

RENEWAL THEORY FOR EMBEDDED REGENERATIVE SETS

BY JEAN BERTOIN

Université Pierre et Marie Curie

We consider the age processes $A^{(1)} \geq \dots \geq A^{(n)}$ associated to a monotone sequence $\mathcal{R}^{(1)} \subseteq \dots \subseteq \mathcal{R}^{(n)}$ of regenerative sets. We obtain limit theorems in distribution for $(A_t^{(1)}, \dots, A_t^{(n)})$ and for $((1/t)A_t^{(1)}, \dots, (1/t)A_t^{(n)})$, which correspond to multivariate versions of the renewal theorem and of the Dynkin–Lamperti theorem, respectively. Dirichlet distributions play a key role in the latter.

1. Introduction. The range \mathcal{R} of a renewal process (i.e., of an increasing random walk) forms a discrete regenerative set which can be viewed as the set of renewal epochs of some recurrent event in the sense of Feller [4, 5]. From the point of view of the present work, renewal theory is concerned with the asymptotic behavior as time goes to ∞ of the so-called age process, $A_t = \inf\{s \geq 0: t - s \in \mathcal{R}\}$. The main results in this field are the renewal theorem, which is the key to the limit theorem for A_t in the case when the renewal process has a finite mean, and the Dynkin–Lamperti theorem, which reveals the role of generalized arcsine laws in the study of the normalized age $(1/t)A_t$.

The purpose of this work is to develop a multivariate renewal theory for a monotone sequence of regenerative sets,

$$\mathcal{R}^{(1)} \subseteq \dots \subseteq \mathcal{R}^{(n)},$$

where the embedding is compatible with the regenerative property (the meaning of “compatible” will be made precise in the next section). Specifically, we will give an explicit formula for the joint Laplace transform of the age processes $A^{(1)}, \dots, A^{(n)}$ and present a multivariate extension of the renewal theorem and of the Dynkin–Lamperti theorem. In the latter, the generalized arcsine (i.e., beta) law which appears as a limit in the standard situation is replaced by a Dirichlet distribution in the multivariate version.

A noticeable feature in our results is that they depend only on the individual distributions of the regenerative sets $\mathcal{R}^{(1)}, \dots, \mathcal{R}^{(n)}$ and not on their joint distribution as one might have expected. In particular, they do not seem to be directly related to the multidimensional version of the renewal theorem in [8], as the latter requires the knowledge of the joint dynamic of the age processes. For the sake of completeness, we refer the reader to [7] and the references therein for multidimensional extensions of the renewal theorem in a different vein.

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The next section contains the precise statements and proofs in this discrete setting. Section 3 deals with the extension to continuous (i.e., nondiscrete) regenerative sets.

2. Discrete setting.

2.1. *Preliminary.* We first recall some basic features in regenerative sets and introduce relevant notation. For the sake of simplicity, we implicitly only discuss unbounded regenerative sets, as bounded regenerative sets are not relevant to the scope of this work.

A closed unbounded random set $\mathcal{R} \subseteq [0, \infty)$ is called a regenerative set if it fulfills the regenerative property. That is, if $(\mathcal{F}_t)_{t \geq 0}$ denotes the filtration induced by the characteristic function $\mathbf{1}_{\mathcal{R}}$, then for every (\mathcal{F}_t) -stopping time T such that $T \in \mathcal{R}$ a.s., the right-hand portion of \mathcal{R} as viewed from T ,

$$\mathcal{R} \circ \theta_T = \{s \geq 0 : s + T \in \mathcal{R}\},$$

is independent of \mathcal{F}_T and has the same law as \mathcal{R} .

When the regenerative set \mathcal{R} is discrete, which we assume from now on throughout this section, there exists a unique renewal process $S = (S_0, S_1, \dots)$ such that $\mathcal{R} = \{S_k : k \in \mathbb{N}\}$. More precisely, $S_k = \xi_1 + \dots + \xi_k$ where the i.i.d. random variables ξ_1, \dots correspond to the waiting times between successive occurrences of \mathcal{R} (for the sake of simplicity, we shall write ξ for a random variable distributed as ξ_i). The distribution of \mathcal{R} is then characterized by the function

$$\Phi(q) = \mathbb{E}(1 - \exp(-qS_1)) = \mathbb{E}(1 - e^{-q\xi}), \quad q \geq 0.$$

The renewal measure associated with \mathcal{R} is given by

$$U(B) = \mathbb{E}\left(\sum_{t \in \mathcal{R}} \mathbf{1}_{\{t \in B\}}\right) = \sum_{k=0}^{\infty} \mathbb{P}(S_k \in B), \quad B \in \mathcal{B}([0, \infty));$$

its Laplace transform is simply

$$\int_0^{\infty} e^{-qx} U(dx) = \frac{1}{\Phi(q)}, \quad q > 0.$$

The age process

$$A_t = \inf\{s \geq 0 : t - s \in \mathcal{R}\}, \quad t \geq 0$$

is a strong Markov process which vanishes exactly on \mathcal{R} . Its one-dimensional distribution is given by

$$(1) \quad \mathbb{P}(t - A_t \in dx) = \mathbb{P}(\xi > t - x)U(dx), \quad 0 \leq x \leq t.$$

When the renewal process is nonlattice (i.e., there is no $r > 0$ such that $\mathcal{R} \subseteq r\mathbb{N}$ with probability 1) and has finite mean $m = \mathbb{E}(\xi)$, the standard renewal theorem combined with (1) yields that A_t converges in law as $t \rightarrow \infty$ towards the probability measure $m^{-1}\mathbb{P}(\xi > x) dx$. In the infinite mean case, $\mathbb{E}(\xi) = \infty$, there is a limit theorem for the normalized age $t^{-1}A_t$ provided that

the step distribution ξ belongs to the domain of attraction of some stable law. More precisely, Tauberian theorems and (1) are the key to the Dynkin–Lamperti theorem, which roughly states that $(1/t)A_t$ converges in distribution if and only if its mean converges. We stress that the renewal theorem and the Dynkin–Lamperti theorem are the only nondegenerate limit theorems for the age process; in particular, one would get nothing interesting by normalizing the age process by other powers of t .

Now consider a sequence of $n \geq 2$ embedded regenerative sets,

$$(2) \quad \mathcal{R}^{(1)} \subseteq \dots \subseteq \mathcal{R}^{(n)}.$$

Notation using superscripts such as $\Phi^{(i)}$ should be clear from the context. For every $i = 1, \dots, n$, we introduce $(\mathcal{F}_t^{(i)})_{t \geq 0}$, the filtration generated by the characteristic functions $\mathbf{1}_{\mathcal{R}^{(i)}}, \dots, \mathbf{1}_{\mathcal{R}^{(n)}}$, after the usual completions.

DEFINITION. We say that the embedding (2) is compatible with the regenerative property if for every $i = 1, \dots, n$ and every $(\mathcal{F}_t^{(i)})$ -stopping time T with $T \in \mathcal{R}^{(i)}$ a.s., the shifted sets

$$\mathcal{R}^{(j)} \circ \theta_T = \{s \geq 0 : s + T \in \mathcal{R}^{(j)}\}, \quad j = i, \dots, n$$

are jointly independent of $\mathcal{F}_T^{(i)}$ and have jointly the same law as $\mathcal{R}^{(i)}, \dots, \mathcal{R}^{(n)}$.

This notion has been introduced for $n = 2$ (in the more general continuous setting) in [1] where it is called “regenerative embedding.” Let us present a few examples in which (2) holds and is compatible with the regenerative property, and refer to [1] for more.

1. Consider a monotone sequence $\mathcal{L}^{(1)} \subseteq \dots \subseteq \mathcal{L}^{(n)}$ of sublattices of \mathbb{Z}^d and a random walk $W = (W_k, k \in \mathbb{N})$ valued in \mathbb{Z}^d . Then take

$$\mathcal{R}^{(i)} = \{k \in \mathbb{N} : W_k \in \mathcal{L}^{(i)}\}.$$

2. Suppose that $\tilde{S}^{(1)}, \dots, \tilde{S}^{(n)}$ are n independent integer valued renewal processes, and consider the compound processes $S^{(i)} = \tilde{S}^{(n)} \circ \dots \circ \tilde{S}^{(i)}, i = 1, \dots, n$. Then each $S^{(i)}$ is a renewal process (this is a discrete version of Bochner’s subordination) and then we can take $\mathcal{R}^{(i)} = \{S_n^{(i)}, n \in \mathbb{N}\}$.
3. Suppose that $\tilde{\mathcal{R}}^{(1)}, \dots, \tilde{\mathcal{R}}^{(n)}$ are independent regenerative sets, and set $\mathcal{R}^{(i)} = \tilde{\mathcal{R}}^{(i)} \cap \dots \cap \tilde{\mathcal{R}}^{(n)}$ for $i = 1, \dots, n$.
4. Consider $(W^{(1)}, \dots, W^{(n)})$, an n -dimensional random walk such that for $j = 1, \dots, n - 1$, the real-valued random walk $W^{(j+1)} - W^{(j)}$ increases (i.e., is a renewal process). Then take for $\mathcal{R}^{(i)}$ the set of ascending ladder epochs of $W^{(i)}$, that is, the set of times when $W^{(i)}$ reaches a new maximum.

More generally, given age processes $A^{(1)}, \dots, A^{(n)}$, (2) holds if $A^{(n)} \leq \dots \leq A^{(1)}$, and is compatible with the regenerative property if the $(n + 1 - i)$ -tuple $(A^{(i)}, \dots, A^{(n)})$ is Markovian for each $i = 1, \dots, n$. We point out that the condition that the age processes are jointly Markovian appears in [8]. On the other hand, it is easy to construct examples where (2) holds and is compat-

ible with the regenerative property though the age processes are not jointly Markovian. See, for instance, the counter-example after the proof of Lemma 1.

We assume from now on that the embedding (2) is compatible with the regenerative property. It is clear that this assumption imposes some restriction on the functions $\Phi^{(1)}, \dots, \Phi^{(n)}$, and more precisely, one can state the following lemma.

LEMMA 1. *For every $i = 1, \dots, n - 1$, the ratios $\Phi^{(i)}/\Phi^{(i+1)}$ are completely monotone functions.*

We stress that, conversely, if the ratio $\Phi^{(1)}/\Phi^{(2)}, \dots, \Phi^{(n-1)}/\Phi^{(n)}$ are completely monotone, then there exists regenerative sets $\mathcal{R}^{(1)}, \dots, \mathcal{R}^{(n)}$ whose individual distributions are characterized by the functions $\Phi^{(1)}, \dots, \Phi^{(n)}$, respectively, such that (2) holds and is compatible with the regenerative property. As we shall not need this feature in the sequel, we just refer to [1] for a complete argument.

PROOF OF LEMMA 1. This is essentially the direct part of Theorem 1 in [1]; for the sake of completeness we present here the proof in the discrete setting. We may suppose that $n = 2$. Fix an arbitrary $q > 0$ and recall that

$$\frac{1}{\Phi^{(2)}(q)} = \int_{[0, \infty)} e^{-qx} U^{(2)}(dx) = \mathbb{E}\left(\sum_{t \in \mathcal{R}^{(2)}} e^{-qt}\right).$$

We introduce the partition generated by $\mathcal{R}^{(1)}$, so that the right-hand side can be expressed as

$$\sum_{k=0}^{\infty} \mathbb{E}\left(\sum_{t \in \mathcal{R}^{(2)}} \mathbf{1}_{\{S_k^{(1)} \leq t < S_{k+1}^{(1)}\}} \exp(-qS_k^{(1)}) \exp(-q(t - S_k^{(1)}))\right).$$

We then apply the regenerative property at $S_k^{(1)}$ to get

$$\begin{aligned} & \mathbb{E}\left(\sum_{t \in \mathcal{R}^{(2)}} \mathbf{1}_{\{S_k^{(1)} \leq t < S_{k+1}^{(1)}\}} \exp(-qS_k^{(1)}) \exp(-q(t - S_k^{(1)}))\right) \\ &= \mathbb{E}\left(\sum_{t \in \mathcal{R}^{(2)} \circ \theta_{S_k^{(1)}}} e^{-qt} \mathbf{1}_{\{t < S_{k+1}^{(1)} - S_k^{(1)}\}} \exp(-qS_k^{(1)})\right) \\ &= \mathbb{E}(\exp(-qS_k^{(1)})) \mathbb{E}\left(\sum_{t \in \mathcal{R}^{(2)}} \mathbf{1}_{\{t < S_1^{(1)}\}} e^{-qt}\right). \end{aligned}$$

Taking the sum over $k \in \mathbb{N}$, we finally get

$$\frac{1}{\Phi^{(2)}(q)} = \frac{1}{\Phi^{(1)}(q)} \mathbb{E}\left(\sum_{t \in \mathcal{R}^{(2)}} \mathbf{1}_{\{t < S_1^{(1)}\}} e^{-qt}\right).$$

The ratio $\Phi^{(1)}/\Phi^{(2)}$ is thus the Laplace transform of some measure on $[0, \infty)$, which proves our claim. \square

At this point, it is also important to stress that the functions $\Phi^{(1)}, \dots, \Phi^{(n)}$ do not determine the joint law of $\mathcal{R}^{(1)}, \dots, \mathcal{R}^{(n)}$; let us present a simple counterexample. Suppose that the law of the generic step $\xi^{(2)}$ of $S^{(2)}$ is $\mathbb{P}(\xi^{(2)} = a) = \mathbb{P}(\xi^{(2)} = b) = 1/2$ for some $a \neq b$. Next, consider the smallest even integer $2k$ such that $\xi_{2k-1}^{(2)} = a$ and $\xi_{2k}^{(2)} = b$, and put $S_1^{(1)} = S_{2k}^{(2)}$. Iterating the construction in an obvious (i.e., regenerative) way, we obtain a renewal process $S^{(1)}$ with range $\mathcal{R}^{(1)} \subseteq \mathcal{R}^{(2)}$ where the embedding is compatible with the regenerative property. Plainly, the next to last and the last waiting times between occurrence times of $\mathcal{R}^{(2)}$ before an occurrence of $\mathcal{R}^{(1)}$ have respective durations a and b . Then do the same construction after exchanging the role of a and b . We get another regenerative set $\mathcal{R}^{(1')} \subseteq \mathcal{R}^{(2)}$, where the embedding is compatible with the regenerative property, but now the last-but-one and the last waiting times between occurrence times of $\mathcal{R}^{(2)}$ before an occurrence of $\mathcal{R}^{(1')}$ have respective durations b and a . The joint distributions of $(\mathcal{R}^{(1)}, \mathcal{R}^{(2)})$ and of $(\mathcal{R}^{(1')}, \mathcal{R}^{(2)})$ thus differ. Nonetheless, it is obvious that the variables $S_1^{(1)}$ and $S_1^{(1')}$ have the same law, so the functions $\Phi^{(1)}$ and $\Phi^{(1')}$ are identical.

Unfortunately, it seems there is no simple analogue of the identity (1) for the joint one-dimensional distribution for the age processes $A^{(1)}, \dots, A^{(n)}$. However, we point out that some joint Laplace transform has a simple expression, which is the key technical point of this study.

LEMMA 2. *For every $q, \lambda_1, \dots, \lambda_n \geq 0$, we have*

$$\begin{aligned} & \mathbb{E} \left(\int_0^\infty dt e^{-qt} \exp\{-\lambda_1(A_t^{(1)} - A_t^{(2)}) - \dots \right. \\ & \qquad \qquad \qquad \left. - \lambda_{n-1}(A_t^{(n-1)} - A_t^{(n)}) - \lambda_n A_t^{(n)}\} \right) \\ &= \frac{\Phi^{(1)}(q + \lambda_1)}{\Phi^{(1)}(q)} \frac{\Phi^{(2)}(q + \lambda_2)}{\Phi^{(2)}(q + \lambda_1)} \dots \frac{\Phi^{(n)}(q + \lambda_n)}{\Phi^{(n)}(q + \lambda_{n-1})} \frac{1}{q + \lambda_n}. \end{aligned}$$

PROOF. We first suppose that $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ and denote the left-hand side in the formula of the statement by $F_q(\lambda_1, \dots, \lambda_n)$. We decompose the integral over $[0, \infty)$ according to the regenerative set $\mathcal{R}^{(1)} = \{S_0^{(1)}, S_1^{(1)}, \dots\}$; that is, we write

$$\int_0^\infty dt e^{-qt} \dots = \sum_{k=0}^\infty \exp(-qS_k^{(1)}) \int_{S_k^{(1)}}^{S_{k+1}^{(1)}} dt \exp(-q(t - S_k^{(1)})) \dots.$$

In the sum, we apply the regenerative property at $S_k^{(1)}$. As $A^{(1)}, \dots, A^{(n)}$ all vanish at each $S_k^{(1)}$ and as $A^{(1)}$ grows linearly on the interval $[S_k^{(1)}, S_{k+1}^{(1)})$, we get

$$\begin{aligned} & F_q(\lambda_1, \dots, \lambda_n) \\ &= \mathbb{E} \left(\sum_{k=0}^\infty \exp(-qS_k^{(1)}) \right) \mathbb{E} \left(\int_0^{S_1^{(1)}} dt e^{-qt} \exp\{-\lambda_1(t - A_t^{(2)}) - \dots - \lambda_n A_t^{(n)}\} \right). \end{aligned}$$

The first term in the product equals $1/\Phi^{(1)}(q)$. Then observe that the second term in the product is identical to the one which appears in the same formula for $F_{q+\lambda_1}(0, \lambda_2 - \lambda_1, \dots, \lambda_n - \lambda_1)$. Specifically, the same calculation gives

$$\begin{aligned} &F_{q+\lambda_1}(0, \lambda_2 - \lambda_1, \dots, \lambda_n - \lambda_1) \\ &= \mathbb{E} \left(\sum_{k=0}^{\infty} \exp(-(q + \lambda_1)S_k^{(1)}) \right) \\ &\quad \times \mathbb{E} \left(\int_0^{S_1^{(1)}} dt \exp(-(q + \lambda_1)t) \right. \\ &\quad \left. \times \exp(-(\lambda_2 - \lambda_1)(A_t^{(2)} - A_t^{(3)}) - \dots - (\lambda_n - \lambda_1)A_t^{(n)}) \right) \\ &= \frac{1}{\Phi^{(1)}(q + \lambda_1)} \mathbb{E} \left(\int_0^{S_1^{(1)}} dt e^{-qt} \exp\{-\lambda_1(t - A_t^{(2)}) - \dots - \lambda_n A_t^{(n)}\} \right). \end{aligned}$$

We deduce that

$$F_q(\lambda_1, \dots, \lambda_n) = \frac{\Phi^{(1)}(q + \lambda_1)}{\Phi^{(1)}(q)} F_{q+\lambda_1}(0, \lambda_2 - \lambda_1, \dots, \lambda_n - \lambda_1).$$

Iterating the argument, we obtain

$$\begin{aligned} F_q(\lambda_1, \dots, \lambda_n) &= \frac{\Phi^{(1)}(q + \lambda_1)}{\Phi^{(1)}(q)} \frac{\Phi^{(2)}(q + \lambda_2)}{\Phi^{(2)}(q + \lambda_1)} \\ &\quad \dots \frac{\Phi^{(n)}(q + \lambda_n)}{\Phi^{(n)}(q + \lambda_{n-1})} F_{q+\lambda_n}(0, \dots, 0). \end{aligned}$$

This is the desired result, as obviously $F_{q+\lambda_n}(0, \dots, 0) = 1/(q + \lambda_n)$. Finally, the general case when the sequence $\lambda_1, \dots, \lambda_n$ is not necessarily monotone follows by analytic continuation. \square

REMARK. There is no similar formula for the joint Laplace transform of residual lifetimes $R_i^{(i)} = \inf\{s > 0: s + t \in \mathcal{R}^{(i)}\}$. The reason is that their joint distribution depends on the joint law of the regenerative sets $\mathcal{R}^{(1)}, \dots, \mathcal{R}^{(n)}$ and not merely on the individual laws.

2.2. *A multivariate renewal theorem.* We first develop some material to state the multivariate version of renewal theorem.

Recall from Lemma 1 that the ratios $\Phi^{(1)}/\Phi^{(2)}, \dots, \Phi^{(n-1)}/\Phi^{(n)}$ are completely monotone, and introduce the corresponding measures μ_1, \dots, μ_{n-1} on $[0, \infty)$; that is,

$$\int_{[0, \infty)} e^{-qx} \mu_i(dx) = \frac{\Phi^{(i)}(q)}{\Phi^{(i+1)}(q)}, \quad q > 0.$$

For $i = n$, we define μ_n by

$$\int_{[0, \infty)} e^{-qx} \mu_n(dx) = \Phi^{(n)}(q)/q, \quad q > 0,$$

or equivalently,

$$\mu_n(dx) = \mathbb{P}(S_1^{(n)} > x) dx, \quad x \in [0, \infty).$$

We also introduce the inverse of the mass

$$c_i = \frac{1}{\mu_i([0, \infty))} = \lim_{q \rightarrow 0+} \frac{\Phi^{(i+1)}(q)}{\Phi^{(i)}(q)}, \quad i = 1, \dots, n,$$

where we agree that $\Phi^{(n+1)}(q) = q$. Observe that

$$c_1 \cdots c_n = \lim_{q \rightarrow 0+} \frac{q}{\Phi^{(1)}(q)} = \frac{1}{\mathbb{E}(S_1^{(1)})},$$

so all the coefficients c_1, \dots, c_n are positive if and only if $\mathbb{E}(S_1^{(1)}) < \infty$, and then $c_1 \mu_1, \dots, c_n \mu_n$ are probability measures.

We now state the multivariate renewal theorem.

THEOREM 1. *Suppose that the renewal process $S^{(1)}$ is nonlattice and has finite mean. Then the n -tuple*

$$(A_t^{(1)} - A_t^{(2)}, \dots, A_t^{(n-1)} - A_t^{(n)}, A_t^{(n)})$$

converges in distribution as $t \rightarrow \infty$ towards the product measure

$$c_1 \mu_1 \otimes \cdots \otimes c_n \mu_n.$$

PROOF. Let T be an independent exponential time with parameter 1 and $q > 0$. According to Lemma 2, the joint Laplace transform of $(A_{T/q}^{(1)} - A_{T/q}^{(2)}, \dots, A_{T/q}^{(n-1)} - A_{T/q}^{(n)}, A_{T/q}^{(n)})$ evaluated at $\lambda_1, \dots, \lambda_n$ is

$$\frac{q}{\Phi^{(1)}(q)} \frac{\Phi^{(1)}(q + \lambda_1)}{\Phi^{(2)}(q + \lambda_1)} \cdots \frac{\Phi^{(n)}(q + \lambda_n)}{q + \lambda_n}.$$

When $q \rightarrow 0 +$, this quantity converges towards the Laplace transform of $c_1 \mu_1 \otimes \cdots \otimes c_n \mu_n$.

Thus, all that we need is to check the convergence in distribution of the n -tuple $(A_t^{(1)} - A_t^{(2)}, \dots, A_t^{(n-1)} - A_t^{(n)}, A_t^{(n)})$, as we have identified the only possible limit. To this end, take real numbers $x_1, \dots, x_n \geq 0$ and consider

$$\begin{aligned} \rho_t &= \mathbb{P}(A_t^{(1)} - A_t^{(2)} \leq x_1, \dots, A_t^{(n-1)} - A_t^{(n)} \leq x_{n-1}, A_t^{(n)} \leq x_n) \\ &= \mathbb{E} \left(\sum_{k=0}^{\infty} \mathbf{1}_{\{S_k^{(1)} \leq t\}} \mathbf{1}_{\{t - S_k^{(1)} < S_{k+1}^{(1)} - S_k^{(1)}\}} \mathbf{1}_{\{A_t^{(1)} - A_t^{(2)} \leq x_1, \dots, A_t^{(n)} \leq x_n\}} \right). \end{aligned}$$

Just as in the proof of Lemma 2, we apply the regenerative property at $S_k^{(1)}$ to rewrite this quantity as

$$\mathbb{E} \left(\sum_{k=0}^{\infty} \mathbf{1}_{\{S_k^{(1)} \leq t\}} f(t - S_k^{(1)}) g(t - S_k^{(1)}) \right)$$

where for $s \geq 0$,

$$f(s) = \mathbb{P}(A_s^{(1)} - A_s^{(2)} \leq x_1, \dots, A_s^{(n-1)} - A_s^{(n)} \leq x_{n-1}, A_s^{(n)} \leq x_n | S_1^{(1)} > s)$$

and $g(s) = \mathbb{P}(S_1^{(1)} > s)$. In other words, if we set $f \equiv 0$ on $(-\infty, 0)$, then we have

$$\rho_t = \int_{[0, \infty)} f(t - s) g(t - s) U^{(1)}(ds),$$

where $U^{(1)}$ is the renewal measure of the renewal process $S^{(1)}$. We now see from (1) that $\rho_t = \mathbb{E}(f(A_t^{(1)}))$.

The standard renewal theorem entails that $A_t^{(1)}$ converges in law as $t \rightarrow \infty$ towards some absolutely continuous distribution. Writing the conditional probability $f(s)$ as a quotient of unconditional probabilities, it is easily shown that the function f is bounded and has at most countably many discontinuities, so we see that the desired convergence holds. \square

Of course, there also exists a version of Theorem 1 in the lattice case which we now state.

THEOREM 2. *Suppose that the renewal processes $S^{(1)}, \dots, S^{(n)}$ are lattice with unit span (in the sense that the group generated by the support of the step distribution of $S_1^{(i)}$ is \mathbb{Z} for every i). If $S^{(1)}$ has finite mean, then for all nonnegative integers a_1, \dots, a_n ,*

$$\mathbb{P}(A_k^{(1)} - A_k^{(2)} = a_1, \dots, A_k^{(n-1)} - A_k^{(n)} = a_{n-1}, A_k^{(n)} = a_n)$$

converges as the integer k goes to infinity toward

$$c \mu^{(1)}(a_1) \cdots \mu^{(n)}(a_n),$$

where

$$\sum_{k=0}^{\infty} e^{-qk} \mu^{(i)}(k) = \frac{\Phi^{(i)}(q)}{\Phi^{(i+1)}(q)} \quad \text{for } i = 1, \dots, n - 1,$$

$$\sum_{k=0}^{\infty} e^{-qk} \mu^{(n)}(k) = \frac{\Phi^{(n)}(q)}{1 - e^{-q}}$$

and

$$c = 1/\mathbb{E}(S_1^{(1)}) = \lim_{q \rightarrow 0^+} q/\Phi^{(1)}(q).$$

The argument for the proof follows the same line as in the nonlattice case, using generating functions instead of Laplace transforms. We skip the details.

2.3. *A multivariate Dynkin–Lamperti theorem.* We now turn our attention to the asymptotic behavior of the renormalized age processes $(1/t)A_t^{(1)}, \dots, (1/t)A_t^{(n)}$. By the standard Dynkin–Lamperti theorem, $(1/t)A_t^{(i)}$ converges in distribution as $t \rightarrow \infty$ if and only if the following limit exists:

$$(3) \quad \lim_{t \rightarrow \infty} t^{-1} \mathbb{E}(A_t^{(i)}) := \alpha_i \in [0, 1].$$

Then $\Phi^{(i)}$ is regularly varying at $0+$ with index $1 - \alpha_i$ and the limit distribution of $(1/t)A_t^{(i)}$ is the so-called generalized arcsine distribution with parameter α_i [i.e., beta with parameters $(\alpha_i, 1 - \alpha_i)$]. See, for instance, Section 8.6.2 in [3].

Therefore, we may (and will) assume from now on that (3) holds for $i = 1, \dots, n$. As the embedding hypothesis $\mathcal{R}^{(1)} \subseteq \dots \subseteq \mathcal{R}^{(n)}$ implies that $A_t^{(1)} \geq \dots \geq A_t^{(n)}$, we must have

$$0 \leq \alpha_n \leq \dots \leq \alpha_1 \leq 1.$$

In order to focus on the most interesting case, we shall assume that these inequalities are strict,

$$(4) \quad 0 < \alpha_n < \dots < \alpha_1 < 1.$$

Indeed, if we had, for example, $\alpha_1 = \alpha_2$, then $(1/t)(A_t^{(1)} - A_t^{(2)})$ would converge in probability to 0 and we might just as well ignore $A_t^{(2)}$.

The multidimensional analogues of the beta distributions are the Dirichlet distributions. More precisely, recall that an n -dimensional random variable (X_1, \dots, X_n) has a Dirichlet distribution with parameters $(\beta_1, \dots, \beta_{n+1}) \in (0, 1)^{n+1}$ if

$$\begin{aligned} \mathbb{P}(X_1 \in dx_1, \dots, X_n \in dx_n) \\ = \frac{\Gamma(\beta_1 + \dots + \beta_{n+1})}{\Gamma(\beta_1) \dots \Gamma(\beta_{n+1})} (1 - (x_1 + \dots + x_n))^{\beta_{n+1}-1} \prod_{i=1}^n x_i^{\beta_i-1} dx_i \end{aligned}$$

for (x_1, \dots, x_n) in the n -dimensional simplex (i.e., $x_i > 0$ and $x_1 + \dots + x_n < 1$).

We are now able to state the multivariate version of the Dynkin–Lamperti theorem.

THEOREM 3. *Suppose that (3) and (4) hold. Then*

$$\left(\frac{1}{t}(A_t^{(1)} - A_t^{(2)}), \dots, \frac{1}{t}(A_t^{(n-1)} - A_t^{(n)}), \frac{1}{t}A_t^{(n)} \right)$$

converges in distribution as $t \rightarrow \infty$ towards an n -dimensional Dirichlet distribution with parameters

$$(\alpha_1 - \alpha_2, \dots, \alpha_{n-1} - \alpha_n, \alpha_n, 1 - \alpha_1).$$

In order to prove the result, it is more convenient to work with the nondecreasing processes

$$G_t^{(i)} = t - A_t^{(i)} = \sup\{s \leq t : s \in \mathcal{R}^{(i)}\}.$$

First, we investigate the limit behavior of some joint Laplace transform of $G^{(1)}, \dots, G^{(n)}$.

LEMMA 3. *Fix $q, \lambda_1, \dots, \lambda_n > 0$. Then*

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \varepsilon \int_0^\infty dt e^{-\varepsilon qt} \mathbb{E}(\exp\{-\lambda_1 \varepsilon G_t^{(1)} - \dots - \lambda_n \varepsilon G_t^{(n)}\}) \\ &= (q + \lambda_1 + \dots + \lambda_n)^{\alpha_1 - 1} (q + \lambda_2 + \dots + \lambda_n)^{\alpha_2 - \alpha_1} \\ & \quad \dots (q + \lambda_n)^{\alpha_n - \alpha_{n-1}} q^{-\alpha_n}. \end{aligned}$$

PROOF. Introduce the degenerate regenerative set $\mathcal{A}^{(0)} = \{0\}$ which corresponds to $A_t^{(0)} = t$ and $\Phi^{(0)}(\lambda) = 1$. We deduce from Lemma 2 (beware of the changes of indices) that

$$\begin{aligned} & \int_0^\infty dt e^{qt} \mathbb{E}(\exp\{-\lambda_1 G_t^{(1)} - \dots - \lambda_n G_t^{(n)}\}) \\ &= \frac{\Phi^{(0)}(q + \lambda_1 + \dots + \lambda_n)}{\Phi^{(0)}(q)} \frac{\Phi^{(1)}(q + \lambda_2 + \dots + \lambda_n)}{\Phi^{(1)}(q + \lambda_1 + \dots + \lambda_n)} \dots \frac{\Phi^{(n)}(q)}{\Phi^{(n)}(q + \lambda_n)} \frac{1}{q} \end{aligned}$$

(of course, the first term in the product simply equals 1).

Multiply $q, \lambda_1, \dots, \lambda_n$ by ε and recall from the hypothesis (3), that $\Phi^{(i)}$ is regularly varying at $0 +$ with index $1 - \alpha_i$. This yields our statement. \square

Next, we invert the multivariate Laplace transform which appears in Lemma 3.

LEMMA 4. *For every $q, \lambda_1, \dots, \lambda_n > 0$, we have*

$$\begin{aligned} & (q + \lambda_1 + \dots + \lambda_n)^{\alpha_1 - 1} (q + \lambda_2 + \dots + \lambda_n)^{\alpha_2 - \alpha_1} \dots (q + \lambda_n)^{\alpha_n - \alpha_{n-1}} q^{-\alpha_n} \\ &= \int_0^\infty dt_1 e^{-\lambda_1 t_1} \frac{t_1^{-\alpha_1}}{\Gamma(1 - \alpha_1)} \int_{t_1}^\infty dt_2 \exp(-\lambda_2 t_2) \frac{(t_2 - t_1)^{\alpha_1 - \alpha_2 - 1}}{\Gamma(\alpha_1 - \alpha_2)} \\ & \quad \dots \int_{t_n}^\infty dt e^{-qt} \frac{(t - t_n)^{\alpha_n - 1}}{\Gamma(\alpha_n)}. \end{aligned}$$

PROOF. To start with, recall that

$$\begin{aligned} (q + \lambda_1 + \dots + \lambda_n)^{\alpha_1 - 1} &= \int_0^\infty dt_1 \exp(-\lambda_1 t_1) \frac{t_1^{-\alpha_1}}{\Gamma(1 - \alpha_1)} \\ & \quad \times \exp(-(q + \lambda_2 + \dots + \lambda_n) t_1). \end{aligned}$$

This yields

$$\begin{aligned} & (q + \lambda_1 + \dots + \lambda_n)^{\alpha_1 - 1} (q + \lambda_2 + \dots + \lambda_n)^{\alpha_2 - \alpha_1} \\ &= \int_0^\infty dt_1 \exp(-\lambda_1 t_1) \frac{t_1^{-\alpha_1}}{\Gamma(1 - \alpha_1)} \\ & \quad \times \int_0^\infty ds_2 \exp(-(q + \lambda_2 + \dots + \lambda_n)(t_1 + s_2)) \frac{s_2^{\alpha_1 - \alpha_2 - 1}}{\Gamma(\alpha_1 - \alpha_2)} \\ &= \int_0^\infty dt_1 \exp(-\lambda_1 t_1) \frac{t_1^{-\alpha_1}}{\Gamma(1 - \alpha_1)} \\ & \quad \times \int_{t_1}^\infty dt_2 \exp(-\lambda_2 t_2) \frac{(t_2 - t_1)^{\alpha_1 - \alpha_2 - 1}}{\Gamma(\alpha_1 - \alpha_2)} \exp(-(q + \lambda_3 + \dots + \lambda_n)t_2), \end{aligned}$$

and we iterate the calculation. \square

We are now able to characterize the asymptotic behavior in distribution of $(G_t^{(1)}, \dots, G_t^{(n)})$.

COROLLARY 1. *Suppose that (3) and (4) hold. The probability measure*

$$\mathbb{P}(\varepsilon G_{1/\varepsilon}^{(1)} \in dt_1, \dots, \varepsilon G_{1/\varepsilon}^{(n)} \in dt_n),$$

on $\{0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq 1\}$ converges weakly as $\varepsilon \rightarrow 0+$ toward

$$\frac{t^{-\alpha_1}}{\Gamma(1 - \alpha_1)} \frac{(t_2 - t_1)^{\alpha_1 - \alpha_2 - 1}}{\Gamma(\alpha_1 - \alpha_2)} \dots \frac{(t_n - t_{n-1})^{\alpha_{n-1} - \alpha_n - 1}}{\Gamma(\alpha_{n-1} - \alpha_n)} \frac{(1 - t_n)^{\alpha_n - 1}}{\Gamma(\alpha_n)} dt_1 \dots dt_n.$$

PROOF. For each $\varepsilon > 0$, the processes $t \rightarrow G_{t/\varepsilon}^{(i)}$ are nondecreasing, so the function

$$t \rightarrow \mathbb{E}(\exp\{-\lambda_1 \varepsilon G_{t/\varepsilon}^{(1)} - \dots - \lambda_n \varepsilon G_{t/\varepsilon}^{(n)}\})$$

is nonincreasing. We then obtain from Lemmas 3 and 4 and by an integration by parts that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0+} \int_0^\infty e^{-qt} d\left(1 - \mathbb{E}(\exp\{-\lambda_1 \varepsilon G_{t/\varepsilon}^{(1)} - \dots - \lambda_n \varepsilon G_{t/\varepsilon}^{(n)}\})\right) \\ &= \int_0^\infty e^{-qt} d\left(1 - \int_0^t dt_n \exp(-\lambda_n t_n) \frac{(t - t_n)^{\alpha_n - 1}}{\Gamma(\alpha_n)} \right. \\ & \quad \left. \dots \int_0^{t_2} dt_1 \exp(-\lambda_1 t_1) \frac{(t_2 - t_1)^{\alpha_1 - \alpha_2 - 1}}{\Gamma(\alpha_1 - \alpha_2)} \frac{t_1^{-\alpha_1}}{\Gamma(1 - \alpha_1)}\right). \end{aligned}$$

As this holds for all $q > 0$, it implies that for almost every $t > 0$,

$$\lim_{\varepsilon \rightarrow 0+} \mathbb{E}(\exp\{-\lambda_1 \varepsilon G_{t/\varepsilon}^{(1)} - \dots - \lambda_n \varepsilon G_{t/\varepsilon}^{(n)}\})$$

$$\begin{aligned}
 &= \int_0^t dt_n \exp(-\lambda_n t_n) \frac{(t - t_n)^{\alpha_n - 1}}{\Gamma(\alpha_n)} \\
 &\quad \cdots \int_0^{t_2} dt_1 \exp(-\lambda_1 t_1) \frac{(t_2 - t_1)^{\alpha_1 - \alpha_2 - 1}}{\Gamma(\alpha_1 - \alpha_2)} \frac{t_1^{-\alpha_1}}{\Gamma(1 - \alpha_1)},
 \end{aligned}$$

which readily entails our claim. \square

Theorem 3 follows now from Corollary 1 by an immediate change of variables.

3. Continuous setting. We now turn our attention to the continuous setting; let us first briefly present basic notions in that field and refer to Fristedt [6] or [2] for details.

The continuous-time analogue of a renewal process is a so-called subordinator (that is, an increasing Lévy process), which will be denoted generically by $\sigma = (\sigma_t, t \geq 0)$. The closed range $\mathcal{R} = \{\sigma_t, t \geq 0\}^{\text{cl}}$ of a subordinator is a regenerative set in the sense given at the beginning of Section 2, and conversely, any discrete or nondiscrete regenerative set can be viewed as the closed range of some subordinator.

The distribution of σ is characterized by its Laplace exponent Φ ,

$$\mathbb{E}(\exp(-q\sigma_t)) = \exp\{-t\Phi(q)\}, \quad t, q \geq 0,$$

and the Laplace exponent is given by the Lévy–Khintchine formula

$$\Phi(q) = dq + \int_{(0, \infty)} (1 - e^{-qx}) \Pi(dx).$$

Here $d \geq 0$ is the drift coefficient, Π the Lévy measure; loosely speaking, the Lévy measure describes the distribution of gaps in \mathcal{R} and can be thought of as the analogue of the step distribution of the renewal process (the absence of killing terms is because we implicitly focus on unbounded regenerative sets). The renewal measure U of σ is its potential measure; that is, $U(dx) = \int_0^\infty \mathbb{P}(\sigma_t \in dx) dt$, and its Laplace transform is $\int_0^\infty e^{-qx} U(dx) = 1/\Phi(q)$.

Plainly, the notion of compatibility with the regenerative property for a sequence of embedded regenerative sets also makes sense in the continuous setting. Then the multivariate versions of the renewal theorem and of the Dynkin–Lamperti theorem still hold true. The proofs follow essentially the same route as in the discrete setting. Technically, Maisonneuve’s exit system formula and excursion theory provide the right tools for adapting the arguments given in Section 2 to the continuous setting; we skip the details (the interested reader is referred to the proof of Theorem 1 in [1] for the continuous version of Lemma 1; the modifications which are needed for the other steps are in the same vein).

On the other hand, we point out that (the continuous version of) Theorem 3 also holds for small times, that is, when $t \rightarrow \infty$ is replaced by $t \rightarrow 0 +$ in the statement. Moreover, there is a version at fixed time for self-similar (i.e., stable) regenerative sets.

PROPOSITION 1. *Suppose that (4) holds and that $\Phi^{(i)}(q) = q^{1-\alpha_i}$ for every $q > 0$ and $i = 1, \dots, n$. Then for every $t > 0$,*

$$\left(\frac{1}{t} (A_t^{(1)} - A_t^{(2)}), \dots, \frac{1}{t} (A_t^{(n-1)} - A_t^{(n)}), \frac{1}{t} A_t^{(n)} \right)$$

has an n -dimensional Dirichlet distribution with parameters

$$(\alpha_1 - \alpha_2 \cdots \alpha_{n-1} - \alpha_n, \alpha_n, 1 - \alpha_1).$$

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LABORATOIRE DE PROBABILITÉS, UMR 7599
 UNIVERSITÉ PIERRE ET MARIE CURIE
 PARIS
 FRANCE F-75252
 E-MAIL: jbe@ccr.jussieu.fr