/4}}(L_T^x - L_T^0)\right\},$$

viewed as a continuous function on R^3; compare Theorem 5, below.

Our methods are not restricted to Brownian motion. We also show similar results for the superprocess $\mu_{\beta,d,t}$ and critical branching process $N_{\beta,d,t}$ built over the symmetric stable process of index β in R^d, with $d < 2\beta < 2+d$. Let $\{L_T^x,\ T \geq 0,\ x \in R^d\}$ be the local time of $\mu_{\beta,d,T}$. Set

$$\Lambda_{\beta,d}(\theta) = \log E_\mu[\exp(\theta L_1^0)]. \tag{1.8}$$

Let

$$\sigma_{\beta,d}(h) = -c_{\beta,d} \int\int |x-y|^{2\beta-d}\, h(x)h(y)\,dx\,dy, \tag{1.9}$$

where

$$c_{\beta,d} = \int_0^\infty s(p(s,0)-p(s,u))\,ds < \infty, \tag{1.10}$$

and $p(s, y)$ denotes the transition density for the symmetric stable process of index β in R^d and $u \in R^d$ is an arbitrary unit vector. Let

$$\rho_{\beta,d}(h) = \frac{\Gamma((d-\beta)/2)}{2^\beta \pi^{d/2}\Gamma(\beta/2)} \int\int |x-y|^{-(d-\beta)}\, h(x)h(y)\,dx\,dy. \tag{1.11}$$

Introduce the rate functions $\Lambda^{\mu,*}_{\beta,d,h}$, $\Lambda^{N,*}_{\beta,d,h}$: $R \to [0,\infty)$:

$$\Lambda^{\mu,*}_{\beta,d,h}(x) = \sup_{\theta\in R}\{x\theta - \Lambda_{\beta,d}(\sigma_{\beta,d}(h)\theta^2)\},$$
$$\Lambda^{N,*}_{\beta,d,h}(x) = \sup_{\theta\in R}\{x\theta - \Lambda_{\beta,d}((\sigma_{\beta,d}(h)+\rho_{\beta,d}(h))\theta^2)\}.$$

Let

$$\bar N_{\beta,d,T} \equiv \frac{1}{T}\int_0^T N_{\beta,d,s}\,ds \quad \text{and} \quad \bar\mu_{\beta,d,T} \equiv \frac{1}{T}\int_0^T \mu_{\beta,d,s}\,ds,$$

the *occupation time functionals* of $N_{\beta,d}$ and $\mu_{\beta,d}$.

Our main result for stable processes is the following large deviation principle.

THEOREM 2. *Let $d < 2\beta < 2+d$. If $h \in \mathscr{C}_0$, then for each $x \geq 0$,*

$$\lim_{T\to\infty} \frac{1}{T^{d/\beta-1}} \log P(T^{1-d/(2\beta)}\bar\mu_{\beta,d,T}(h) \geq x) = -\Lambda^{\mu,*}_{\beta,d,h}(x), \tag{1.12}$$

$$\lim_{T\to\infty} \frac{1}{T^{d/\beta-1}} \log P(T^{1-d/(2\beta)}\bar N_{\beta,d,T}(h) \geq x) = -\Lambda^{N,*}_{\beta,d,h}(x). \tag{1.13}$$

Finally, a simple consequence of our methods gives the following central limit theorem, which should be contrasted with Theorem 6.1 of [9] and Theorem 1 of [2] for the case where $\langle h, \lambda\rangle > 0$ (see also [3]).

Let $\mathscr{S}_0(R^d) = \mathscr{S}(R^d) \cap \mathscr{C}_0$ be the set of rapidly decreasing test functions with 0 integral over R^d. We view $\{\sqrt{T}\bar\mu_T(h),\ h \in \mathscr{S}_0(R^d)\}$ and $\{\sqrt{T}\bar N_T(h),\ h \in \mathscr{S}_0(R^d)\}$ as distribution valued processes. Next, let $\{W_i(h),\ h \in \mathscr{S}_0(R^d)\},\ i = 1, 2$ be centered Gaussian processes with covariance

$$E[W_1^2(h)] = 2\sigma(h), \qquad E[W_2^2(h)] = 2(\sigma(h)+\rho(h)).$$

THEOREM 3. *In the sense of weak convergence, we have*

$$\lim_{T\to\infty} \sqrt{T}\bar\mu_T = W_1 \tag{1.14}$$

and

$$\lim_{T\to\infty} \sqrt{T}\bar N_T = W_2. \tag{1.15}$$

Note that (1.14) is shown in Main Theorem of [11], which also contains a moderate deviation result for the super-Brownian motion. Similarly, we can show the following central limit theorem for the processes built over stable processes: let $\{W_{\beta,i}(h),\ h \in \mathscr{S}_0(R^d)\},\ i = 1, 2$ be centered Gaussian processes with covariances

$$E\big[W^2_{\beta,1}(h)\big] = 2\sigma_{\beta,d}(h), \qquad E\big[W^2_{\beta,2}(h)\big] = 2(\sigma_{\beta,d}(h) + \rho_{\beta,d}(h)).$$

THEOREM 4. *Let $d < 2\beta < d + 2$, then, in the sense of weak convergence,*

$$\lim_{T\to\infty} \sqrt{T}\bar{\mu}_{\beta,d,T} = W_{\beta,1} \tag{1.16}$$

and

$$\lim_{T\to\infty} \sqrt{T}\bar{N}_{\beta,d,T} = W_{\beta,2}. \tag{1.17}$$

The rest of this paper is divided into five sections. Section 2 deals with the large deviation principle. Section 3 shows the convergence of the moment generating function for differences of local times $L^x_T - L^0_T$ for super-Brownian motion. This is a preliminary step for the convergence in (1.6) which is proved in Section 4. We present a proof of (1.7) in Section 5 and in Section 6 we show the convergence in the stable case.

2. The large deviation principle.

Our first step is a proof of Theorem 1. Note that the proof of Theorem 2 is completely analogous.

PROOF OF THEOREM 1. The upper bounds

$$\limsup_{T\to\infty} \frac{1}{T^{1/2}} \log P(T^{1/4}\bar{\mu}_T(h) \geq x) \leq -\Lambda^{\mu,*}_h(x),$$
$$\limsup_{T\to\infty} \frac{1}{T^{1/2}} \log P(T^{1/4}\bar{N}_T(h) \geq x) \leq -\Lambda^{N,*}_h(x),$$

are simple consequences of (1.6) and (1.7) and Chebyshev's inequality. The lower bounds

$$\liminf_{T\to\infty} \frac{1}{T^{1/2}} \log P(T^{1/4}\bar{\mu}_T(h) \geq x) \geq -\Lambda^{\mu,*}_h(x),$$
$$\liminf_{T\to\infty} \frac{1}{T^{1/2}} \log P(T^{1/4}\bar{N}_T(h) \geq x) \geq -\Lambda^{N,*}_h(x),$$

follow from the Gärtner–Ellis theorem, once the the steepness of $\theta \to \Lambda(\sigma(h)\theta^2)$ at $\theta = \theta_{c,h}$, respectively, of $\theta \to \Lambda((\sigma(h) + \rho(h))\theta^2)$ at $\theta = \theta_{cb,h}$ is verified, (cf. Section 2.3 of [5]). The steepness follows easily from the fact that $\Lambda \in C^1$ on $(-\infty, \theta^*)$ with

$$\lim_{\theta \nearrow \theta^*} \Lambda'(\theta) = \infty, \tag{2.1}$$

where $\Lambda'(\theta) = (d/d\theta)\Lambda(\theta)$; compare [10]. This concludes the proof of Theorem 1. $\square$

Our next purpose is to derive a large deviation principle simultanuously for all h. We will concentrate on the super-Brownian motion, which is simpler since we can use the local time. Let $C_0 = \{f \in C(R^3; R)\colon f(0) = 0\}$ endowed with the topology of uniform convergence on compact sets. Let C_0^* be the dual of C_0, that is, the set of bounded signed measures ν with compact support in R^3 such that $\nu(R^3) = 0$, and write $\langle f, \nu\rangle$ for the duality relation.

Next, for each $T > 0$ define $x\colon R^3 \to \bar{L}_T^x \in C_0$ by

$$\bar{L}_T^x \equiv \frac{1}{T}(L_T^x - L_T^0), \qquad x \in R^3.$$

Note that

$$\langle \bar{L}_T, \nu\rangle = \int_{R^3} \bar{L}_T^x\, \nu(dx)$$

so that by Theorem 10 below we have

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T^{1/2}} \log E\big[\exp\big(T^{3/4}\langle \bar{L}_T, \nu\rangle\big)\big] \\
(2.2)\qquad &= \lim_{T\to\infty} \frac{1}{T^{1/2}} \log E\left[\exp\left(T^{-1/4}\int_{R^3} L_T^x\, \nu(dx)\right)\right] \\
&= \Lambda(\sigma(\nu)),
\end{aligned}$$

where

$$\sigma(\nu) = -\frac{1}{2\pi}\int\int |x-y|\, \nu(dx)\, \nu(dy).$$

Define the rate function $I\colon C_0 \to [0, \infty]$,

$$I(\phi) = \sup_{\nu \in C_0^*}\big\{\langle \phi, \nu\rangle - \Lambda(\sigma(\nu))\big\}.$$

Our main result in this section is the following large deviation principle.

THEOREM 5. *For each closed $F \in C_0$ and open $G \in C_0$,*

$$\liminf_{T\to\infty} \frac{1}{T^{1/2}} \log P(T^{1/4}\bar{L}_T \in G) \geq -\inf_G I, \tag{2.3}$$

$$\limsup_{T\to\infty} \frac{1}{T^{1/2}} \log P(T^{1/4}\bar{L}_T \in F) \leq -\inf_F I. \tag{2.4}$$

PROOF. The first step in the proof is the *exponential tightness*. For a compact set $K \subseteq R^3$, $0 \in K$ and $\alpha \in (0, \frac{1}{2})$, let

$$\|f\|_{K,\alpha} \equiv \sup\left\{\frac{|f(x) - f(y)|}{|x-y|^\alpha}\colon x \neq y \in K,\ |x-y| \leq 1\right\}, \qquad f \in C_0$$

be the α-Hölder norm of f in K. We claim the existence of $\delta > 0$ such that

$$\limsup_{T\to\infty} \frac{1}{T^{1/2}} \log E\big[\exp(\delta T^{3/4}\|\bar{L}_T\|_{K,\alpha})\big] < \infty. \tag{2.5}$$

Let $\beta\in(0,\frac{1}{2})$. For fixed $x\neq y\in K$, $|x-y|\le 1$, set $\nu(dz)=(\delta/|x-y|^{\beta})(\delta_x(dz)-\delta_y(dz))$, then $\langle L_T,\nu\rangle=\delta((L_T^x-L_T^y)/|x-y|^{\beta})$ with $\sigma(\nu)=(\delta^2/\pi)|x-y|^{1-2\beta}\le\delta^2$. Thus for $\delta<\sqrt{\theta^*}$ we have

$$\begin{aligned}
&\limsup_{T\to\infty}\frac{1}{T^{1/2}}\log E\biggl[\exp\biggl(T^{3/4}\delta\frac{|\bar L_T^x-\bar L_T^y|}{|x-y|^{\beta}}\biggr)\biggr]\\
&\qquad=\limsup_{T\to\infty}\frac{1}{T^{1/2}}\log E\biggl[\exp\biggl(T^{-1/4}\delta\frac{L_T^x-L_T^y}{|x-y|^{\beta}}\biggr)\biggr]\\
&\qquad\le\Lambda(\sigma(\nu))\le\Lambda(\delta^2)<\infty.
\end{aligned}$$

Now (2.5) follows from the Garsia–Rodemich–Rumsey inequality [1].

Next, let $\mathscr{K}_{L,K}\equiv\{f\in C_0\colon \|f\|_{\alpha,K}\le L\}$, which is a compact subset of C_0. Equation (2.5) implies the exponential tightness

$$\lim_{L\to\infty}\limsup_{T\to\infty}\frac{1}{T^{1/2}}\log P\bigl(T^{1/4}\bar L_T\notin\mathscr{K}_{L,K}\bigr)=-\infty. \tag{2.6}$$

This together with (2.2) implies the upper bound (2.4).

As far as the lower bound is concerned, let us first prove a finite-dimensional result. For $d\in N$ and fixed distinct $x_1,\dots,x_d\in R^3$, let

$$\bar L_{T,d}=(\bar L_T^{x_1},\dots,\bar L_T^{x_d})\in R^d.$$

Next let

$$\mathscr{C}_0^{d,*}=\biggl\{\nu=\sum_{i=1}^d\nu_i\delta_{x_i},\ \text{with}\ \sum_{i=1}^d\nu_i=0\biggr\}$$

and then for $\nu\in\mathscr{C}_0^{d,*}$ we have $\sigma(\nu)=-(1/\pi)\sum_{i,j=1}^d|x_i-x_j|\,\nu_i\,\nu_j$. Set $\Lambda_d(\nu)\equiv\Lambda(\sigma(\nu))$, $\nu\in\mathscr{C}_0^{d,*}$ and define $I_d(f)=\sup\{\langle f,\nu\rangle-\Lambda_d(\nu)\colon \nu\in\mathscr{C}_0^{d,*}\}$. Of course $I_d(f)\le I(f)$. We then prove a lower bound for $T^{1/4}\bar L_{T,d}$ with rate I_d. In view of the Ellis–Gärtner theorem, it suffices to show the steepness of Λ_d. Let

$$\begin{aligned}
D&=\biggl\{\nu=(\nu_1,\dots,\nu_d)\colon \nu_d=-\sum_{i=1}^{d-1}\nu_i,\ \sigma(\nu)<\theta^*\biggr\},\\
\partial D&=\biggl\{\nu=(\nu_1,\dots,\nu_d)\colon \nu_d=-\sum_{i=1}^{d-1}\nu_i,\ \sigma(\nu)=\theta^*\biggr\}.
\end{aligned}$$

We view D as a subset of R^{d-1} and set $\nabla_i\sigma(\nu)=(d/d\nu_i)\sigma(\nu)$, $i=1,\dots,d-1$. Then Λ_d is C^1 in D with $\nabla\Lambda_d(\nu)=\Lambda'(\sigma(\nu))\nabla\sigma(\nu)$. Thus, let $\{\nu_n,\ n\in N\}\subseteq D$ be such that $\lim_{n\to\infty}\nu_n=\nu\in\partial D$. Then, in view of (2.1), we have the following steepness result:

$$\lim_{n\to\infty}|\nabla\Lambda_d(\nu_n)|=\lim_{n\to\infty}|\nabla\sigma(\nu_n)|\,|\Lambda'(\sigma(\nu_n))|=\infty.$$

Now an application of the Ellis–Gärtner theorem implies

$$\liminf_{T\to\infty} \frac{1}{T^{1/2}} \log P\big(T^{1/4}\bar{L}_{T,d} \in G_d\big) \geq -\inf_{G_d} I_d, \tag{2.7}$$

for all open $G_d \in R^d$.

Once the lower bound is verified for each finite-dimensional cylinder set, the lower bound (2.3) follows from the exponential tightness by a routine argument, (cf., e.g., [6] or Section 5.1 of [5]). Let $f \in G$ be such that $I(f) < \infty$. We may assume that $B_K(f;\varepsilon) = \{g \in C_0\colon \sup_{x\in K} |f(x) - g(x)| < \varepsilon\} \subseteq G$ for some $\varepsilon > 0$ and compact set $K \subseteq R^3$. For each $L > 0$, we can then find $d = d(L,\varepsilon) \in N$, and $x_1,\dots,x_d \in R^3$ such that $B_{K,d}(f;\varepsilon/2) \cap \mathscr{K}_{L,K} \subseteq B_K(f;\varepsilon) \cap \mathscr{K}_{L,K}$ where $B_{K,d}(f;\varepsilon/2) = \{g \in C_0\colon \max_{i=1}^d |f(x_i) - g(x_i)| < \varepsilon/2\}$. In view of the exponential tightness (2.6) we then have

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{1}{T^{1/2}} \log P(T^{1/4}\bar{L}_T \in B_K(f;\varepsilon)) \\
&\quad \geq \lim_{L\to\infty} \liminf_{T\to\infty} \frac{1}{T^{1/2}} \log P\left(T^{1/4}\bar{L}_{T,d} \in B_{K,d}\left(f;\frac{\varepsilon}{2}\right)\right) \\
&\quad \geq -I_d(f) \geq -I(f).
\end{aligned}$$

This completes the proof of Theorem 5. □

Before concluding this section, let us briefly indicate how the above results can be extended to stable processes in R^d. Let $\{L_T^x,\ T \geq 0,\ x \in R^d\}$ be the local time of the symmetric superstable process of index β in R^d when $d < 2\beta < 2 + d$. Set

$$\Lambda_{\beta,d}(\theta) = \log E_\mu[\exp(\theta L_1^0)]. \tag{2.8}$$

Following [10], we can show the existence of $0 < \theta^*_{\beta,d} < \infty$ such that $\Lambda_{\beta,d}(\theta) < \infty$ for $\theta < \theta^*_{\beta,d}$ and $\lim_{\theta\uparrow\theta^*_{\beta,d}} (d\Lambda_{\beta,d}(\theta)/d\theta) = \infty$. Also by rescaling, we have for each $\theta < \theta^*_{\beta,d}$,

$$\frac{1}{T^{d/\beta-1}} \log E_\mu[\exp(T^{-(2-d/\beta)}\theta L_T^0)] = \log E_\mu[\exp(\theta L_1^0)] = \Lambda_{\beta,d}(\theta); \tag{2.9}$$

compare (6.9) below.

Let ν be a signed measure with compact support in R^d, such that

$$\int_{R^d} d\nu(x) = 0. \tag{2.10}$$

Next set

$$\sigma_{\beta,d}(\nu) = -c_{\beta,d} \int\!\!\int |x-y|^{2\beta-d}\, d\nu(x)\, d\nu(y), \tag{2.11}$$

where $c_{\beta,d}$ is defined in (1.10), and assume that $\sigma_{\beta,d}(\nu) < \theta^*_{\beta,d}$. Then we show below in Theorem 13 that

$$\lim_{T\to\infty} \frac{1}{T^{d/\beta-1}} \log E_\mu\left(\exp\left(T^{d/(2\beta)} \int \frac{1}{T} L_T^y\, d\nu(y)\right)\right) = \Lambda_{\beta,d}(\sigma_{\beta,d}(\nu)). \tag{2.12}$$

Our first result in this context is a large deviation principle for $\{(1/T)L_T^x: x \in R^d\}$ viewed as an element of $C = C(R^d; R)$ endowed with the topology of uniform convergence on compact sets.

THEOREM 6. *Let $\Lambda^*_{\beta,d}$ be the Legendre transform of $\Lambda_{\beta,d}$ and define the good rate function $J_{\beta,d}: C \to [0,\infty]$,*

$$J_{\beta,d}(\phi) = \Lambda^*_{\beta,d}(c), \qquad \phi(x) \equiv c, x \in R^d$$

and ∞ otherwise. Then for each closed $F \in C$ and open $G \in C$,

$$\liminf_{T\to\infty} \frac{1}{T^{d/\beta-1}} \log P\left(\frac{1}{T} L_T \in G\right) \geq -\inf_G J_{\beta,d}, \tag{2.13}$$

$$\limsup_{T\to\infty} \frac{1}{T^{d/\beta-1}} \log P\left(\frac{1}{T} \bar{L}_T \in F\right) \leq -\inf_F J_{\beta,d}. \tag{2.14}$$

PROOF. Using (2.12), we see that for each $x, y \in \mathrm{R}^d$ and $\theta \geq 0$,

$$\lim_{T\to\infty} \frac{1}{T^{d/\beta-1}} \log E_\mu\left[\exp(T^{d/\beta}\theta(L_T^x - L_T^y))\right] = 0. \tag{2.15}$$

This shows via the Garsia–Rodemich–Rumsey inequality [1] that, for each $\alpha \geq 0$, $\varepsilon > 0$ and compact $K \ni 0$,

$$\lim_{T\to\infty} \frac{1}{T^{d/\beta-1}} \log P_\mu\left(\left\| \frac{1}{T} L_T \right\|_{K,\alpha} \geq \varepsilon\right) = -\infty \tag{2.16}$$

and implies exponential tightness. Thus in order to prove the upper bound, all we need to show is that

$$\lim_{\varepsilon\searrow 0} \limsup_{T\to\infty} \frac{1}{T^{d/\beta-1}} \log P_\mu\left(\frac{1}{T} L_T \in B_K(\phi;\varepsilon)\right) \leq -J_{\beta,d}(\phi). \tag{2.17}$$

Note that by (2.16), the right-hand side of (2.17) is $-\infty$, unless $\phi(x) \equiv c$, $x \in R^d$. In this case $P_\mu((1/T)L_T \in B_K(\phi;\varepsilon)) \leq P_\mu(|(1/T)L_T^0 - c| < \varepsilon)$ with

$$\lim_{\varepsilon\searrow 0} \limsup_{T\to\infty} \frac{1}{T^{d/\beta-1}} \log P_\mu\left(\left|\frac{1}{T} L_T^0 - c\right| < \varepsilon\right) \leq -\Lambda^*_{\beta,d}(c) = -J_{\beta,d}(\phi), \tag{2.18}$$

which follows from (2.9). As for the lower bound, we may assume that the open set G contains the ball $B_K(\phi;\varepsilon)$ for some constant $\phi \equiv c$ and $\varepsilon > 0$. Then $\{|(1/T)L_T^0 - c| < \varepsilon/2\} \cap \{\|(1/T)L_T\|_{K,0} < \varepsilon/2\} \subseteq B_K(\phi;\varepsilon)$ and therefore, using (2.16), (2.9) and the steepness of $\Lambda_{\beta,d}$,

$$\begin{aligned}\liminf_{T\to\infty} \frac{1}{T^{d/\beta-1}} \log P_\mu\left(\frac{1}{T} L_T \in G\right) &\geq \liminf_{T\to\infty} \frac{1}{T^{d/\beta-1}} \log P_\mu\left(\left|\frac{1}{T} L_T^0 - c\right| < \varepsilon/2\right) \\ &\geq -\Lambda^*_{\beta,d}(c) = -J_{\beta,d}(\phi).\end{aligned}$$

Finally let $I_{\beta,d}\colon C_0 \to [0,\infty]$ be given by

$$I_{\beta,d}(\phi) = \sup_{\nu\in C_0^*}\big\{\langle\phi,\nu\rangle - \Lambda_{\beta,d}(\sigma(\nu))\big\},$$

then, in view of (2.12), using precisely the same argument as in the proof of Theorem 5, we get the following.

THEOREM 7. *Let* $\{\bar L_T^x = (1/T)(L_T^x - L_T^0),\ x \in R^d\} \in C_0$. *For each closed* $F \in C_0$ *and open* $G \in C_0$,

$$\liminf_{T\to\infty} \frac{1}{T^{d/\beta-1}} \log P(T^{1-d/(2\beta)}\bar L_T \in G) \ge -\inf_G I_{\beta,d}, \tag{2.19}$$

$$\limsup_{T\to\infty} \frac{1}{T^{d/\beta-1}} \log P(T^{1-d/(2\beta)}\bar L_T \in F) \le -\inf_F I_{\beta,d}. \tag{2.20}$$

3. Local time. Let L_t^x denote the local time of super Brownian motion in R^3, and let μ denote the Lebesgue measure on R^3. Set

$$\Lambda(\theta) = \log E_\mu(\exp(\theta L_1^0)). \tag{3.1}$$

Iscoe and Lee [10] have shown the existence of $0 < \theta^* < \infty$ such that $\Lambda(\theta) < \infty$ for $\theta < \theta^*$ and $\lim_{\theta\uparrow\theta^*}(d\Lambda(\theta)/d\theta) = \infty$. Let θ_c be defined by

$$\theta^* = |y|\theta_c^2/\pi.$$

THEOREM 8. *Let L_t^x denote the local time of super-Brownian motion in R^3, and let μ denote Lebesgue measure on R^3. Then*

$$\lim_{T\to\infty} \frac{1}{T^{1/2}} \log E_\mu\bigg(\exp\bigg(\frac{\theta}{T^{1/4}}(L_T^0 - L_T^y)\bigg)\bigg) = \Lambda(|y|\theta^2/\pi) \tag{3.2}$$

for all $\theta < \theta_c$.

PROOF. We begin with some preliminaries. We let $\mathscr{G}$ denote the set of finite planar rooted unlabeled binary trees G. Here G consists of vertices and arrows, that is, directed bonds. When G has more than one vertex, the vertices of G consist of three types; the unique root r has no incoming arrows but two outgoing arrows, the exits have one incoming but no outgoing arrows, while the internal vertices have one incoming and two outgoing arrows. We use A_G to denote the arrows of G and $E(G)$ to denote the exits of G. We use $\mathscr{G}_n$ to denote the set of $G \in \mathscr{G}$ with n exits. Figure 1 contains an example of a $G \in \mathscr{G}_{11}$ with $E(G) = \{e_j;\ j = 1,\dots,11\}$.

Also included in $\mathscr{G}$ is the set $\widehat{G} \in \mathscr{G}_1$ with a single vertex, which we consider both as root and exit.

Alternatively, $\mathscr{G}$ can be described as the set of finite family trees with a single progenitor, r, where each individual can have either zero or two children. Exits are precisely those individuals with zero children. The requirement that

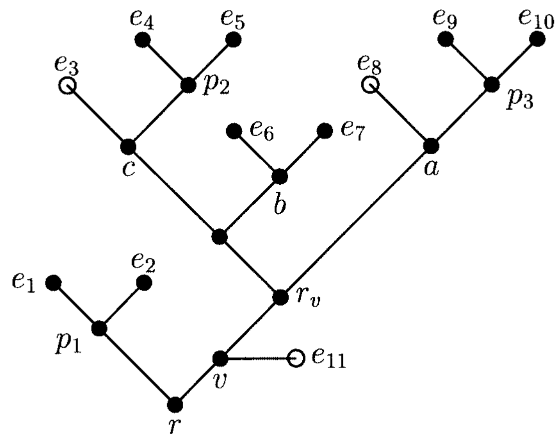

FIG. 1. G.

our trees be planar is equivalent to saying that whenever an individual has two children, there is a natural order to their birth.

Let $p(t,x)$, $t > 0$ be the transition density for Brownian motion in R^3. By convention we set $p(t,x) = 0$ for $t \le 0$. According to Dynkin's moment formula [8] (see also [12]), which is itself derived from the Laplace transform and is equivalent to the expansion used in [10], we have

$$\log E_\mu(\exp(\theta L_T^0)) = \sum_{n=1}^{\infty} \theta^n c_n(T) \tag{3.3}$$

with

$$c_n(T) = \sum_{\mathscr{G}_n} c(G,T), \tag{3.4}$$

where

$$\begin{aligned} c(G,T) = \int\int_{\Delta_G^T} p(t_r, x_r - w) \prod_{a \in A_G} p(t_{a_f} - t_{a_i}, x_{a_f} - x_{a_i}) \\ \times \prod_{e \in E(G)} \delta_0(x_e)\, dt\, dx\, dw. \end{aligned} \tag{3.5}$$

In the above formula, for each $a \in A_G$ we use a_i, a_f to denote the initial and final vertices of a. To each vertex $v \in G$ is associated the pair of variables $(t_v, x_v) \in R_+ \times R^3$. We set

$$\Delta_G^T = \{t_v;\ v \in G \,|\, 0 \le t_v \le T,\ \forall\, v \in G \text{ and } t_{a_i} < t_{a_f},\ \forall\, a \in A_G\}.$$

For notational convenience we have used the convention that $\delta_y(x)\,dx$ stands for the probability measure with unit mass at the point y. With our conventions

$$c(\widehat{G}, T) = \int\int_{\Delta^T_{\widehat{G}}} p(t_r, x_r - w)\delta_0(x_r)\,dt\,dx\,dw = T.$$

It is easily seen by induction that for each $G \in \mathscr{G}_n$ we have $|G| = 2n-1$ and $|A_G| = |G| - 1$. Hence by scaling we see that

$$\begin{aligned} c(G,T) &= T^{|G|}T^{-(3/2)(|A_G|+1)}T^{(3/2)(|G|-n+1)}c(G,1) \\ &= T^{2n-1}T^{-(3/2)(2n-1)}T^{(3/2)n}c(G,1) \\ &= T^{n/2+1/2}c(G,1). \end{aligned} \tag{3.6}$$

We now show how to bound $c(G,1)$, or more generally,

$$\begin{aligned} c(G,B;z_e,\ e\in E(G)) = \int\int_B p(t_r, x_r - w) &\prod_{a\in A^0_G} p(t_{a_f} - t_{a_i}, x_{a_f} - x_{a_i}) \\ &\times \prod_{a\in A^e_G} p(t_{a_f} - t_{a_i}, z_{a_f} - x_{a_i})\,dt\,dx\,dw, \end{aligned} \tag{3.7}$$

where $B \subseteq \Delta^1_G$, A^0_G denotes the set of internal arrows, that is, all arrows which do not lead to an exit, while A^e_G denotes the set of n arrows leading to the n exits. [We note that (3.7) with $B = \Delta^1_G$ is the contribution from the (now labelled) graph G to $E_\mu(\prod_e L_1^{z_e})$.] When $z_e = 0$, $\forall\ e \in E(G)$, we write $c(G,B)$ for the expression in (3.7).

We use

$$\int_0^1 p(t,x)\,dt \le e\int_0^\infty e^{-t}p(t,x)\,dt = e\,u^1(x), \tag{3.8}$$

where

$$u^1(x) = \frac{e^{-|x|}}{2\pi|x|}$$

to bound (3.7) by

$$\begin{aligned} c(G,B;z_e,\ e\in E(G)) \le e^{2n}\int u^1(x_r - w) &\prod_{a\in A^0_G} u^1(x_{a_f} - x_{a_i}) \\ &\times \prod_{a\in A^e_G} u^1(z_{a_f} - x_{a_i})\,dx\,dw. \end{aligned} \tag{3.9}$$

To obtain our bound we proceed step by step, moving upwards along the tree from its root. The dw integral over the first factor, $u^1(x_r - w)$, just gives 1. At the next step, we use the fact that $u^1(x) \in L^2$ to bound the dx_r integral and then keep repeating the procedure.

Using the dominated convergence theorem, it is now easy to show that $c(G,B;z_e,\ e\in E(G))$ is continuous in $B, z_e,\ e \in E(G)$ in the obvious sense.

In particular, to see the continuity in z_e, $e \in E(G)$ we use the mean-value theorem in the form

$$|p(t,x) - p(t,y)| \le \frac{c|x-y|^\delta}{t^{\delta/2}}(p(t,x) + p(t,y)).$$

Furthermore, we easily see from these considerations that

$$c(G,B; z_e,\ e \in E(G)) \le c^n \tag{3.10}$$

for some $c < \infty$ independent of n, $G \in \mathscr{G}_n$, $B \subseteq \Delta_G^1$, and z_e, $e \in E(G)$.

When $B = \Delta_G^T$ we write $c(G,T; z_e,\ e \in E(G))$ for $c(G,B; z_e,\ e \in E(G))$. We note for future reference that

$$c(G,B; z_e,\ e \in E(G)) \le c(G,B) \tag{3.11}$$

for all z_e, $e \in E(G)$. This follows easily from the definitions by writing the kernel $p(t,x)$ in terms of its Fourier transform.

After these preliminaries we now come to our theorem. As in (3.3)–(3.5) we have

$$\frac{1}{T^{1/2}} \log E_\mu\left(\exp\left(\frac{\theta}{T^{1/4}}(L_T^0 - L_T^y)\right)\right) = \sum_{n=1}^\infty \frac{\theta^n}{T^{n/4+1/2}} d_n(T,y), \tag{3.12}$$

where

$$d_n(T,y) = \sum_{\mathscr{G}_n} d(G,T,y) \tag{3.13}$$

and

$$\begin{aligned} d(G,T,y) = \int\int_{\Delta_G^T} & p(t_r, x_r - w) \prod_{a \in A_G^0} p(t_{a_f} - t_{a_i}, x_{a_f} - x_{a_i}) \\ & \times \prod_{a \in A_G^e} \left\{ p(t_{a_f} - t_{a_i}, -x_{a_i}) - p(t_{a_f} - t_{a_i}, y - x_{a_i}) \right\} dt\, dx\, dw. \end{aligned} \tag{3.14}$$

As before, scaling leads to

$$d(G,T,y) = T^{n/2+1/2} d(G,1,y/\sqrt{T}) \tag{3.15}$$

so that

$$\frac{d(G,T,y)}{T^{n/4+1/2}} = T^{n/4} d(G,1,y/\sqrt{T}) \tag{3.16}$$

and consequently

$$\frac{1}{T^{1/2}} \log E_\mu\left(\exp\left(\frac{\theta}{T^{1/4}}(L_T^0 - L_T^y)\right)\right) = \sum_{n=1}^\infty \theta^n T^{n/4} \sum_{\mathscr{G}_n} d(G,1,y/\sqrt{T}). \tag{3.17}$$

Fix $\theta_0 < \theta_c$ so that

$$\Lambda(|y|\theta_0^2/\pi) = \sum_{n=1}^\infty (|y|\theta_0^2/\pi)^n \sum_{\mathscr{G}_n} c(G,1) < \infty \tag{3.18}$$

and choose $\theta < \theta_0$. The following lemma will be proven later in this section.

LEMMA 1. *We can find $T', C' < \infty$ such that*

$$\sum_{n=1}^{\infty} \theta^n T^{n/4} \sum_{\mathscr{G}_n} |d(G, 1, y/\sqrt{T})| \leq C' \tag{3.19}$$

for all $T \geq T'$.

Then, to establish our theorem for such a θ, it will suffice to study the $T \to \infty$ limit term by term.

Let us say that an exit e of G is twinned if there is a vertex v, the immediate predecessor of e, such that the two vertices which branch directly from v are e and another exit e'. Otherwise we say that the exit e is untwinned. In Figure 1, $\{e_1, e_2, e_4, e_5, e_6, e_7, e_9, e_{10}\}$ is the set of twinned exits, and $\{e_3, e_8, e_{11}\}$ is the set of untwinned exits. Following our proof below of Lemma 1 we will be able to show that in the $T \to \infty$ limit the only graphs which contribute to (3.12) are the graphs G in which all exits are twinned. If $G \in \mathscr{G}_n$ is a graph in which all exits are twinned, we have $n = 2m$ and there are m vertices $u_1, \dots, u_m$, (the pre-exits) such that from each u_j branch two exits which we denote by v_{2j-1}, v_{2j}. Let $G_0 \in \mathscr{G}_m$ be the graph obtained from G by removing all exits, and all arrows leading to those exits. Thus, the exits of G_0 are the pre-exits of G, which we continue to denote by $u_1, \dots, u_m$. We will soon prove that

$$\lim_{T\to\infty} \frac{d(G, T, y)}{T^{n/4+1/2}} = c(G_0, 1)\left(\frac{|y|}{\pi}\right)^m. \tag{3.20}$$

In view of (3.16)–(3.18) and our remarks following Lemma 1 this establishes our theorem. We now prove (3.20).

If $G \in \mathscr{G}_n$ is a graph in which all exits are twinned, with the notation of the last paragraph,

$$\begin{aligned} \frac{d(G, T, y)}{T^{n/4+1/2}} &= \int \int_{[0,1]^m} \int_{\Delta^1_{G_0}} p(t_r, x_r - w) \prod_{a \in A_{G_0}} p(t_{a_f} - t_{a_i}, x_{a_f} - x_{a_i}) \\ &\qquad \times \prod_{j=1}^{m} \delta_{s_j}(t_{u_j}) \delta_{z_j}(x_{u_j})\, dt\, dx\, dw \\ &\qquad \times \prod_{j=1}^{m} T^{1/2}\left(\int_{s_j}^{1} (p(r - s_j, z) - p(r - s_j, z - y/\sqrt{T}))\, dr\right)^2 ds\, dz \\ &= \int \int_{[0,1]^m} \int_{\Delta^1_{G_0}} p(t_r, x_r - w) \prod_{a \in A_{G_0}} p(t_{a_f} - t_{a_i}, x_{a_f} - x_{a_i}) \\ &\qquad \times \prod_{j=1}^{m} \delta_{s_j}(t_{u_j}) \delta_{z_j}(x_{u_j})\, dt\, dx\, dw \prod_{j=1}^{m} f_T(1 - s_j, z_j, y)\, ds\, dz, \end{aligned} \tag{3.21}$$

where

$$
\begin{aligned}
f_T(t,z,y) &\stackrel{\text{def}}{=} T^{1/2}\biggl(\int_0^t (p(r,z)-p(r,z-y/\sqrt{T}))\,dr\biggr)^2 \\
&= T^{1/2}\int_0^t\int_0^t (p(r,z)-p(r,z-y/\sqrt{T})) \\
&\qquad\times (p(s,z)-p(s,z-y/\sqrt{T}))\,dr\,ds.
\end{aligned}
\tag{3.22}
$$

We note that

$$
p(t,x) > p(t,x') \quad \Leftrightarrow \quad |x| < |x'|, \tag{3.23}
$$

which implies that $f_T(t,z,y)$ is increasing in t, and therefore for any t,

$$
\int f_T(t,z,y)\,dz \le \int f_T(\infty,z,y)\,dz. \tag{3.24}
$$

We can compute

$$
\begin{aligned}
\int f_T(\infty,z,y)\,dz &= 2T^{1/2}\int_0^\infty\int_0^\infty (p(r+s,0)-p(r+s,y/\sqrt{T}))\,dr\,ds \\
&= 2T^{1/2}\int_0^\infty s(p(s,0)-p(s,y/\sqrt{T}))\,ds \\
&= |y|/\pi.
\end{aligned}
\tag{3.25}
$$

We will soon show the following.

LEMMA 2.

$$
\int f_T(\infty,z,y)\,dz = \int f_T(T^{-1/2},z,y)\,dz + O(T^{-1/4}), \tag{3.26}
$$

and that for any $\delta > 0$

$$
\lim_{T\to\infty}\int_{|z|\ge\delta} f_T(T^{-1/2},z,y)\,dz = 0. \tag{3.27}
$$

We will use the notation $I(B)$ for the integral similar to the integral in (3.21), but in which the ds integration is over the region $B \subseteq [0,1]^m$ rather than $[0,1]^m$. Thus we have

$$
\frac{d(G,T,y)}{T^{n/4+1/2}} = I([0,1]^m) = I([0,1-T^{-1/2}]^m) + I(B_T), \tag{3.28}
$$

where

$$
B_T = [0,1]^m - [0,1-T^{-1/2}]^m.
$$

We first write

$$
\begin{aligned}
&I([0,1-T^{-1/2}]^m)\\
&\quad=\int\int_{[0,1-T^{-1/2}]^m}\int_{\Delta^1_{G_0}} p(t_r,x_r-w)\prod_{a\in A_{G_0}} p(t_{a_f}-t_{a_i},x_{a_f}-x_{a_i})\\
&\qquad\qquad\times\prod_{j=1}^m \delta_{s_j}(t_{u_j})\delta_{z_j}(x_{u_j})\,dt\,dx\,dw\prod_{j=1}^m f_T(T^{-1/2},z_j,y)\,ds\,dz\\
&\qquad+\int\int_{[0,1-T^{-1/2}]^m}\int_{\Delta^1_{G_0}} p(t_r,x_r-w)\prod_{a\in A_{G_0}} p(t_{a_f}-t_{a_i},x_{a_f}-x_{a_i})\\
&\qquad\qquad\times\prod_{j=1}^m \delta_{s_j}(t_{u_j})\delta_{z_j}(x_{u_j})\,dt\,dx\,dw\\
&\qquad\times\left\{\prod_{j=1}^m f_T(1-s_j,z_j,y)-\prod_{j=1}^m f_T(T^{-1/2},z_j,y)\right\}ds\,dz.
\end{aligned}
\tag{3.29}
$$

Since, as noted above, $f_T(t,z,y)$ is positive and monotone increasing in t, using (3.24)–(3.26) in conjunction with (3.10) and (3.11), we see that the last integral on the right-hand side of (3.29) is $O(T^{-1/4})$. Using once again (3.25) and (3.26) in conjunction with (3.10) and (3.11), we see that

$$
\begin{aligned}
I(B_T)&=\int\int_{B_T}\int_{\Delta^1_{G_0}} p(t_r,x_r-w)\prod_{a\in A_{G_0}} p(t_{a_f}-t_{a_i},x_{a_f}-x_{a_i})\\
&\qquad\times\prod_{j=1}^m \delta_{s_j}(t_{u_j})\delta_{z_j}(x_{u_j})\,dt\,dx\,dw\prod_{j=1}^m f_T(1-s_j,z_j,y)\,ds\,dz\\
&\le O(c(G_0,B_T)).
\end{aligned}
\tag{3.30}
$$

Similarly, again decomposing $[0,1]^m$ into $[0,1-T^{-1/2}]^m$ and B_T and bounding the integral over B_T as in (3.30) we see that for the first integral on the right-hand side of (3.29) we have

$$
\begin{aligned}
&\int\int_{[0,1-T^{-1/2}]^m}\int_{\Delta^1_{G_0}} p(t_r,x_r-w)\prod_{a\in A_{G_0}} p(t_{a_f}-t_{a_i},x_{a_f}-x_{a_i})\\
&\qquad\times\prod_{j=1}^m \delta_{s_j}(t_{u_j})\delta_{z_j}(x_{u_j})\,dt\,dx\,dw\prod_{j=1}^m f_T(T^{-1/2},z_j,y)\,ds\,dz\\
&\quad=\int\int_{[0,1]^m}\int_{\Delta^1_{G_0}} p(t_r,x_r-w)\prod_{a\in A_{G_0}} p(t_{a_f}-t_{a_i},x_{a_f}-x_{a_i})\\
&\qquad\qquad\times\prod_{j=1}^m \delta_{s_j}(t_{u_j})\delta_{z_j}(x_{u_j})\,dt\,dx\,dw\\
&\qquad\qquad\times\prod_{j=1}^m f_T(T^{-1/2},z_j,y)\,ds\,dz+O(c(G_0,B_T)).
\end{aligned}
\tag{3.31}
$$

The continuity of $c(G_0, B_T)$ in B_T, as described following (3.9), implies that $c(G_0, B_T) \to 0$ as $T \to \infty$. Putting this all together and using (3.26) and (3.27) establishes (3.20).

PROOF OF LEMMA 2. We now prove (3.26). Note that by (3.23) the integrands which appear in the following display are all positive:

$$\begin{aligned}
&T^{1/2} \int \int_{T^{-1/2}}^{\infty} \int_0^{\infty} (p(r,z) - p(r, z - y/\sqrt{T})) \\
&\qquad\qquad \times (p(s,z) - p(s, z - y/\sqrt{T}))\, dr\, ds\, dz \\
&\quad = T^{1/2} \int_{T^{-1/2}}^{\infty} \int_0^{\infty} \big(2p(r+s, 0) - p(r+s, y/\sqrt{T}) \\
&\qquad\qquad\qquad - p(r+s, -y/\sqrt{T})\big)\, dr\, ds \\
&\quad = 2T^{1/2} \int \bigg(\int_{T^{-1/2}}^{\infty} \int_0^{\infty} (1 - \cos(py/\sqrt{T})) \\
&\qquad\qquad \times \exp(-(r+s)|p|^2/2)\, dr\, ds \bigg)\, d^3 p \qquad (3.32) \\
&\quad = 8T^{1/2} \int \frac{(1 - \cos(py/\sqrt{T}))}{|p|^4} \exp(-|p|^2/(2T^{1/2}))\, d^3 p \\
&\quad \le |y|^2 \frac{T^{1/2}}{T} \int \frac{1}{|p|^2} \exp(-|p|^2/(2T^{1/2}))\, d^3 p \\
&\quad = c|y|^2 T^{-1/4},
\end{aligned}$$

with an analogous bound for the integral over the region

$$\{0 \le r \le T^{-1/2},\ T^{-1/2} \le s \le \infty\}.$$

To see (3.27), we note that for $|y|/\sqrt{T} < \delta/2$,

$$\begin{aligned}
&\int_{|z| \ge \delta} f_T(T^{-1/2}, z, y)\, dz \\
&\quad \le T^{1/2} \int_0^{T^{-1/2}} \int_0^{T^{-1/2}} \int_{|x| \ge \delta/2} p(r,x) p(s,x)\, dx\, dr\, ds \\
&\quad \le cT^{1/2} \int_0^{T^{-1/2}} \int_0^{T^{-1/2}} \exp(-\delta^2/(4r)) \int p(r/2, x) p(s,x)\, dx\, dr\, ds \qquad (3.33) \\
&\quad \le cT^{1/2} \exp(-T^{1/2}\delta^2/4) \int_0^{T^{-1/2}} \int_0^{T^{-1/2}} \frac{1}{(r+s)^{3/2}}\, dr\, ds \\
&\quad \le cT^{1/2} \exp(-T^{1/2}\delta^2/4) \to 0
\end{aligned}$$

as $T \to \infty$. This completes the proof of Lemma 2. $\square$

PROOF OF LEMMA 1. A vertex $v \in G$ will be called a pre-exit if it is the immediate predecessor of an exit. We use $PT(G)$ to denote the set of pre-exits for twinned exits, and $PU(G)$ to denote the set of pre-exits for untwinned exits. In Figure 1, $PT(G) = \{p_1, p_2, b, p_3\}$, and $PU(G) = \{c, v, a\}$. To each rooted binary tree $G \in \mathscr{G}$ we now associate a new rooted tree, $F(G)$, called the frame of G as follows. The vertices of $F(G)$ consist of the root r of G together with $PU(G)$, all pre-exits for untwinned exits in G. Two vertices $v, v' \in F(G)$ are connected by a bond in $F(G)$ if there is a path from v to v' in G which does not pass through any other vertices in $PU(G)$. In Figure 2 we give $F(G)$ for the G of Figure 1. Note that $F(G)$ is a rooted tree but not necessarily a binary tree. Let $\mathscr{T}$ denote the set of finite rooted trees.

If $F \in \mathscr{T}$ and $0 < \varepsilon < 1$, let $p_\varepsilon(F)$ be the probability that F is the family tree in a geometric Galton–Watson process with parameter ε. In other words, if $S(v)$ denotes the set of successors of the vertex v,

$$p_\varepsilon(F) = \prod_{v \in F} (1-\varepsilon)\varepsilon^{|S(v)|}. \tag{3.34}$$

We intend to prove (3.19) by showing that for any ε sufficiently small we can find $T_\varepsilon < \infty$ and $C_\varepsilon < \infty$ such that

$$\sum_{G:\, F(G)=F} \theta^{|E(G)|} T^{|E(G)|/4} |d(G, 1, y/\sqrt{T})| \le C_\varepsilon p_\varepsilon(F) \tag{3.35}$$

for all $T \ge T_\varepsilon$ and all $F \in \mathscr{T}$. Lemma 1 then follows since

$$\sum_{F \in \mathscr{T}} p_\varepsilon(F) \le 1, \tag{3.36}$$

because this sum is the extinction probability of our Galton–Watson process.

In order to prove (3.35) we will use a specific decomposition of $G \in \mathscr{G}$ which we refer to as the frame decomposition of G. For each $v \in F(G)$, $v \ne r$ we let G_v denote the rooted binary subtree of G consisting of all non-exits $w \ne v$ of G for which there exists a path in G from v to w which does not pass through any vertices in $PU(G)$. In other words, the condition for a non-exit $w \ne v$ to be in G_v is that $u \notin PU(G)$ for every $u \ne v$, $u \ne w$ in the path from v to w. See Figure 3 for the G_v corresponding to the G of Figure 1. Now p_2 and p_3 are not in G_v since, for example, the path from v to p_2 passes through $c \in PU(G)$. Note that the root of G_v, denoted by r_v, is the vertex in G which is the unique non-exit successor to v. The exits of G_v are of two types. The first set of exits, which we denote by $D(G_v)$, consists of the vertices in G which were pre-exits

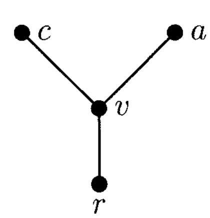

FIG. 2. $F(G)$.

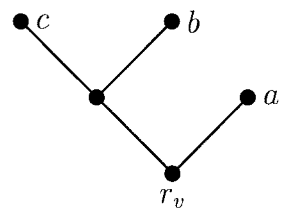

FIG. 3. G_v.

for twinned exits. The second set of exits, which we denote by $S(G_v)$ consists of the vertices in $S(v)$, the successors of v in F. [As a set, $S(G_v)$ is the same as $S(v)$ but we use the notation $S(G_v)$ when we think of it as a subset of the binary tree G_v.] In Figure 3, $D(G_v) = \{b\}$ and $S(G_v) = \{a, c\}$. When $v = r$ we take G_r as above except that we also include r. See Figure 4. In our notation, $r_r = r$.

Note that the data $(F, G_v,\ v \in F)$ does not determine G, that is, does not allow us to reconstruct G. We also need to specify $S(G_v)$, $v \in F$ to tell us which of the exits of G_v are the elements of $S(v)$. Hence there are

$$\prod_{v\in F} \binom{|E(G_v)|}{|S(v)|} \tag{3.37}$$

binary trees $G \in \mathscr{G}$ giving rise to the same data $(F, G_v,\ v \in F)$. We refer to $(F, G_v, S(G_v),\ v \in F)$ as the frame decomposition of G.

With the notation

$$\begin{aligned} \widetilde{c}(G, T, u, s) = \int\int_{\Delta_G^T} \prod_{a\in A_G} p(t_{a_f} - t_{a_i}, x_{a_f} - x_{a_i}) \delta_u(x_r)\delta_s(t_r) \\ \times \prod_{e\in E(G)} \delta_0(x_e)\, dt\, dx \end{aligned} \tag{3.38}$$

for $G \neq \widehat{G}$ and

$$\widetilde{c}(\widehat{G}, T, u, s) = 1_{\{s\le T\}}\delta_0(u),$$

we have the following lemma.

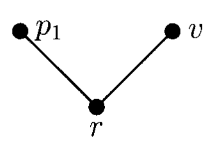

FIG. 4. G_r.

LEMMA 3.

$$T^{|E(G)|/4}|d(G,1,y/\sqrt{T})| \le \prod_{v\in F}\left(\int \tilde{c}(G_v,1,u,0)\,du\right)(|y|/\pi)^{|D(G_v)|} \times \left(\tilde{Q}_{T,y}\int_0^1 \|p(t,\cdot)\|_{5/2}\,dt\right)^{|S(v)|}, \tag{3.39}$$

where

$$\tilde{Q}_{T,y} = T^{1/4}\sup_{0\le t\le 1}\left\|\int_0^t (p(b,\cdot)-p(b,\cdot-y/\sqrt{T}))\,db\right\|_{5/3}$$

and

$$\lim_{T\to\infty}\tilde{Q}_{T,y} = 0. \tag{3.40}$$

We prove Lemma 3 at the end of this section.

Choose $0<\varepsilon_0<1$ so that $(1+\varepsilon_0)^2\theta \le \theta_0$ and consequently

$$\Lambda((1+\varepsilon_0)^2|y|\theta^2/\pi) = \sum_{n=1}^{\infty}((1+\varepsilon_0)^2|y|\theta^2/\pi)^n\sum_{\mathscr{G}_n}c(G,1) < \infty. \tag{3.41}$$

It is easily seen that $\int_0^1\|p(t,\cdot)\|_{5/2}\,dt<\infty$. Hence, by (3.40), for any $\varepsilon\le\varepsilon_0$ we can find $T_\varepsilon<\infty$ such that

$$\tilde{Q}_{T,y}\int_0^1\|p(t,\cdot)\|_{5/2}\,dt \le (1-\varepsilon)\varepsilon^3(|y|\theta^2/\pi)(\Lambda((1+\varepsilon)^2|y|\theta^2/\pi))^{-1} \tag{3.42}$$

for all $T\ge T_\varepsilon$.

Note that

$$\begin{aligned} c(G,T) &= \int p(s,u-w)\tilde{c}(G,T,u,s)\,dw\,du\,ds \\ &= \int \tilde{c}(G,T,u,s)\,du\,ds. \end{aligned} \tag{3.43}$$

Since $\tilde{c}(G_v,1,x,0)=\tilde{c}(G_v,1+\varepsilon,x,\varepsilon)$ for any ε, as follows from translation invariance, we have, using (3.43) and (3.6), that

$$\begin{aligned} \int \tilde{c}(G_v,1,x,0)\,dx &= \int \tilde{c}(G_v,1+\varepsilon,x,\varepsilon)\,dx \\ &\le \varepsilon^{-1}\int_0^{\varepsilon}\int \tilde{c}(G_v,1+\varepsilon,x,t)\,dx\,dt \\ &\le \varepsilon^{-1}\int_0^{1}\int \tilde{c}(G_v,1+\varepsilon,x,t)\,dx\,dt \\ &= \varepsilon^{-1}c(G_v,1+\varepsilon) \\ &\le \varepsilon^{-1}(1+\varepsilon)^{|E(G_v)|}c(G_v,1). \end{aligned} \tag{3.44}$$

Thus (3.39) gives the estimate

$$
\begin{aligned}
&\theta^{|E(G)|}T^{|E(G)|/4}|d(G,1,\,y/\sqrt{T})| \\
&\quad\le \prod_{v\in F}\varepsilon^{-1}c(G_v,1)((1+\varepsilon)|y|\theta^2/\pi)^{|E(G_v)|}\left(\frac{(1-\varepsilon)\varepsilon^3}{\Lambda((1+\varepsilon)^2|y|\theta^2/\pi)}\right)^{|S(v)|}.
\end{aligned}
\tag{3.45}
$$

Hence, using (3.37) we see that for fixed $F, G_v;\ v\in F$,

$$
\begin{aligned}
&\sum_{G\mapsto(F,\,G_v;\,v\in F)}\theta^{|E(G)|}T^{|E(G)|/4}|d(G,1,\,y/\sqrt{T})| \\
&\quad\le \prod_{v\in F}\varepsilon^{-1}c(G_v,1)((1+\varepsilon)|y|\theta^2/\pi)^{|E(G_v)|}\binom{|E(G_v)|}{|S(v)|}\varepsilon^{|S(v)|} \\
&\qquad\qquad\times\left(\frac{(1-\varepsilon)\varepsilon^2}{\Lambda((1+\varepsilon)^2|y|\theta^2/\pi)}\right)^{|S(v)|} \\
&\quad\le \prod_{v\in F}\varepsilon^{-1}c(G_v,1)((1+\varepsilon)|y|\theta^2/\pi)^{|E(G_v)|}(1+\varepsilon)^{|E(G_v)|} \\
&\qquad\qquad\times\left(\frac{(1-\varepsilon)\varepsilon^2}{\Lambda((1+\varepsilon)^2|y|\theta^2/\pi)}\right)^{|S(v)|} \\
&\quad= \prod_{v\in F}\varepsilon^{-1}c(G_v,1)((1+\varepsilon)^2|y|\theta^2/\pi)^{|E(G_v)|}\left(\frac{(1-\varepsilon)\varepsilon^2}{\Lambda((1+\varepsilon)^2|y|\theta^2/\pi)}\right)^{|S(v)|}.
\end{aligned}
\tag{3.46}
$$

Therefore, letting $S(F)=F-\{r\}$ be the set of successors in F, we have

$$
\begin{aligned}
&\sum_{G:\,F(G)=F}\theta^{|E(G)|}T^{|E(G)|/4}|d(G,1,\,y/\sqrt{T})| \\
&\quad\le \prod_{v\in F}\varepsilon^{-1}\left(\sum_{G\in\mathscr{G}}c(G,1)((1+\varepsilon)^2|y|\theta^2/\pi)^{|E(G)|}\right) \\
&\qquad\qquad\times\left(\frac{(1-\varepsilon)\varepsilon^2}{\Lambda((1+\varepsilon)^2|y|\theta^2/\pi)}\right)^{|S(v)|} \\
&\quad= \prod_{v\in F}\varepsilon^{-1}\Lambda((1+\varepsilon)^2|y|\theta^2/\pi)\left(\frac{(1-\varepsilon)\varepsilon^2}{\Lambda((1+\varepsilon)^2|y|\theta^2/\pi)}\right)^{|S(v)|} \\
&\quad= \left(\prod_{v\in F}\varepsilon^{-1}\Lambda((1+\varepsilon)^2|y|\theta^2/\pi)\varepsilon^{|S(v)|}\right) \\
&\qquad\qquad\times\left(\prod_{v\in S(F)}\frac{(1-\varepsilon)\varepsilon}{\Lambda((1+\varepsilon)^2|y|\theta^2/\pi)}\right) \\
&\quad\le \varepsilon^{-1}(1-\varepsilon)^{-1}\Lambda((1+\varepsilon)^2|y|\theta^2/\pi)\prod_{v\in F}(1-\varepsilon)\varepsilon^{|S(v)|}
\end{aligned}
\tag{3.47}
$$

and (3.35) now follows. This completes the proof of Lemma 1. $\square$

It remains to justify the assertion we made in the paragraph following Lemma 1 that in the $T\to\infty$ limit the only graphs which contribute to (3.12) are the graphs G in which all exits are twinned. This follows easily from our proof of (3.19), in particular from (3.39) if we use (3.44) with $\varepsilon=1/2$ to bound the $\widetilde{c}$ integrals and then use (3.40).

PROOF OF LEMMA 3. Using the frame decomposition we can rewrite

$$\begin{aligned}
&T^{|E(G)|/4}d(G,1,y/\sqrt{T})\\
&\quad=\int\int p(t_r,x_r-w)\\
&\qquad\times\prod_{v\in F}\biggl(\int\widetilde{c}(G_v,x_{r_v},t_{r_v};(z_e,T_e),\ e\in E(G_v))\prod_{e\in D(G_v)}q_{T,y}(T_e,z_e)\\
&\qquad\qquad\times\prod_{e\in S(G_v)}\widetilde{q}_{T,y}(T_e,z_e,t_{r_e},x_{r_e})\,dT_e\,dz_e\biggr)\,dt_{r_v}\,dx_{r_v}\,dw.
\end{aligned}\tag{3.48}$$

Here

$$\begin{aligned}
q_{T,y}(t,z)&\stackrel{\text{def}}{=}T^{1/2}\int_0^1\int_0^1(p(r-t,z)-p(r-t,z-y/\sqrt{T}))\\
&\qquad\qquad\times(p(s-t,z)-p(s-t,z-y/\sqrt{T}))\,dr\,ds,\\
&=T^{1/2}\biggl(\int_0^1(p(r-t,z)-p(r-t,z-y/\sqrt{T}))\,dr\biggr)^2\\
&=f_T(1-t,z,y),
\end{aligned}\tag{3.49}$$

$$\begin{aligned}
&\widetilde{q}_{T,y}(t,z,s,x)\\
&\quad=T^{1/4}p(s-t,x-z)\int_0^1(p(b-t,z)-p(b-t,z-y/\sqrt{T}))\,db
\end{aligned}\tag{3.50}$$

and

$$\begin{aligned}
&\widetilde{c}(G,u,s;(z_e,T_e),\ e\in E(G))\\
&\quad=\int\int\prod_{a\in A_G^0}p(t_{a_f}-t_{a_i},x_{a_f}-x_{a_i})\\
&\qquad\times\prod_{a\in A_G^e}p(T_{a_f}-t_{a_i},z_{a_f}-x_{a_i})1_{\{T_{a_f}\le 1\}}\delta_u(x_r)\delta_s(t_r)\,dt\,dx
\end{aligned}\tag{3.51}$$

for $G\neq\widehat{G}$ and

$$\widetilde{c}(\widehat{G},u,s;z,T)=\delta_u(z)\delta_s(T)1_{\{T\le 1\}}.$$

(Note, e.g., in the G of Figure 1 we have that $G_a=\{p_3\}$.)

We will use the notation

$$Q_{T,y}=\sup_t\int q_{T,y}(t,x)\,dx.$$

We have seen in (3.24), (3.25) that

$$Q_{T,\,y} \le |y|/\pi. \tag{3.52}$$

We note by Hölder's inequality that

$$\sup_x \int |\widetilde{q}_{T,\,y}(t,z,s,x)|\,dz \le \widetilde{Q}_{T,\,y}\|p(s-t,\cdot)\|_{5/2} \tag{3.53}$$

We now explain how to obtain the following bound for (3.48).

$$\begin{aligned}
&T^{|E(G)|/4}|d(G,1,y/\sqrt{T})|\\
&\quad\le \int\int p(t_r, x_r - w)\\
&\qquad\times \prod_{v\in F}\Bigg(\int \widetilde{c}(G_v, x_{r_v}, t_{r_v};\ (z_e, T_e),\ e\in E(G_v))\\
&\qquad\qquad\times \prod_{e\in D(G_v)} q_{T,\,y}(T_e, z_e)\\
&\qquad\qquad\times \prod_{e\in S(G_v)} |\widetilde{q}_{T,\,y}(T_e, z_e, t_{r_e}, x_{r_e})|\,dT_e\,dz_e\Bigg)\,dt_{r_v}\,dx_{r_v}\,dw\\
&\quad\le \int\int \prod_{v\in F}\Bigg(\int \widetilde{c}(G_v, x_{r_v}, t_{r_v};\ (z_e, T_e),\ e\in E(G_v))\\
&\qquad\qquad\times \prod_{e\in D(G_v)} q_{T,\,y}(T_e, z_e)\\
&\qquad\qquad\times \prod_{e\in S(G_v)} |\widetilde{q}_{T,\,y}(T_e, z_e, t_{r_e}, x_{r_e})|\,dT_e\,dz_e\Bigg)\,dt_{r_v}\,dx_{r_v}\\
&\quad\le \int \prod_{v\in F}\Bigg(\int\int \widetilde{c}(G_v, x_{r_v}, t_{r_v};\ (0, T_e),\ e\in E(G_v))\,dx_{r_v}\\
&\qquad\qquad\times \prod_{e\in D(G_v)} Q_{T,\,y} \prod_{e\in S(G_v)} \widetilde{Q}_{T,\,y}\|p(t_{r_e}-T_e,\cdot)\|_{5/2}\,dT_e\Bigg)\,dt_{r_v}.
\end{aligned} \tag{3.54}$$

In the first inequality we use the fact that $\widetilde{c}$ and $q_{T,\,y}(T_e, z_e)$ are positive, while the second inequality simply comes from integrating out w [so that now x_r occurs only in the $\widetilde{c}(G_r,\cdot)$ term]. To explain the last inequality we begin by noting as in (3.11) that

$$\int \widetilde{c}(G,x,t;(z_e,T_e),\ e\in E(G))\,dx \le \int \widetilde{c}(G,x,t;(0,T_e),\ e\in E(G))\,dx \tag{3.55}$$

for all (z_e, T_e), $e\in E(G)$. We then apply this repeatedly, moving upwards from the root in $F(G)$. Thus, we first estimate

$$\begin{aligned}
&\int\Bigg(\int \widetilde{c}(G_r, x_r, t_r;\ (z_e, T_e),\ e\in E(G_r))\,dx_r\Bigg)\\
&\qquad\times \prod_{e\in D(G_r)} q_{T,\,y}(T_e, z_e) \prod_{e\in S(G_r)} |\widetilde{q}_{T,\,y}(T_e, z_e, t_{r_e}, x_{r_e})|\,dT_e\,dz_e
\end{aligned}$$

$$
\begin{aligned}
&\le \int\biggl(\int \tilde{c}(G_r, x_r, t_r; (0, T_e),\ e \in E(G_r))\,dx_r\biggr)\\
&\qquad \times \prod_{e\in D(G_r)} q_{T,y}(T_e, z_e) \prod_{e\in S(G_r)} |\tilde{q}_{T,y}(T_e, z_e, t_{r_e}, x_{r_e})|\,dT_e\,dz_e\\
&= \int\biggl(\int \tilde{c}(G_r, x_r, t_r; (0, T_e),\ e \in E(G_r))\,dx_r\biggr)\\
&\qquad \times \prod_{e\in D(G_r)} \biggl(\int q_{T,y}(T_e, z_e)\,dz_e\biggr)\\
&\qquad \times \prod_{e\in S(G_r)} \biggl(\int |\tilde{q}_{T,y}(T_e, z_e, t_{r_e}, x_{r_e})|\,dz_e\biggr)\,dT_e\\
&= \int\biggl(\int \tilde{c}(G_r, x_r, t_r; (0, T_e),\ e \in E(G_r))\,dx_r\biggr)\\
&\qquad \times \prod_{e\in D(G_r)} Q_{T,y} \prod_{e\in S(G_r)} \tilde{Q}_{T,y}\|p(t_{r_e} - T_e, \cdot)\|_{5/2}\,dT_e.
\end{aligned}
$$

At this stage, for each $v \in S(G_r)$ we see that x_{r_v} occurs only in the $\tilde{c}(G_v, \cdot)$ term, hence we can repeat the last series of inequalities with G_r replaced by G_v for such v, that is, for the immediate successors of r in $F(G)$. Continuing in this manner, we obtain the last inequality of (3.54).

We now explain how to obtain the following bound for the right-hand side of (3.54):

$$
\begin{aligned}
&\int \prod_{v\in F}\biggl(\int\int \tilde{c}(G_v, x_{r_v}, t_{r_v}; (0, T_e),\ e \in E(G_v))\,dx_{r_v}\\
&\qquad \times \prod_{e\in D(G_v)} Q_{T,y} \prod_{e\in S(G_v)} \tilde{Q}_{T,y}\|p(t_{r_e} - T_e, \cdot)\|_{5/2}\,dT_e\biggr)\,dt_{r_v}\\
&\quad \le \prod_{v\in F}\biggl(\int\int \tilde{c}(G_v, x_{r_v}, 0; (0, T_e),\ e \in E(G_v))\,dx_{r_v}\,dT_e\biggr)\\
&\qquad \times \prod_{e\in D(G_v)} Q_{T,y} \prod_{e\in S(G_v)} \tilde{Q}_{T,y}\biggl(\int_0^1 \|p(t, \cdot)\|_{5/2}\,dt\biggr).
\end{aligned} \tag{3.56}
$$

To explain the last inequality we begin by noting the simple monotonicity bounds

$$
\begin{aligned}
&\int\int \tilde{c}(G_v, x_{r_v}, t_{r_v}; (0, T_e),\ e \in E(G_v))\,dx_{r_v}\,dT_e\\
&\quad \le \int\int \tilde{c}(G_v, x_{r_v}, 0; (0, T_e),\ e \in E(G_v))\,dx_{r_v}\,dT_e
\end{aligned} \tag{3.57}
$$

and

$$
\int_0^1 \|p(t - s, \cdot)\|_{5/2}\,dt \le \int_0^1 \|p(t, \cdot)\|_{5/2}\,dt. \tag{3.58}
$$

We will apply these repeatedly, this time working downwards from the top of our tree $F(G)$. Thus, to begin, if v is a vertex in $F(G)$ with no successor, $S(G_v)$ is necessarily empty, so that T_e, $e \in E(G_v)$ occurs only in the $\widetilde{c}(G_v, \cdot)$ term and we can apply (3.57). Following this, t_{r_v} now occurs only in the $\| p(t_{r_e} - T_e, \cdot)\|_{5/2}$ term where $v = e \in S(G_{v'})$ and v' is the predecessor of v in $F(G)$. We then apply (3.58), and after doing this for all $e \in S(G_{v'})$ we find that T_e, $e \in E(G_{v'})$ occurs only in the $\widetilde{c}(G_{v'}, \cdot)$ term. Continuing in this manner we obtain (3.56). In view of (3.38) this proves (3.39).

We now prove (3.40). Using (3.23) again, we see that for any $t \le 1$,

$$\begin{aligned} &\int_0^t |(p(s,x) - p(s, x - y/\sqrt{T}))|\,ds \\ &\qquad \le e\int_0^\infty e^{-s}|(p(s,x) - p(s, x - y/\sqrt{T}))|\,ds \\ &\qquad = e|u^1(x) - u^1(x - y/\sqrt{T})|. \end{aligned} \tag{3.59}$$

The proof of (3.40) will be complete once we show that

$$\|u^1(\cdot) - u^1(\cdot - y/\sqrt{T})\|_{5/3} \le cT^{-3/8}\,|y|^{3/4}. \tag{3.60}$$

To prove (3.60) we note that

$$\begin{aligned} &\left|\frac{e^{-|x|}}{|x|} - \frac{\exp(-|x - y/\sqrt{T}|)}{|x - y/\sqrt{T}|}\right| \\ &\qquad \le \frac{|y|}{\sqrt{T}}\left(\frac{\exp(-|x|/2)}{|x|^2} + \frac{\exp(-|x - y/\sqrt{T}|/2)}{|x - y/\sqrt{T}|^2}\right) \end{aligned} \tag{3.61}$$

to see by interpolating that

$$\begin{aligned} &\left|\frac{e^{-|x|}}{|x|} - \frac{\exp(-|x - y/\sqrt{T}|)}{|x - y/\sqrt{T}|}\right| \\ &\qquad \le \frac{|y|^{3/4}}{T^{3/8}}\left(\frac{\exp(-|x|/4)}{|x|^{7/4}} + \frac{\exp(-|x - y/\sqrt{T}|/4)}{|x - y/\sqrt{T}|^{7/4}}\right) \end{aligned} \tag{3.62}$$

and

$$\left\|\frac{\exp(-|x|/4)}{|x|^{7/4}}\right\|_{5/3} < \infty.$$

This completes the proof of Lemma 3. □

4. Functionals. If h is a function on R^3, let

$$\sigma(h) = -\frac{1}{2\pi}\int\int |x - y|\, h(x)h(y)\,dx\,dy. \tag{4.1}$$

With the notation of Theorem 8, let $\theta_{c,h}$ be defined by

$$\theta^* = \sigma(h)\theta_{c,\,h}^2.$$

THEOREM 9. *Let μ_t denote super-Brownian motion in R^3, and let μ denote the Lebesgue measure on R^3. If h is a function on R^3 with*

$$\int_{R^3} (1+|x|)|h(x)|\,dx < \infty \tag{4.2}$$

and

$$\int_{R^3} h(x)\,dx = 0 \tag{4.3}$$

then with the notation of Theorem 8,

$$\lim_{T\to\infty} \frac{1}{T^{1/2}} \log E_\mu\biggl(\exp\biggl(\frac{\theta}{T^{1/4}}\int_0^T \mu_s(h)\,ds\biggr)\biggr) = \Lambda(\sigma(h)\theta^2) \tag{4.4}$$

for all $\theta < \theta_{c,h}$.

PROOF. The proof follows along the lines of the proof of Theorem 8. As in (3.12) we have

$$\frac{1}{T^{1/2}} \log E_\mu\biggl(\exp\biggl(\frac{\theta}{T^{1/4}}\int_0^T \mu_s(h)\,ds\biggr)\biggr) = \sum_{n=1}^{\infty} \frac{\theta^n}{T^{n/4+1/2}} d_n(T,h), \tag{4.5}$$

where now

$$d_n(T,h) = \sum_{\mathscr{G}_n} d(G,T,h) \tag{4.6}$$

and

$$\begin{aligned} d(G,T,h) = \int\int_{\Delta_G^T} p(t_r, x_r - w) \prod_{a\in A_G^0} p(t_{a_f} - t_{a_i}, x_{a_f} - x_{a_i}) \\ \times \prod_{a\in A_G^e} \int p(t_{a_f} - t_{a_i}, y - x_{a_i}) h(y)\,dy\,dt\,dx\,dw. \end{aligned} \tag{4.7}$$

Using (4.3), we can rewrite this as

$$\begin{aligned} d(G,T,h) = \int\int_{\Delta_G^T} p(t_r, x_r - w) \prod_{a\in A_G^0} p(t_{a_f} - t_{a_i}, x_{a_f} - x_{a_i}) \\ \times \prod_{a\in A_G^e} \int \{p(t_{a_f} - t_{a_i}, -x_{a_i}) - p(t_{a_f} - t_{a_i}, y - x_{a_i})\} \\ \times h(y)\,dy\,dt\,dx\,dw. \end{aligned} \tag{4.8}$$

As before, scaling leads to

$$d(G,T,h) = T^{n/2+1/2} d(G,1,h_T), \tag{4.9}$$

where $h_T(y) = T^{3/2} h(y\sqrt{T})$ so that

$$\int g(y) h_T(y)\,dy = \int g(y/\sqrt{T}) h(y)\,dy. \tag{4.10}$$

Thus

$$\frac{d(G,T,h)}{T^{n/4+1/2}} = T^{n/4} d(G,1,h_T). \tag{4.11}$$

As in (3.21), when all exits in G are twinned we can write (4.7) as

$$\begin{aligned}\frac{d(G,T,h)}{T^{n/4+1/2}} &= \int\int_{[0,1]^m}\int_{\Delta^1_{G_0}} p(t_r, x_r - w)\prod_{a\in A_{G_0}} p(t_{a_f}-t_{a_i}, x_{a_f}-x_{a_i}) \\ &\quad\times \prod_{j=1}^m \delta_{s_j}(t_{u_j})\delta_{z_j}(x_{u_j})\,dt\,dx\,dw \prod_{j=1}^m f_T(1-s_j, z_j, h)\,ds\,dz,\end{aligned} \tag{4.12}$$

where

$$\begin{aligned} f_T(t,z,h) &= T^{1/2}\int\int\int_0^t\int_0^t p(r, z - x/\sqrt{T})p(s, z-y/\sqrt{T})) \\ &\qquad\times h(x)h(y)\,dx\,dy\,dr\,ds, \\ &= T^{1/2}\left(\int\int_0^t p(r, z-y/\sqrt{T})h(y)\,dy\,dr\right)^2 \\ &= T^{1/2}\left(\int\int_0^t (p(r,z-y/\sqrt{T}) - p(r,z))h(y)\,dy\,dr\right)^2. \end{aligned} \tag{4.13}$$

It is easy to check that $\int f_T(t,z,h)\,dz < \infty$ for fixed $t, T < \infty$.

We also note that

$$\begin{aligned} &\int f_T(t,z,h)\,dz \\ &\quad = T^{1/2}\int\int\int_0^t\int_0^t p(r+s, (x-y)/\sqrt{T})h(x)h(y)\,dx\,dy\,dr\,ds, \\ &\quad = T^{1/2}\int\int\int\int_0^t\int_0^t p((r+s)/2, z-x/\sqrt{T})p((r+s)/2, z-y/\sqrt{T})) \\ &\qquad\qquad \times h(x)h(y)\,dx\,dy\,dz\,dr\,ds, \\ &\quad = T^{1/2}\int\int_0^t\int_0^t\left(\int p((r+s)/2, z-y/\sqrt{T})h(y)\,dy\right)^2 dr\,ds\,dz, \end{aligned} \tag{4.14}$$

which shows that $\int f_T(t,z,h)\,dz$ is positive and monotone increasing in t. Because of (4.3),

$$\begin{aligned} &\lim_{t\to\infty}\int f_T(t,z,h)\,dz \\ &\quad = T^{1/2}\lim_{t\to\infty}\int\int\int_0^t\int_0^t p(r+s, (x-y)/\sqrt{T})h(x)h(y)\,dr\,ds\,dx\,dy \\ &\quad = -T^{1/2}\lim_{t\to\infty}\int\int\left(\int_0^t\int_0^t (p(r+s,0) - p(r+s,(x-y)/\sqrt{T}))\,dr\,ds\right) \\ &\qquad\qquad \times h(x)h(y)\,dx\,dy \end{aligned} \tag{4.15}$$

$$= -T^{1/2} \int\int \left(\int_0^\infty s(p(s,0) - p(s,(x-y)/\sqrt{T}))\, ds \right) h(x)h(y)\, dx\, dy$$
$$= \sigma(h).$$

We next show that uniformly in $t \geq T^{-1/2} \geq 1$,

$$\int |f_T(t,z,h) - f_T(T^{-1/2},z,h)|\, dz = O(T^{-1/8}), \tag{4.16}$$

and for any $\delta > 0$,

$$\lim_{T\to\infty} \int_{|z|\geq\delta} f_T(T^{-1/2},z,h)\, dz = 0. \tag{4.17}$$

To prove (4.16) we use (4.13), (3.23)–(3.25) and the analogue of (3.32) to see that

$$\begin{aligned}
&T^{1/2} \int \left| \left(\int\int_{T^{-1/2}}^{t} (p(r, z - x/\sqrt{T}) - p(r,z))h(x)\, dx\, dr \right) \right. \\
&\qquad \left. \times \left(\int\int_0^{T^{-1/2}} (p(s, z - y/\sqrt{T}) - p(s,z))h(y)\, dy\, ds \right) \right| dz \\
&\leq T^{1/2} \int \left(\int\int_{T^{-1/2}}^{\infty} |p(r, z - x/\sqrt{T}) - p(r,z)|\, |h(x)|\, dx\, dr \right) \\
&\qquad \times \left(\int\int_0^1 |p(s, z - y/\sqrt{T}) - p(s,z)|\, |h(y)|\, dy\, ds \right) dz \\
&\leq T^{1/2} \int \left\| \int_{T^{-1/2}}^{\infty} |p(r, z - x/\sqrt{T}) - p(r,z)|\, dr \right\|_2 |h(x)|\, dx \\
&\qquad \times \int \left\| \int_0^1 |p(s, z - y/\sqrt{T}) - p(s,z)|\, ds \right\|_2 |h(y)|\, dy \\
&\leq cT^{-1/8} \left(\int |x||h(x)|\, dx \right)^{1/2} \left(\int |y|^{1/2} |h(y)|\, dy \right)^{1/2} \\
&\leq cT^{-1/8}.
\end{aligned} \tag{4.18}$$

There is an analogous bound for the integrals over the regions

$$\{0 \leq r \leq T^{-1/2},\ T^{-1/2} \leq s \leq t\},\ \{T^{-1/2} \leq r \leq t,\ T^{-1/2} \leq s \leq t\}.$$

The proof of (4.17) is similar to that of (3.27) once we note that by (4.2),

$$\int_{\{|y|/\sqrt{T} \geq \delta/2\}} |h(y)|\, dy \to 0 \tag{4.19}$$

as $T \to \infty$.

Putting all this together as in the proof of Theorem 8, we see that for any graph $G \in \mathscr{G}_n$ with $n = 2m$ and all exits twinned we have

$$\lim_{T\to\infty} \frac{d(G,T,h)}{T^{n/4+1/2}} = c(G_0,1)\sigma^m(h). \tag{4.20}$$

Our theorem will then follow as in the proof of Theorem 8 once we establish the analogue of (3.19). All that is really needed is to show that

$$\lim_{T\to 0} \widetilde{Q}_{T,h} = 0, \tag{4.21}$$

where now, without risk of confusion,

$$\widetilde{Q}_{T,h} = T^{1/4} \sup_{0\le t\le 1} \left\| \int_0^t \int (p(b,\cdot) - p(b,\cdot - y/\sqrt{T}))h(y)\,dy\,db \right\|_{5/3}.$$

Using (3.23) again, we see that for any $t \le 1$,

$$\begin{aligned}
&\int_0^t \int |(p(s,x) - p(s,x-y/\sqrt{T}))||h(y)|\,dy\,ds \\
&\quad \le e\int\int_0^\infty e^{-s}|(p(s,x) - p(s,x-y/\sqrt{T}))|\,ds|h(y)|\,dy \\
&\quad = e\int |u^1(x) - u^1(x-y/\sqrt{T})||h(y)|\,dy,
\end{aligned} \tag{4.22}$$

and using (3.60) we have

$$\begin{aligned}
&\left\| \int |u^1(\cdot) - u^1(\cdot - y/\sqrt{T})||h(y)|\,dy \right\|_{5/3} \\
&\quad \le \int \|u^1(\cdot) - u^1(\cdot - y/\sqrt{T})\|_{5/3}|h(y)|\,dy \\
&\quad \le cT^{-3/8}\int |y|^{3/4}|h(y)|\,dy.
\end{aligned} \tag{4.23}$$

This completes the proof of Theorem 9. □

The same proof will also lead to the following theorem. If ν is a signed measure with on R^3, let

$$\sigma(\nu) = -\frac{1}{2\pi}\int\int |x-y|\,d\nu(x)\,d\nu(y). \tag{4.24}$$

With the notation of Theorem 8, let $\theta_{c,\nu}$ be defined by

$$\theta^* = \sigma(\nu)\theta_{c,\nu}^2.$$

THEOREM 10. *Let μ_t denote super-Brownian motion in R^3, and let μ denote the Lebesgue measure on R^3. If ν is a signed measure on R^3 with*

$$\int_{R^3} (1+|x|)\,d|\nu|(x) < \infty \tag{4.25}$$

and

$$\int_{R^3} d\nu(x) = 0 \tag{4.26}$$

then with the notation of Theorem 8,

$$\lim_{T\to\infty} \frac{1}{T^{1/2}} \log E_\mu\left(\exp\left(\frac{\theta}{T^{1/4}} \int L_T^y \, d\nu(y)\right)\right) = \Lambda(\sigma(\nu)\theta^2) \tag{4.27}$$

for all $\theta < \theta_{c,\nu}$.

PROOF OF THEOREM 3. We first prove (1.14). We will show that for all θ,

$$\lim_{T\to\infty} \log E\left[\exp\left(\theta \frac{\bar{\mu}_T(h)}{\sqrt{T}}\right)\right] = \sigma(h)\theta^2. \tag{4.28}$$

As in (4.5) we have

$$\log E\left[\exp\left(\theta \frac{\bar{\mu}_T(h)}{\sqrt{T}}\right)\right] = \sum_{n=1}^{\infty} \frac{\theta^n}{T^{n/2}} d_n(T, h), \tag{4.29}$$

where

$$d_n(T, h) = \sum_{\mathscr{G}_n} d(G, T, h) \tag{4.30}$$

and $d(G, T, h)$ is defined in (4.7). We note that

$$d_1(T, h) = d(\widehat{G}, T, h) = \langle h, \lambda \rangle = 0 \tag{4.31}$$

so that the sum in (4.29) is actually over $n \geq 2$.

From (3.20) we see that for any graph $G \in \mathscr{G}_n$ with $n = 2m$ and all exits twinned we have

$$\lim_{T\to\infty} \frac{d(G, T, h)}{T^{n/4+1/2}} = c(G_0, 1)\sigma^m(h), \tag{4.32}$$

while for any other $G \in \mathscr{G}_n$ the limit is 0. We note that $n/4 + 1/2 < n/2$ for all $n > 2$, while for the unique graph $G \in \mathscr{G}_2$ we have $c(G_0, 1) = 1$. Hence

$$\lim_{T\to\infty} \frac{d_n(T, h)}{T^{n/2}} = \begin{cases} \sigma(h), & \text{if } n = 2, \\ 0, & \text{otherwise.} \end{cases} \tag{4.33}$$

Equation (1.14) now follows from the proof of Theorem 9, which shows that we can take the limit in (4.29) term by term.

Equation (1.15) is proven similarly, which completes the proof of Theorem 3. □

5. Critical branching Brownian motion in R^3. Let $\xi_t = \xi_t^1$ denote the critical branching Brownian motion in R^3 described in the Introduction with $\tau = 1$. Similarly, we will write $N_t(h)$ for $N_t^1(h)$.

Recall the notation

$$\sigma(h) = -\frac{1}{2\pi} \int\int |x - y| \, h(x)h(y) \, dx \, dy. \tag{5.1}$$

Let

$$\rho(h) = \frac{1}{2\pi} \int\int |x - y|^{-1} \, h(x)h(y) \, dx \, dy. \tag{5.2}$$

With the notation of Theorem 8, let $\theta_{cb,\,h}$ be defined by

$$\theta^* = (\sigma(h) + \rho(h))\theta^2_{cb,\,h}.$$

THEOREM 11. *Let h be a bounded function on R^3 with*

$$\int_{R^3} (1 + |x|)|h(x)|\,dx < \infty \tag{5.3}$$

and

$$\int_{R^3} h(x)\,dx = 0. \tag{5.4}$$

Then with the notation of Theorem 8,

$$\lim_{T\to\infty} \frac{1}{T^{1/2}} \log E\biggl(\exp\biggl(\frac{\theta}{T^{1/4}} \int_0^T N_s(h)\,ds\biggr)\biggr) = \Lambda((\sigma(h) + \rho(h))\theta^2) \tag{5.5}$$

for all $\theta < \theta_{bc,\,h}$.

PROOF. As explained in Section 2 of [2] we have the Campbell formula

$$\begin{aligned} &\log E\biggl(\exp\biggl(\frac{\theta}{T^{1/4}} \int_0^T N_s(h)\,ds\biggr)\biggr) \\ &\qquad = \int_{R^3} \biggl\{E\biggl(\exp\biggl(\frac{\theta}{T^{1/4}} \int_0^T N^y_s(h)\,ds\biggr)\biggr) - 1\biggr\}\,dy, \end{aligned} \tag{5.6}$$

where

$$N^y_t(h) = \sum_{x\in\xi^y_t} h(x), \tag{5.7}$$

with ξ^y_t denoting critical branching Brownian motion starting with a single particle at $y \in R^3$. From the formulas of Section 2 [2] we can easily develop the following graphical representation for the right-hand side of (5.6) analogous to (4.6). Write $\mathscr{G}^n = \bigcup_{i=0}^n \mathscr{G}^n_i$, where $\mathscr{G}^n_i$ denotes the set of unlabeled directed graphs with one root r, $n - i$ exits and a distinguished subset $\mathscr{I}$ of internal vertices with $|\mathscr{I}| = i$, and such that each vertex in $\mathscr{I}$ has one incoming and one outgoing arrow, while all other internal vertices in G have one incoming and two outgoing arrows. We then have

$$\frac{1}{T^{1/2}} \int_{R^3} \biggl\{E\biggl(\exp\biggl(\frac{\theta}{T^{1/4}} \int_0^T N^y_s(h)\,ds\biggr)\biggr) - 1\biggr\}\,dy = \sum_{n=1}^{\infty} \frac{\theta^n}{T^{n/4+1/2}} g_n(T, h), \tag{5.8}$$

where now

$$g_n(T, h) = \sum_{\mathscr{G}^n} g(G, T, h) \tag{5.9}$$

and for $G \in \mathscr{G}_i^n$,

$$g(G,T,h) = \int\int_{\Delta_G^T} p(t_r, x_r - w) \prod_{a\in A_G^0} p(t_{a_f} - t_{a_i}, x_{a_f} - x_{a_i}) \prod_{v\in\mathscr{I}} h(x_v) \\ \times \prod_{a\in A_G^e} \int p(t_{a_f} - t_{a_i}, y - x_{a_i}) h(y)\, dy\, dt\, dx\, dw. \tag{5.10}$$

We can think of $G \in \mathscr{G}_i^n$ as obtained from a graph $\widetilde{G} \in \mathscr{G}_{n-i}$ by placing i vertices on the arrows of $\widetilde{G}$. ($\widetilde{G}$ is obtained fron G by removing the distinguished vertices $\mathscr{I}$ and filling in the gaps, that is, joining the incoming and outgoing arrow for each such vertex into a single arrow.) It is then easily seen that $|G| = 2(n-i) + i - 1$ and $|A_G| = |G| - 1 = 2(n-i) + i - 2$. Hence by scaling we see that

$$\begin{aligned} g(G,T,h) &= T^{|G|} T^{-(3/2)(|A_G|+1)} T^{(3/2)(|G|-n+1)} g(G,1,h_T) \\ &= T^{2(n-i)+i-1} T^{-(3/2)(2(n-i)+i-1)} T^{(3/2)(n-i)} g(G,1,h_T) \\ &= T^{(1/2)(n-i)+1/2-i/2} g(G,1,h_T) \end{aligned} \tag{5.11}$$

where as before $h_T(y) = T^{3/2} h(y\sqrt{T})$. Thus

$$\frac{g(G,T,h)}{T^{n/4+1/2}} = T^{(1/4)(n-i)-(3/4)i} g(G,1,h_T). \tag{5.12}$$

Let $\mathscr{G}_{i,0}^{2m} \subseteq \mathscr{G}_i^{2m}$ doenote the set of graphs with m pre-exits and i distinguished vertices, such that all i distinguished vertices are pre-exits, (necessarily the predecessors of untwinned exits), and the remaining $m-i$ pre-exits are predecessors of twinned exits. We will explain below how to adapt the methods used in the proof of Theorem 8 to show that it suffices to study the $T \to \infty$ limit of (5.8) term by term, and furthermore, that the only graphs G which contribute to (5.8) in the $T \to \infty$ limit are the graphs in $\mathscr{G}_{i,0}^{2m}$, $i = 0, 1, \ldots m$.

As in (4.12), when $G \in \mathscr{G}_{i,0}^{2m}$ we have

$$\frac{g(G,T,h)}{T^{(2m)/4+1/2}} = \int\int_{[0,1]^m} \int_{\Delta_{G_0}^1} p(t_r, x_r - w) \prod_{a\in A_{G_0}} p(t_{a_f} - t_{a_i}, x_{a_f} - x_{a_i}) \\ \times \prod_{j=1}^m \delta_{s_j}(t_{u_j}) \delta_{z_j}(x_{u_j})\, dt\, dx\, dw \prod_{j=1}^m \bar{f}_T(1 - s_j, z_j, h) \\ \times \prod_{j=1}^{m-i} f_T(1 - s_j, z_j, h)\, ds\, dz, \tag{5.13}$$

where $f_T(t,z,h)$ is as in (4.13) and

$$\bar{f}_T(t,z,h) = T^{-1/2} \int\int_0^t p(r, z-y) h_T(z) h_T(y)\, dy\, dr. \tag{5.14}$$

If we now scale out T from the h_T factors in $\bar{f}_T(1-s_j, z_j, h)$ then $\bar{f}_T(1-s_j, z_j, h)$ becomes $\bar{f}_{T,0}(1-s_j, z_j, h)$ where

$$\bar{f}_{T,0}(t,z,h) = T^{-1/2}\int\int_0^t p(r,(z-y)/\sqrt{T})h(z)h(y)\,dy\,dr, \tag{5.15}$$

but also each factor in the integral in (5.13) of the form

$$p(t_{a_f}-t_{a_i}, z_j - x_{a_i})$$

is changed to

$$p(t_{a_f}-t_{a_i}, z_j/\sqrt{T} - x_{a_i}). \tag{5.16}$$

It follows easily from (3.59)–(3.62) and the bounds of this section that the error introduced in replacing (5.16) by

$$p(t_{a_f}-t_{a_i}, -x_{a_i}) \tag{5.17}$$

can be bounded by a factor which goes to 0 as $T\to\infty$. Thus we can can remove all z_j's from the inner integral in (5.13) and study

$$\int \bar{f}_{T,0}(t,z,h)\,dz = T^{-1/2}\int_0^t\int\int p(r,(z-y)/\sqrt{T})h(z)h(y)\,dz\,dy\,dr. \tag{5.18}$$

Using the fact that $p(r,x-y)$ is positive definite [see (4.14)], we can see that $\int \bar{f}_{T,0}(t,z,h)\,dz$ is positive and monotone increasing in t. Using (5.3) and (5.4) to get the estimate

$$|\hat{h}(p)| \le |p|\|\nabla\hat{h}\|_\infty \le c|p|\int |x||h(x)|\,dx \le c|p|, \tag{5.19}$$

where $\hat{h}$ denotes the Fourier transform of h, we see that

$$\begin{aligned} T^{-1/2}&\int_{T^{-1/2}}^\infty\int\int p(s,(z-y)/\sqrt{T})h(z)h(y)\,dz\,dy\,dr \\ &= T^{-1/2}\int\left(\int_{T^{-1/2}}^\infty |\hat{h}(p/\sqrt{T})|^2\exp(-r|p|^2/2)\,dr\right)d^3p \\ &= 2T^{-1/2}\int\frac{|\hat{h}(p/\sqrt{T})|^2}{|p|^2}\exp(-|p|^2/(2T^{1/2}))\,d^3p \\ &\le cT^{-3/2}\int\exp(-|p|^2/(2T^{1/2}))\,d^3p \\ &= O(T^{-3/4}) \end{aligned} \tag{5.20}$$

so that arguing as in Sections 3 and 4 we see that for each $G\in\mathscr{G}_{i,0}^{2m}$ we have

$$\lim_{T\to\infty}\frac{g(G,T,h)}{T^{(2m)/4+1/2}} = (\rho(h))^i(\sigma(h))^{m-i}c(G_0,1), \tag{5.21}$$

where

$$\begin{aligned}\rho(h) &= \lim_{t\to\infty}\int \bar f_{T,0}(t,z,h)\,dz\\ &= T^{-1/2}\int_0^\infty\int\int p(r,(z-y)/\sqrt{T})h(z)h(y)\,dz\,dy\,dr\\ &= T^{-1/2}\int\int u^0((z-y)/\sqrt{T})h(z)h(y)\,dz\,dy \qquad (5.22)\\ &= T^{-1/2}\int\int \frac{1}{2\pi}|(z-y)/\sqrt{T}|^{-1}h(z)h(y)\,dz\,dy\\ &= \frac{1}{2\pi}\int\int |z-y|^{-1}\,h(z)h(y)\,dz\,dy.\end{aligned}$$

Since for each $G_0\in\mathscr{G}_m$ there are $\binom{m}{i}$ graphs in $\mathscr{G}^{2m}_{i,0}$ for which G_0 is the corressponding graph of pre-exits (we choose i exits of G_0 to be distinguished vertices and the remaining $m-i$ exits of G_0 will be pre-exits for twinned exits), by (5.21) the contribution of all such graphs will be

$$\sum_{i=0}^m\binom{m}{i}(\rho(h))^i(\sigma(h))^{m-i}c(G_0,1)=(\rho(h)+\sigma(h))^m c(G_0,1),$$

and this will complete the proof of Theorem 11 as soon as we explain how to adapt the methods used in the proof of Theorem 8 to get the analogue of (3.19).

Before doing this, we explain the main technical difficulty in trying to adapt the methods used in the proof of Theorem 8. In the very first inequality of (3.54) we bounded the factors in the frame decomposition by positive factors. The expression $q_{T,y}$ is itself positive, while $\widetilde{q}_{T,y}$ was bounded by its absolute value $|\widetilde{q}_{T,y}|$. Since each factor of $|\widetilde{q}_{T,y}|$ gives rise to a small error term which goes to zero as $T\to\infty$, nothing was lost in replacing $\widetilde{q}_{T,y}$ by $|\widetilde{q}_{T,y}|$. However, in the present situation, whenever there are distinguished pre-exits, we have factors $\bar f_T$ which are not positive, but which give rise to contributions $\bar f_{T,0}$ which are nonzero in the limit as $T\to\infty$. Replacing $\bar f_T$ by $|\bar f_T|$ would increase that contribution, so that we would never be able to prove convergence all the way up to $\theta<\theta_{bc,h}$. On the other hand, the methods used in the proof of Theorem 8 were based on working with positive factors. The somewhat complicated approach outlined below is designed to cope with this difficulty.

For each distinguished pre-exit with coordinates (r,y), let (r',y') denote the coordinates of its successor, which is an exit, and (t,z) the coordinates of its predecessor. Using

$$p(r-t,y/\sqrt{T}-z)=p(r-t,-z)+\{p(r-t,y/\sqrt{T}-z)-p(r-t,-z)\}$$

we then rewrite

$$\begin{aligned}&T^{-1/2}\int p(r-t,y-z)h_T(y)p(r'-r,y'-y)h_T(y')\,dr'\,dy'\,dr\,dy\\ &\quad= T^{-1/2}\int p(r-t,y/\sqrt{T}-z) \qquad (5.23)\\ &\qquad\times h(y)p(r'-r,(y'-y)/\sqrt{T})h(y')\,dr'\,dy'\,dr\,dy\end{aligned}$$

as

$$\int p(r-t,-z)\int q^*_{T,h}(r,y)\,dy\,dr + \bar f^*_{T,h}(t,z), \tag{5.24}$$

where

$$q^*_{T,h}(r,y) = T^{-1/2}h(y)\int p(r'-r,(y'-y)/\sqrt{T})h(y')\,dr'\,dy' \tag{5.25}$$

and

$$\begin{aligned}\bar f^*_{T,h}(t,z) = T^{-1/2}\int &\{p(r-t,y/\sqrt{T}-z) - p(r-t,-z)\}\\ &\times h(y)p(r'-r,(y'-y)/\sqrt{T})h(y')\,dr'\,dy'\,dr\,dy.\end{aligned} \tag{5.26}$$

We note that $\int q^*_{T,h}(r,y)\,dy \ge 0$ for any r, and

$$Q^*_{T,h} \stackrel{\text{def}}{=} \sup_r \int q^*_{T,h}(r,y)\,dy \le \rho(h). \tag{5.27}$$

See (5.22). Set

$$\widetilde{q}^*_{T,h}(t,z,s,x) = p(s-t,x-z)\bar f^*_{T,h}(t,z).$$

We have

$$\sup_x \int |\widetilde{q}^*_{T,h}(t,z,s,x)|\,dz \le \widetilde{Q}^*_{T,h}\|p(s-t,\cdot)\|_{5/2}, \tag{5.28}$$

where [see (5.22), (3.53) and (3.60)]

$$\begin{aligned}\widetilde{Q}^*_{T,h} &\stackrel{\text{def}}{=} \sup_{0\le t\le 1}\|\bar f^*_{T,h}(t,\cdot)\|_{5/3}\\ &\le \frac{1}{2\pi}\int \sup_{0\le t\le 1}\left\|\int_0^t (p(r,y/\sqrt{T}-\cdot) - p(r,-\cdot))\,dr\right\|_{5/3}\\ &\qquad\times |y-y'|^{-1}|h(y)|\,|h(y')|\,dy\,dy'\\ &\le cT^{-3/8}\int |y|^{3/4}|y-y'|^{-1}|h(y)|\,|h(y')|\,dy\,dy'\\ &\le cT^{-3/8}.\end{aligned} \tag{5.29}$$

We shall refer to the complex which consists of a distinguished pre-exit, the exit which is its successor and the arrow connecting them, as a $*$-exit. The predecessor of a $*$-exit will be called a pre$*$-exit. In the previous paragraph, the pre$*$-exit was the vertex with coordinates (t,z). With the above notation, each $*$-exit contributes a sum of two factors, one which involves $q^*_{T,h}$ and the other $\widetilde{q}^*_{T,h}$. In comparison with the proof of Theorem 1, $q^*_{T,h}$ should be associated with the sort of analysis attached to twinned exits, while $\widetilde{q}^*_{T,h}$ should be associated with the sort of analysis attached to untwinned exits. These different exits naturally lead to different frame decompositions. In order to deal with the combinatorics in a systematic way, we will create new "mirror image" graphs for each $*$-exit. Here are the details.

Let $\mathscr{G}^{*,n}_i$ denote the set of decorated $\mathscr{G}^n_i$ graphs. A decorated $\mathscr{G}^n_i$ graph consists of a graph $G \in \mathscr{G}^n_i$ together with a decoration in which each $*$-exit is assigned a label 0 or 1, in which case the $*$-exit, as well as the associated pre$*$-exit, will be said to be of type 0 or 1. Thus, if $G \in \mathscr{G}_i^n$ has k $*$-exits, there will be 2^k graphs in $\mathscr{G}_i^{*,n}$ corresponding to the various possible decorations of G. The set of all possible decorations of $G \in \mathscr{G}_i^n$ will be denoted $L(G)$. Let $G^* \in \mathscr{G}_i^{*,n}$. We define $F(G^*)$ in a manner similar to the way we defined $F(G)$, except that in addition to all vertices in $PU(G)$, we also include all distinguished vertices which are not pre-exits, as well as all pre$*$-exits of type 1. In defining G_v^*, each $*$-exit will be considered as a single vertex; G_v^* is defined analogously to G_v except that G_v^* has five possible types of exits. In addition to $D(G_v^*)$ and $S(G_v^*)$, we also have $D_0(G_v^*)$, the set of $*$-exits of type 0, $S_1(G_v^*)$, the set of pre$*$-exits of type 1, and $S_2(G_v^*)$, the set of distinguished vertices which are not pre-exits. In analogy with the frame decomposition (3.48) we have

$$d(G,1,h_T) = \sum_{\{G^* \in L(G)\}} d(G^*,1,h_T), \tag{5.30}$$

where

$$\begin{aligned}
&T^{|E(G)|/4} d(G^*,1,h_T)\\
&\quad = \int\int p(t_r, x_r - w)\\
&\qquad \times \prod_{v \in F(G^*)} \Bigg(\int \widetilde{c}(G_v^*, x_{r_v}, t_{r_v}; (z_e^*, T_e),\ e \in E(G_v^*))\\
&\qquad\qquad \times \prod_{e \in D(G_v^*)} q_{T,h}(T_e, z_e) \prod_{e \in S(G_v^*)} \widetilde{q}_{T,h}(T_e, z_e, t_{r_e}, x_{r_e})\\
&\qquad\qquad \times \prod_{e \in D_0(G_v^*)} q^*_{T,h}(T_e, z_e) \prod_{e \in S_1(G_v^*)} \widetilde{q}^*_{T,h}(T_e, z_e, t_{r_e}, x_{r_e})\\
&\qquad\qquad\qquad \times \prod_{e \in S_2(G_v^*)} \check{q}_{T,h}(T_e, z_e, t_{r_e}, x_{r_e})\, dT_e\, dz_e \Bigg)\\
&\qquad \times dt_{r_v}\, dx_{r_v}\, dw.
\end{aligned} \tag{5.31}$$

Here $z_e^* = 0$ if $e \in D_0(G_v^*)$ and $z_e^* = z_e$ otherwise, to take (5.24) into account, and

$$\check{q}_{T,h}(t,z,s,x) = T^{-3/4} h_T(z) p(s-t, x-z). \tag{5.32}$$

In analogy with (3.40) we see that

$$\begin{aligned}
\check{Q}_{T,h} &\overset{\text{def}}{=} \sup_{t,x} \int\int |\check{q}_{T,h}(t,z,s,x)|\, ds\, dz\\
&\le cT^{-3/4} \int u^1(x-z)\, |h_T(z)|\, dz\\
&\le cT^{-3/4} \int \frac{1}{|x-z|} |h_T(z)|\, dz\\
&= cT^{-1/4} \int \frac{1}{|z - \sqrt{T}x|} |h(z)|\, dz\\
&\le cT^{-1/4}.
\end{aligned} \tag{5.33}$$

Note that $e \in S(G_v^*) \cap S_1(G_v^*)$ is possible. In this case we can assign e arbitrarily to either $S(G_v^*)$ or $S_1(G_v^*)$. Since either case leads to an arbitrarily small contribution, the choice will not be important.

In view of all these bounds, it is now a straightforward matter to proceed along the path mapped out in the proof of Theorem 8 and establish the needed analogue of (3.19). We only wish to point out that the multiplicity of graphs introduced by "decoration" is easily controlled, since all decorations of type 1 can be controlled using (5.28) systematically, as we did for untwinned exits in the proof of Theorem 8. This completes the proof of our theorem. □

6. Superprocesses and critical branching processes over stable processes. In this section we show how the methods developed in previous sections to study large deviations for superprocesses and critical branching processes based on Brownian motion in R^3 can be easily adapted to prove analogous results for superprocesses and critical branching processes based on the symmeteric stable process of index β in R^d when $d < 2\beta < 2 + d$. We first state the analogue of Theorem 8.

Let L_t^x denote the local time of the symmetric superstable process of index β in R^d, with $d < 2\beta < 2 + d$, and let μ denote the Lebesgue measure on R^d. As in the Introduction, set

$$\Lambda_{\beta,d}(\theta) = \log E_\mu(\exp(\theta L_1^0)). \tag{6.1}$$

Following [10], we can show the existence of $0 < \theta^*_{\beta,d} < \infty$ such that $\Lambda_{\beta,d}(\theta) < \infty$ for $\theta < \theta^*_{\beta,d}$ and $\lim_{\theta \uparrow \theta^*_{\beta,d}}((d\Lambda_{\beta,d}(\theta))/d\theta) = \infty$. Let $\theta_{\beta,d,c}$ be defined by

$$\theta^*_{\beta,d} = 2c_{\beta,d}|y|^{2\beta-d}\theta^2_{\beta,d,c},$$

where

$$c_{\beta,d} = \int_0^\infty s(p(s,0) - p(s,u))\,ds < \infty, \tag{6.2}$$

and $p(s, y)$ denotes the transition density for the symmetric stable process of index β in R^d and $u \in R^d$ is an arbitrary unit vector.

THEOREM 12. *Let L_t^x denote the local time of the symmetric superstable process of index β in R^d, with $d < 2\beta < 2 + d$. Let μ denote the Lebesgue measure on R^d. Then*

$$\lim_{T\to\infty} \frac{1}{T^{d/\beta-1}} \log E_\mu\left(\exp\left(\frac{\theta}{T^{1-d/(2\beta)}}(L_T^0 - L_T^y)\right)\right) = \Lambda_{\beta,d}(2c_{\beta,d}|y|^{2\beta-d}\theta^2) \tag{6.3}$$

for all $\theta < \theta_{\beta,d,c}$.

We will indicate briefly the necessary modifications in the proofs of previous sections needed to obtain the proof of Theorem 12.

Let $c_{\beta,d}(G,T)$, $d_{\beta,d}(G,T,y)$ be defined just as we defined $c(G,T)$, $d(G,T,y)$ except that now $p(s,x)$ denotes the transition density for the symmetric stable process of index β in R^d. We have

$$\log E_\mu(\exp(\theta L_T^0)) = \sum_{n=1}^{\infty} \theta^n c_{n,\beta,d}(T) \tag{6.4}$$

with

$$c_{n,\beta,d}(T) = \sum_{\mathscr{G}_n} c_{\beta,d}(G,T) \tag{6.5}$$

and

$$\begin{aligned} &\frac{1}{T^{d/\beta-1}} \log E_\mu\left(\exp\left(\frac{\theta}{T^{1-d/(2\beta)}}(L_T^0 - L_T^y)\right)\right) \\ &\qquad = \sum_{n=1}^{\infty} \frac{\theta^n}{T^{(1-d/(2\beta))n+d/\beta-1}} d_{n,\beta,d}(T,y), \end{aligned} \tag{6.6}$$

where

$$d_{n,\beta,d}(T,y) = \sum_{\mathscr{G}_n} d_{\beta,d}(G,T,y). \tag{6.7}$$

The scaling

$$p(sT,x) = T^{-d/\beta} p(s, x/T^{1/\beta}) \tag{6.8}$$

leads to

$$\begin{aligned} c_{\beta,d}(G,T) &= T^{|G|} T^{-d/\beta(|A_G|+1)} T^{d/\beta(|G|-n+1)} c_{\beta,d}(G,1) \\ &= T^{2n-1} T^{-d/\beta(2n-1)} T^{d/\beta n} c_{\beta,d}(G,1) \\ &= T^{(2-d/\beta)n+d/\beta-1} c_{\beta,d}(G,1) \end{aligned} \tag{6.9}$$

and

$$\frac{d_{\beta,d}(G,T,y)}{T^{(1-d/(2\beta))n+d/\beta-1}} = T^{(1-d/(2\beta))n} d_{\beta,d}(G,1,y/T^{1/\beta}). \tag{6.10}$$

With

$$f_T(t,z,y) \stackrel{\text{def}}{=} T^{2-d/\beta}\left(\int_0^t (p(r,z) - p(r, z-y/T^{1/\beta}))\,dr\right)^2, \tag{6.11}$$

we first note that as in (3.25) and using the scaling (6.8) we can compute

$$\begin{aligned} \int f_T(\infty,z,y)\,dz &= 2T^{2-d/\beta} \int_0^\infty \int_0^\infty (p(r+s,0) - p(r+s, y/T^{1/\beta}))\,dr\,ds \\ &= 2T^{2-d/\beta} \int_0^\infty s(p(s,0) - p(s, y/T^{1/\beta}))\,ds \\ &= 2c_{\beta,d}|y|^{2\beta-d}. \end{aligned} \tag{6.12}$$

We also have the analogue of (3.32),

$$
\begin{aligned}
T^{2-d/\beta} &\int_{T^{-1/2}}^{\infty}\int_{0}^{\infty}(p(r,z)-p(r,z-y/T^{1/\beta}))\\
&\qquad\times(p(s,z)-p(s,z-y/T^{1/\beta}))\,dr\,ds\,dz\\
&= T^{2-d/\beta}\int_{T^{-1/2}}^{\infty}\int_{0}^{\infty}\big(2p(r+s,0)-p(r+s,y/T^{1/\beta})\\
&\qquad\qquad -p(r+s,-y/T^{1/\beta})\big)\,dr\,ds\\
&= 2T^{2-d/\beta}\int\bigg(\int_{T^{-1/2}}^{\infty}\int_{0}^{\infty}(1-\cos(py/T^{1/\beta}))\\
&\qquad\qquad\times\exp(-(r+s)|p|^{\beta})\,dr\,ds\bigg)\,d^d p\\
&= 2T^{2-d/\beta}\int\frac{(1-\cos(py/T^{1/\beta}))}{|p|^{2\beta}}\exp(-|p|^{\beta}/(T^{1/2}))\,d^d p\\
&\le |y|^2\,T^{2-d/\beta-2/\beta}\int\frac{1}{|p|^{2\beta-2}}\exp(-|p|^{\beta}/(T^{1/2}))\,d^d p\\
&= c|y|^2\,T^{-1/(2\beta)(2+d-2\beta)}.
\end{aligned}
\tag{6.13}
$$

The last integral is finite because of our condition $2\beta < 2+d$.

It remains to provide the analogues of (3.53) and (3.40). If q, q' denote conjugate indices, so that by $1/q+1/q'=1$, then by Hölder's inequality we have

$$
\begin{aligned}
T^{1-d/(2\beta)}\sup_x &\int p(s-t,x-z)\\
&\times\left|\int_0^1(p(b-t,z)-p(b-t,z-y/T^{1/\beta}))\,db\right|dz\\
&\le \widetilde{Q}_{\beta,d,T,y}\|p(s-t,\cdot)\|_q,
\end{aligned}
\tag{6.14}
$$

where, as in (3.23),

$$
\begin{aligned}
\widetilde{Q}_{\beta,d,T,y} &= T^{1-d/(2\beta)}\sup_{0\le t\le 1}\left\|\int_0^t(p(b,\cdot)-p(b,\cdot-y/T^{1/\beta}))\,db\right\|_{q'}\\
&\le T^{1-d/(2\beta)}\|u^1(x)-u^1(x-y/T^{1/\beta}/)|\|_{q'}.
\end{aligned}
\tag{6.15}
$$

We first note by scaling that

$$
\begin{aligned}
\|p_t(\cdot)\|_q &= t^{-d/\beta}\left(\int p_1(x/t^{1/\beta})\,dx\right)^{1/q}\\
&= t^{-d/\beta(1-1/q)}\|p_1(\cdot)\|_q.
\end{aligned}
\tag{6.16}
$$

The strict inequality in our condition $2\beta > d$ then implies that

$$
\int_0^1\|p_t(\cdot)\|_q\,dt < \infty
\tag{6.17}
$$

for some $q > 2$.

Since, then, $q' < 2$ we can choose $\varepsilon > 0$ such that $q'(1+2\varepsilon) < 2$. Since $\beta - d/2 < 1$ we can choose ε so small that also $\beta - d/2 + \varepsilon < 1$. We claim that

$$\|u^1(x) - u^1(x - y/T^{1/\beta}/)|\|_{q'} \le cT^{-(1-d/(2\beta)+\varepsilon/\beta)}|y|^{\beta-d/2+\varepsilon}, \tag{6.18}$$

which will give us the analogue of (3.40).

For the last inequality we use the analogue of (3.61),

$$|u^1(x) - u^1(x - y/T^{1/\beta})| \le \frac{|y|}{T^{1/\beta}}\left(\frac{u^1(x)}{|x|} + \frac{u^1(x - y/T^{1/\beta})}{|x - y/T^{1/\beta}|}\right), \tag{6.19}$$

to see by interpolating that

$$\begin{aligned} &|u^1(x) - u^1(x - y/T^{1/\beta})| \\ &\qquad \le \frac{|y|^{\beta-d/2+\varepsilon}}{T^{(\beta-d/2+\varepsilon)/\beta}}\left(\frac{u^1(x)}{|x|^{\beta-d/2+\varepsilon}} + \frac{u^1(x - y/T^{1/\beta})}{|x - y/T^{1/\beta}|^{\beta-d/2+\varepsilon}}\right) \end{aligned} \tag{6.20}$$

and $u^1(x)|x|^{-(\beta-d/2+\varepsilon)} \in L^{q'}$ as long as $q'(d - \beta + (\beta - d/2 + \varepsilon)) < d$ which follows from our condition that $q'(1+2\varepsilon) < 2$.

This completes our proof of Theorem 12. $\square$

Similarly we can show the following analogue of Theorem 10: if ν is a signed measure on R^d, let

$$\sigma_{\beta,d}(\nu) = -c_{\beta,d}\int\int |x-y|^{2\beta-d}\, d\nu(x)\, d\nu(y). \tag{6.21}$$

With the notation of Theorem 12, let $\theta_{\beta,d,c,\nu}$ be defined by

$$\theta^*_{\beta,d} = \sigma(\nu)\theta^2_{\beta,d,c,\nu}.$$

THEOREM 13. *Let X_t denote the symmetric superstable process of index β in R^d, with $d < 2\beta < 2 + d$, and let μ denote the Lebesgue measure on R^d. If ν is a signed measure on R^d with*

$$\int_{R^d} (1+|x|)\, d|\nu|(x) < \infty \tag{6.22}$$

and

$$\int_{R^d} d\nu(x) = 0 \tag{6.23}$$

then with the notation of Theorem 12,

$$\lim_{T\to\infty} \frac{1}{T^{d/\beta-1}} \log E_\mu\left(\exp\left(\frac{\theta}{T^{1-d/(2\beta)}}\int L^y_T\, d\nu(y)\right)\right) = \Lambda_{\beta,d}(\sigma(\nu)\theta^2) \tag{6.24}$$

for all $\theta < \theta_{\beta,d,c,\nu}$.

Here is our theorem for critical branching stable processes. As in (6.21), let

$$\sigma_{\beta,d}(h) = -c_{\beta,d} \int\int |x-y|^{2\beta-d}\, h(x)h(y)\, dx\, dy. \tag{6.25}$$

Let

$$\rho_{\beta,d}(h) = \frac{\Gamma((d-\beta)/2)}{2^{\beta}\pi^{d/2}\Gamma(\beta/2)} \int\int |x-y|^{-(d-\beta)}\, h(x)h(y)\, dx\, dy. \tag{6.26}$$

With the notation of Theorem 12, let $\theta_{\beta,d,cb,h}$ be defined by

$$\theta^*_{\beta,d} = (\sigma_{\beta,d}(h) + \rho_{\beta,\ d}(h))\theta^2_{\beta,d,cb,h}.$$

THEOREM 14. *Consider the critical branching symmetric stable process of index β in R^d, with $d < 2\beta < 2+d$, and let μ denote Lebesgue measure on R^d. Let h be a bounded function on R^d with*

$$\int_{R^d} (1+|x|)|h(x)|\, dx < \infty \tag{6.27}$$

and

$$\int_{R^d} h(x)\, dx = 0. \tag{6.28}$$

Then with the notation of Theorem 12,

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T^{d/\beta-1}} \log E\left(\exp\left(\frac{\theta}{T^{1-d/(2\beta)}} \int_0^T N_s(h)\, ds\right)\right) \\ = \Lambda_{\beta,d}((\sigma_{\beta,d}(h)+\rho_{\beta,d}(h))\theta^2)\end{aligned} \tag{6.29}$$

for all $\theta < \theta_{\beta,d,bc,h}$.

This follows as above once we note the analogue of (5.22):

$$\begin{aligned}\rho(h) &= T^{1-d/\beta} \int_0^\infty \int\int p(r,(z-y)/T^{1/\beta}) h(z)h(y)\, dz\, dy\, dr \\ &= T^{1-d/\beta} \int\int u^0((z-y)/T^{1/\beta}) h(z)h(y)\, dz\, dy \\ &= T^{1-d/\beta} \frac{\Gamma((d-\beta)/2)}{2^{\beta}\pi^{d/2}\Gamma(\beta/2)} \int\int |(z-y)/T^{1/\beta}|^{-(d-\beta)} h(z)h(y)\, dz\, dy \\ &= \frac{\Gamma((d-\beta)/2)}{2^{\beta}\pi^{d/2}\Gamma(\beta/2)} \int\int |z-y|^{-(d-\beta)}\, h(z)h(y)\, dz\, dy.\end{aligned} \tag{6.30}$$

Finally, we mention that the proof of Theorem 4 follows along the lines of the proof of Theorem 3.

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