*The Annals of Probability*
1999, Vol. 27, No. 3, 1183–1207

# GIBBS MEASURES RELATIVE TO BROWNIAN MOTION

BY HIROFUMI OSADA AND HERBERT SPOHN

*Nagoya University and Technische Universität München*

We consider Brownian motion perturbed by the exponential of an action. The action is the sum of an external, one-body potential and a two-body interaction potential which depends only on the increments. Under suitable conditions on these potentials, we establish existence and uniqueness of the corresponding Gibbs measure. We also provide an example where uniqueness fails because of a slow decay in the interaction potential.

**1. Introduction.** In its standard form, the theory of Gibbs measures is formulated as a random field over $\mathbb{Z}^d$ with general single site space and a product measure as reference measure [5]. Gibbs measures also arise in the context of the Euclidean version of quantum field theory [6]. In this case, the setup is somewhat modified. The Gibbs measure is defined on $\mathscr{S}'(\mathbb{R}^d)$, the space of tempered distributions over $\mathbb{R}^d$, and the reference measure is a suitable Gaussian measure on $\mathscr{S}'(\mathbb{R}^d)$. Considerable effort has been invested to construct such Gibbs measures, one central problem being the control of the singular behavior at short distances. In our paper we will study a class of Gibbs measures which also originate from quantum mechanics, but are in fact stochastic processes (rather than fields).

The physical context is a single quantum particle subject to a prescribed external potential and coupled to a free bosonic field. For simplicity we consider the case of one-dimensional motion only. Then the position of the particle at time $t$ is $X_t \in \mathbb{R}$ and the external potential is given through a function $\varphi: \mathbb{R} \to \mathbb{R}$. The reference measure is the Brownian bridge, denoted by $\mathscr{W}_{T,\xi}$. Since $X_t$ will be constructed as stationary process, we choose the left end point at $-T$, $X_{-T} = \xi_{-T}$ and the right end point at $T$, $X_T = \xi_T$. Then $\mathscr{W}_{T,\xi}$ is defined on $C([-T, T]; \mathbb{R})$. Let $\phi$ be the $\mathscr{S}'(\mathbb{R}^d)$ valued infinite-dimensional stationary Ornstein–Uhlenbeck process, defined as the Gaussian measure, $\mu_G$, with mean zero and covariance

$$\int \mu_G(d\phi)\phi(g,t)\phi(f,0) = \int \hat{g}(k)^* \omega(k)^{-1} \exp(-\omega(k)|t|)\hat{f}(k)\, d^d k. \tag{1.1}$$

Here $\phi(g,t) = \int \phi(y,t)g(y)\, d^d y$, $\hat{}$ denotes Fourier transform, and $\omega(k) \geq 0$, $\omega(k) = \omega(-k)$. The path measure of physical interest is then given by

$$\begin{aligned} &Z_T^{-1} \exp\biggl[-\int_{|t|\leq T}\biggl\{\varphi(X_t) + \int \lambda(y - X_t)\phi(y,t)\, d^d y\biggr\}\, dt\biggr] \\ &\qquad \times \mu_G(d\phi)\mathscr{W}_{T,\xi}(dX), \end{aligned} \tag{1.2}$$

Received November 1997; revised February 1999.
*AMS* 1991 *subject classifications*. 60K35, 82B21.
*Key words and phrases*. Gibbs measure, Brownian motion, pair potential.

where $Z_T$ is the normalization and $\lambda \in \mathscr{S}(\mathbb{R}^d)$. Clearly, the Gaussian integration can be carried out explicitly with the result

$$Z_T^{-1} \exp\biggl[-\int_{|t|\le T} \varphi(X_t)\,dt - \tfrac{1}{2}\int_{|t|,\,|s|\le T} w(t-s, X_t - X_s)\,dt\,ds\biggr] \mathscr{W}_{T,\,\xi}(dX) \tag{1.3}$$

and

$$w(t,x) = -\int |\hat{\lambda}(k)|^2 \frac{1}{\omega(k)} \exp(-\omega(k)|t|)\cos(kx)\,d^d k. \tag{1.4}$$

In our paper we abstract from the specific physical application and investigate Gibbs measure of the form (1.3) with a single site potential $\varphi$ and a pair potential $w$. This means we study probability measures on $C(\mathbb{R};\mathbb{R})$, whose conditional measures, that is, the distribution of $\{X_t;\ |t| \le T\}$ conditioned on the outside path $\{\xi_t;\ |t| \ge T\}$, have the Gibbsian structure (1.3).

On a theoretical level the basic issues are the *existence* of a Gibbs measure with potentials as in (1.3) and its *uniqueness*. For the existence, besides some regularity on $w$, it should suffice to require that the potential $\varphi$ is sufficiently confining. This can be ensured either by $\varphi(x) \to \infty$ as $|x| \to \infty$ or by $\varphi$ being sufficiently attractive close to the origin. To really prove the existence, only two methods seem to be available: monotonicity, which is used in this paper, and compactness as technically encoded in the Ruelle superstability estimates [10, 7]. This latter method should work for a large class of potentials. It relies on the product structure of the a priori measure, however, and we have not succeeded in generalizing it to the present context.

Monotonicity uses the Fortuin-Kasteleyn-Ginibre (FKG) and Brascamp–Lieb inequalities. They require a rather special pair potential of the form $w(t,x) = \rho(t)v(x)$ with $v$ convex. Clearly, for the physical pair potential (1.4), such a condition cannot be satisfied. Fortunately, in many cases the so-called dipole approximation suffices, which in the functional integral corresponds to the quadratic approximation

$$\begin{aligned} w(t,x) &\simeq -\int |\hat{\lambda}(k)|^2 \frac{1}{\omega(k)} \exp(-\omega(k)|t|)\left(1 - \frac{1}{2}(kx)^2\right) d^d k \\ &= -\sigma(t) + \rho(|t|)x^2, \end{aligned} \tag{1.5}$$

to which our results apply.

For the uniqueness of the Gibbs measure, the standard statistical mechanics intuition is that a bounded interaction energy between right- and left-half lines should provide a sufficient condition no matter what $\varphi$, which under our assumptions means $\int_0^\infty \rho(t)t\,dt < \infty$. This is precisely the criterion under which we establish uniqueness; compare Theorem 2.3 below. To convince ourselves that for a more slowly decaying interaction potential uniqueness may fail, we consider the case where $\varphi$ is a double well potential and $w$ is of the form (1.5) with $\rho(t) \cong t^{-\gamma}$, $1 < \gamma \leq 1$, for large $t$ and show that there exist then at least two distinct Gibbs measures.

To give an outline: in the following section we list the precise assumptions on $\varphi$, $w$ and state our main results. In Sections 3 and 4 we establish existence and in Section 5 uniqueness under suitable conditions. In Section 6 we present our example for nonuniqueness.

## 2. Main results.

We introduce a notion of right- and left-dominators.

DEFINITION 2.1. Let $f = f(x)$ and $g = g(x)$ be functions with value on $\mathbb{R} \cup \{\infty\}$ defined on $\mathbb{R}$. We say $g = g(x)$ is a right-dominator (resp. left-dominator) of $f$ if:

(D1) $g$ is convex and finite on at least two distinct points.

(D2) $f - g$ is nondecreasing (resp. nonincreasing) in $x$, where we use the convention that $\infty - \infty = 0$ in case of $f(x) = g(x) = \infty$.

(D3) There exists a constant $a > 0$ such that $g''(x) \geq 2a$ a.e. $x \in \{g < \infty\}$.

If $f$ and $g$ are differentiable, then a right-dominator satisfies $f'(x) \geq g'(x)$ and a left-dominator $f'(x) \leq g'(x)$ with $g'$ strictly increasing. We say a dominator $g$ is symmetric around $m$ if $g(x-m) = g(|x-m|)$ for all $x$. We remark, if $g$ is a right-dominator (resp. left-dominator) of $f$ symmetric around $m$ (resp. $-m$), then there exist right-dominators (resp. left-dominators) of $f$ symmetric around $n$ (resp. $-n$) for all $n \geq m$. Indeed, we easily see that

$$R_n^+(g)(x) = \begin{cases} g(x), & \text{for } x \geq n, \\ g(2n - x), & \text{for } x \leq n, \end{cases}$$

$$R_n^-(g)(x) = \begin{cases} g(-2n - x), & \text{for } x \geq -n, \\ g(x), & \text{for } x \leq -n \end{cases}$$

are such dominators.

Let $\varphi$: $\mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be the external potential. Let $I = (a_-, a_+)$ be an open interval, $-\infty \leq a_- < a_+ \leq \infty$. We assume that $\varphi(x) = \infty$ for $x \in \mathbb{R} \setminus I$, which simply means that in (1.3) the paths are restricted to $\{X_t \in I,\ |t| \leq T\}$. Let $w$: $\mathbb{R}^2 \to \mathbb{R}$ be interaction potential. We also refer to $(\varphi, w)$ as potential.

(A1) Assumptions on the external potential $\varphi$.

(P1) $\varphi$: $I \to \mathbb{R}$ is locally integrable and bounded from below.

(P2) $\varphi$ has a right-dominator symmetric around $m$ and a left-dominator symmetric around $-m$ for some $m \geq 0$.

(A2) Assumption on the interaction potential $w$.

(W1) $w(t, x) = \rho(t)v(x)$ with $\rho \geq 0$, $\rho(t) = \rho(|t|)$ and $v(x) = v(|x|)$.

(W2) $v(\cdot)$ is convex and piecewise smooth. In addition, for some $p_0 > 1$,

$$u(x) = \operatorname{ess\,sup}_{y\in\mathbb{R}} \frac{|v'(x) - v'(x-y)|}{1+|y|^{p_0}} \tag{2.1}$$

is finite and for each $\varepsilon > 0$ there exists a $b = b(\varepsilon) \geq 0$ such that

$$u(x) \leq \varepsilon\{v'(x) - v'(x-b)\} \quad \text{for all } x \in \mathbb{R}. \tag{2.2}$$

(W3) $\rho(|t|) \leq \rho_0(t)$, where $\rho_0\colon \mathbb{R}^+ \to \mathbb{R}^+$ is an integrable, convex and nonincreasing function such that $\rho_0(t) > 0$ for all $t$ and $\rho_0(0) < \infty$.

We remark $v'' \geq 0$, $v'(x) = -v'(-x)$ is nondecreasing and $u(x) = u(-x)$ by definition. (2.1) implies that there exists $C_v$ such that

$$v(x) \leq C_v(1+|x|)^{p_0+1} \quad \text{for all } x \in \mathbb{R}.$$

We easily see (2.2) is satisfied if $v(x) = |x|^a$ for some $a \geq 1$ or more generally, $v(x) = \sum_{i=1}^N a_i |x|^{p_i}$ with $a_i > 0$, $p_i \geq 1$.

To define a Gibbs measure (DLR measure), we first introduce the energy $H_T(X,\xi)$ of the path $X = \{X_t; |t| \leq T\}$, $X \in C([-T,T];\ \mathbb{R}])$, for given outside path $\xi = \{\xi_t;\ |t| \geq T\}$, $\xi \in C(\{|t| \geq T\};\ \mathbb{R})$. This energy is given by

$$\begin{aligned} H_T(X,\xi) = \int_{|t|\leq T} \varphi(X_t)\,dt + \tfrac{1}{2}\int_{|t|,\,|u|\leq T} w(t-u, X_t - X_u)\,dt\,du \\ + \int_{|t|\leq T<|u|} w(t-u, X_t - \xi_u)\,dt\,du. \end{aligned} \tag{2.3}$$

Note that $H_T$ can take the value $\infty$ but is bounded from below.

Let $\mathscr{W}_{T,\xi}$ be the path measure of the Brownian bridge on $C([-T,T];\mathbb{R})$ with boundary condition $\mathscr{W}_{T,\xi}(X_{\pm T} = \xi_{\pm T}) = 1$ and let $E^{\mathscr{W}}_{T,\xi}$ be the corresponding expectation. We define

$$Z_{T,\xi} = E^{\mathscr{W}}_{T,\xi}[\exp[-H_T(X,\xi)]]. \tag{2.4}$$

Let $\pi_T\colon C(\mathbb{R};\mathbb{R}) \to C([-T,T];\mathbb{R})$ and $\pi_T^*\colon C(\mathbb{R};\mathbb{R}) \to C(\{|t| \geq T\};\mathbb{R})$ be projections. For a probability measure $\mu$ on $C(\mathbb{R};\mathbb{R})$, we denote by $\mu_{T,\xi}$ the probability on $C([-T,T];\mathbb{R})$ given by

$$\mu_{T,\xi} = \mu(\pi_T \in \cdot\,|\pi_T^* = \xi).$$

Here $\mu(\cdot|\pi_T^* = \xi)$ is the regular conditional probability with respect to the $\sigma$-field $\sigma[\pi_T^*]$, evaluated by the value $\pi_T^*(X) = \xi$.

Definition 2.2. A probability measure $\mu$ on $C(\mathbb{R};\mathbb{R})$ is called Gibbs measure with external potential $\varphi$ and interaction potential $w$ if its conditional measures are given by

$$\mu_{T,\xi} = Z_{T,\xi}^{-1} \exp[-H_T(X,\xi)]\, d\mathscr{W}_{T,\xi} \tag{2.5}$$

for $\mu(\pi_T^* \in \cdot)$-a.s. $\xi \in C(\mathbb{R};\mathbb{R})$.

To establish the existence of a Gibbs measure, we have to go through a finite volume construction. For this purpose we define $\mu_{T,\xi}^{\varphi,w}$ by

$$\mu_{T,\xi}^{\varphi,w} = (Z_{T,\xi})^{-1} \exp[-H_T(X,\xi)]\, d\mathscr{W}_{T,\xi}. \tag{2.6}$$

as a measure on $C([-T,T];\mathbb{R})$, where we assume that $\|\xi\|_\infty < \infty$ with $\|\cdot\|_\infty$ the sup-norm on $C(\mathbb{R};\mathbb{R})$. Let $E_{T,\xi}^{\varphi,w}$ denote the expectation with respect to $\mu_{T,\xi}^{\varphi,w}$.

THEOREM 2.1 (Localization). *Assume* (A1), (W1) *and* (W2). *Then there exists $\alpha > 0$ such that, for all $\xi$ with $\|\xi\|_\infty < \infty$, there exists a constant $C_1$, depending only on $a$ in* (D3), *$m$ in* (P2), *$w$ and $\|\xi\|_\infty$, satisfying*

$$E_{T,\xi}^{\varphi,w}[\exp(\alpha|X_t|^2)] \le C_1 \quad \textit{for all } |t| \le T < \infty. \tag{2.7}$$

*In particular, the finite-dimensional distributions of $\{\mu_{T,\xi}^{\varphi,w}\}_T$ are tight.*

REMARK 2.1. The finite volume construction can be carried through also for free boundary condition. In this case, one has the energy

$$H_T(X) = \int_{|t|\le T} \varphi(X_t)\, dt + \frac{1}{2}\int_{|t|,\,|u|\le T} w(t-u, X_t - X_u)\, dt\, du, \tag{2.8}$$

and defines a sequence of probability measures $\mu_{T,\xi,\circ}^{\varphi,w}$ by

$$\mu_{T,\xi,\circ}^{\varphi,w} = (Z_{T,\xi,\circ})^{-1} \exp[-H_T(X)]\, d\mathscr{W}_{T,\xi}, \tag{2.9}$$

with $Z_{T,\xi,\circ}$ the normalization. Theorem 2.1 also holds in this case.

EXAMPLE 2.1.

(i) By Lemma A.3 we see the following satisfy (A1):

(a) $\varphi(x) = |x|^q + p(x) + \ell(x)$, where $q \ge 2$, $p$ is a polynomial whose degree is less than or equal to $q-1$, and $\ell$ is a Lipschitz continuous function.

(b) $\varphi(x) = e^{|x|} + q(x)$, $q$ is a function whose derivative $q'$ is at most polynomial growth order.

(ii) An example of $w$ satisfying (A2) is

$$w(t,x) = (1+|t|)^{-\alpha}|x|^\beta \quad \text{with } \alpha > 1 \text{ and } \beta \ge 1.$$

Using the moment bounds from Theorem 2.1 we prove the following.

THEOREM 2.2 (Existence of Gibbs measure). *Assume* (A1) *and* (A2). *Then there exists a Gibbs measure for* $(\varphi, w)$.

REMARK 2.2. (i) $\varphi$ may depend on time $t$.

(ii) Once Theorem 2.2 is established, it is easy to see that there exists at least one translation invariant Gibbs measure for $(\varphi, w)$.

As for statistical mechanics in one dimension, the uniqueness can be linked to a sufficiently fast decay of the interaction energy.

THEOREM 2.3 (Uniqueness of Gibbs measure). *Let* (A1) *and* (A2) *hold. If*

$$\int_0^\infty t\rho(t)\,dt < \infty, \tag{2.10}$$

*then there exists exactly one translation invariant Gibbs measure for* $(\varphi, w)$ *satisfying for some* $p_2 > p_0 + 1$,

$$\int |X_t|^{p_2}\,d\mu < \infty. \tag{2.11}$$

*Moreover, any limit points of* $\{\mu^{\varphi,w}_{T,\xi,\circ}\}_T$ *or* $\{\mu^{\varphi,w}_{T,\xi}\}_T$ *for* $\xi$ *with* $\|\xi\|_\infty < \infty$ *as* $T \to \infty$ *are unique and, henceforth, translation invariant.*

REMARK 2.3. Theorem 2.3 is a corollary of Theorem 5.4, which states a stronger uniqueness result. There we will prove that a *tame* Gibbs measure is unique; compare Definition 5.3 for the notion of tame.

To discuss nonuniqueness, let us consider the particular potential $\varphi(x) = \beta(x^4 - x^2)$, $\beta > 0$, where $\beta$ is a parameter which controls the depth of the two minima at $x = \pm 1/\sqrt{2}$, and $w(t,x) = \alpha(1+|t|)^{-\gamma}x^2$, where $\alpha > 0$ controls the strength and $\gamma$ the range of the interaction, $\gamma > 1$. For $\alpha = 0$, a typical path will fluctuate around one minimum and then rapidly cross to the other minimum. The average waiting time in one minimum is approximately equal to the inverse spectral gap of the symmetric operator $-(1/2)(d^2/dx^2) + \varphi(x)$ on $L^2(\mathbb{R}, dx)$. As $\alpha$ increases, this waiting time becomes longer. If $\gamma \le 2$ and if $\alpha$ is sufficiently strong, then even for $T \to \infty$ boundary conditions persist: if $\xi_{-T} = 1 = \xi_T$, then in the limit measure, the path will spend more time close to the right minimum.

THEOREM 2.4. *Let* $\varphi(x) = \beta(x^4 - x^2)$ *and* $w(t,x) = \alpha(1+|t|)^{-\gamma}x^2$. *If* $1 < \gamma \le 2$, *then we can choose* $\alpha, \beta > 0$ *such that there are at least two translation invariant Gibbs measures for* $(\varphi, w)$.

**3. Proof of Theorem 2.1.** Let $E^{\varphi,w}_{T,\xi}$ and $E^{\varphi,w}_{T,\xi,\circ}$ be expectations with respect to $\mu^{\varphi,w}_{T,\xi}$ and $\mu^{\varphi,w}_{T,\xi,\circ}$, respectively.

LEMMA 3.1. *Let $\psi$ be a right-dominater of $\varphi$. Let $F$ be a nonnegative, continuous function on $C([-T,T];\mathbb{R})$ such that $F$ is increasing in the sense that $F(X) \le F(Y)$ whenever $X \le Y$ on $[-T,T]$. Suppose $\xi \le \eta$. Then*

$$E^{\varphi,w}_{T,\xi}[F(X)] \le E^{\psi,w}_{T,\eta}[F(X)], \tag{3.1}$$

$$E^{\varphi,w}_{T,\xi,\circ}[F(X)] \le E^{\psi,w}_{T,\eta,\circ}[F(X)]. \tag{3.2}$$

PROOF. We introduce discretizations $\mu^1_n$ and $\mu^2_n$ of $\mu^{\varphi,w}_{T,\xi}$ and $\mu^{\psi,w}_{T,\eta}$, respectively. Let $L(n,T) = [-T,T] \cap \{\mathbb{Z}/n\}$. Let $\mu^1_n$ and $\mu^2_n$ be the probability measures on $\mathbb{R}^{L(n,T)}$ given by

$$\begin{aligned}
\mu^1_n &= (Z_1)^{-1}\exp(-m_0 - m_1)\prod_{|i|<nT} dx_{i/n} \times \delta_{\xi_{-T}}(dx_{-T}) \times \delta_{\xi_T}(dx_T),\\
\mu^2_n &= (Z_2)^{-1}\exp(-m_0 - m_2)\prod_{|i|<nT} dx_{i/n} \times \delta_{\eta_{-T}}(dx_{-T}) \times \delta_{\eta_T}(dx_T),
\end{aligned} \tag{3.3}$$

where $Z_1$ and $Z_2$ are normalizations, and $m_0$, $m_1$ and $m_2$ are given by

$$\begin{aligned}
m_0 &= \frac{1}{2n^2}\sum_{|i|,|j|\le nT} w\left(\frac{i-j}{n}, x_{i/n} - x_{j/n}\right) + \frac{n}{2}\sum_{-nT\le i<nT} |x_{i/n} - x_{(i+1)/n}|^2,\\
m_1 &= \frac{1}{n}\sum_{|i|\le nT}\varphi(x_{i/n}) + \frac{1}{n^2}\sum_{|i|\le nT<|j|} w\left(\frac{i-j}{n}, x_{i/n} - \xi_{j/n}\right),\\
m_2 &= \frac{1}{n}\sum_{|i|\le nT}\psi(x_{i/n}) + \frac{1}{n^2}\sum_{|i|\le nT<|j|} w\left(\frac{i-j}{n}, x_{i/n} - \eta_{j/n}\right).
\end{aligned}$$

We note that the square terms in the definition of $m_0$ come from the Brownian bridge.

Let $\iota\colon \mathbb{R}^{L(n,T)} \to C([-T,T];\mathbb{R})$ denote the map defined by the linear interpolation of $x = (x_{i/n}) \in \mathbb{R}^{L(n,T)}$. Let $\tilde{\mu}^i_n = \mu^i_n \circ \iota^{-1}$ $(i = 1,2)$. Then $\tilde{\mu}^1_n$ and $\tilde{\mu}^2_n$ converge to $\mu^{\varphi,w}_{T,\xi}$ and $\mu^{\psi,w}_{T,\eta}$ weakly in $C([-T,T];\mathbb{R})$, respectively. Hence (3.1) is reduced to

$$\int F(X)\,d\tilde{\mu}^1_n \le \int F(X)\,d\tilde{\mu}^2_n \quad \text{for all } n \tag{3.4}$$

because $F$ is continuous and nonnegative. By definition we see (3.4) is equivalent to

$$\int F(\iota(x))\,d\mu^1_n \le \int F(\iota(x))\,d\mu^2_n \quad \text{for all } n. \tag{3.5}$$

In order to prove (3.5), we use Lemma A.2 in the Appendix and Preston's FKG inequality (see Theorem 3 in [9]). Let

$$a_i(t) = \begin{cases} \dfrac{1}{n}\varphi(t) + \dfrac{1}{n^2}\sum_j w\left(\dfrac{i-j}{n}, t - \xi_{j/n}\right), & \text{for } |i| \le nT - 1, \\ \dfrac{1}{n}\varphi(t) + \dfrac{1}{n^2}\sum_j w\left(\dfrac{i-j}{n}, t - \xi_{j/n}\right) & \\ \quad + \dfrac{n}{2}|t - \xi_{\pm(T+1/n)}|^2, & \text{for } i = \pm nT, \end{cases}$$

$$b_i(t) = \begin{cases} \dfrac{1}{n}\psi(t) + \dfrac{1}{n^2}\sum_j w\left(\dfrac{i-j}{n}, t - \eta_{j/n}\right), & \text{for } |i| \le nT - 1 \\ \dfrac{1}{n}\psi(t) + \dfrac{1}{n^2}\sum_j w\left(\dfrac{i-j}{n}, t - \eta_{j/n}\right) & \\ \quad + \dfrac{n}{2}|t - \eta_{\pm(T+1/n)}|^2, & \text{for } i = \pm nT, \end{cases}$$

$$c_{ij}(t) = \begin{cases} \dfrac{1}{2n^2}w\left(\dfrac{i-j}{n}, t\right) + \dfrac{n}{4}t^2, & \text{for } |i-j| = 1, \\ \dfrac{1}{2n^2}w\left(\dfrac{i-j}{n}, t\right), & \text{for } |i-j| > 1. \end{cases}$$

Here the sum $\sum_j$ is taken over $|j| \ge nT + 1$. Then the assumptions of Lemma A.2 are fulfilled. So we can apply Preston's FKG inequality (Theorem 3 in [9]) to obtain (3.5), which yields (3.1). The proof of (3.2) is similar, so it is omitted.

LEMMA 3.2. *Let $\psi$ be a dominator of some function and $\psi_s = a(x-b)^2$, where $a > 0$ is a constant in* (D3) *and $b \in \mathbb{R}$ is any number. Let $G(x) = |x|^p$ with $p \ge 1$ or $G(x) = e^{\alpha|x|}$ or $G(x) = e^{\alpha|x|^2}$ with $\alpha > 0$. Then*

$$E^{\psi, w}_{T, \eta}[G(X_t - E^{\psi, w}_{T, \eta}[X_t])] \le E^{\psi_s, 0}_{T, \eta}[G(X_t - E^{\psi_s, 0}_{T, \eta}[X_t])], \tag{3.6}$$

$$E^{\psi, w}_{T, \eta, \circ}[G(X_t - E^{\psi, w}_{T, \eta, \circ}[X_t])] \le E^{\psi_s, 0}_{T, \eta, \circ}[G(X_t - E^{\psi_s, 0}_{T, \eta, \circ}[X_t])]. \tag{3.7}$$

PROOF. We first note that the cases of $G(x) = e^{\alpha|x|}$ and $G(x) = \exp(\alpha|x|^2)$ follows from the case of $G(x) = |x|^p$, $p \in \mathbb{N}$, because of $\exp(\alpha|x|) = \sum(\alpha|x|)^p/p!$ and $\exp(\alpha|x|^2) = \sum(\alpha|x|^2)^p/p!$. Now we consider discretization of measures similarly to (3.3). Since $\psi$, $\psi_s$ and $w$ are convex functions, these discretized measures have log-concave densities. Hence these inequalities for these discretized measures follow from the Brascamp–Lieb inequality (see [1], Theorem 5.1).

Hence taking the limit $n \to \infty$ completes the proof. □

Combining these lemmas, we see the proof of Theorem 2.1 is reduced to the bound on expectations. Let

$$C_{2.1} = \sup\{E_{T,\eta}^{\psi_s,0}[\exp \alpha |X_t - E_{T,\eta}^{\psi_s,0}[X_t]|];\ |t| \le T < \infty\},$$
$$C_{2.2} = \sup\{E_{T,\eta}^{\psi_s,0}[\exp 2\alpha |X_t - E_{T,\eta}^{\psi_s,0}[X_t]|^2];\ |t| \le T < \infty\}.$$

Then, since $\mu_{T,\eta}^{\psi_s,0}$ are Ornstein–Uhlenbeck processes, $C_{2.1}$ is finite for all $\alpha > 0$ and $C_{2.2}$ is finite for small $\alpha > 0$. These values depend only on $\alpha$, $\|\eta\|_\infty$ and $\psi_s$.

LEMMA 3.3. *Let $\varphi$, $w$, $\psi$, $\xi$ and $\eta$ be as in Lemma* 3.1.

(i) *Assume*

$$C_{2.3} := \sup\{|E_{T,\eta}^{\psi,w}[X_t]|;\ |t| \le T < \infty\} < \infty. \tag{3.8}$$

*Then for each* $|t| \le T < \infty$,

$$E_{T,\xi}^{\varphi,w}[\exp[\alpha \max\{0, X_t\}]] \le \exp(\alpha C_{2.3}) C_{2.1}, \tag{3.9}$$

$$E_{T,\xi}^{\varphi,w}[\exp[\alpha \max\{0, X_t\}^2]] \le \exp(2\alpha C_{2.3}^2) C_{2.2}. \tag{3.10}$$

(ii) *The same result also holds for free boundary conditions.*

PROOF. Let $F(X) = \exp[\alpha \max\{0, X_t\}]$ and $G(x) = e^{\alpha|x|}$. Then

$$\begin{aligned}
E_{T,\xi}^{\varphi,w}[F(X)] &\le E_{T,\eta}^{\psi,w}[F(X)] && \text{by Lemma 3.1}\\
&\le E_{T,\eta}^{\psi,w}[G(X_t)] && \text{by } F(X) \le G(X_t)\\
&\le \exp(\alpha C_{2.3}) E_{T,\eta}^{\psi,w}[G(X_t - E_{T,\eta}^{\psi,w}[X_t])]\\
&\le \exp(\alpha C_{2.3}) E_{T,\eta}^{\psi_s,0}[G(X_t - E_{T,\eta}^{\psi_s,0}[X_t])] && \text{by Lemma 3.2.}
\end{aligned}$$

This implies (3.9). The proofs of (3.10) and (ii) are similar, so we omit them. □

PROOF OF THEOREM 2.1. Let us take $\alpha$ so small in such a way that $C_{2.2} < \infty$. By (P2) there exists a right-dominator symmetric around $m > 0$. Let us take $b \ge 0$ in such a way that $b = \max\{m, \|\xi\|_\infty\}$. Then by the remark after Definition 2.1, there exists a right-dominator $\psi$ of $\varphi$ symmetric around $b$. Let $\eta \equiv b$. By symmetry,

$$E_{T,\eta}^{\psi,w}[X_t] = b \quad \text{for all } t$$

in both boundary conditions, which means (3.8). So by Lemma 3.3 we have (3.10).

By using a left-dominator we obtain also (3.10) for $-X_t$. So, combining these two inequalities yields Theorem 2.1. □

REMARK 3.1. (i) In the proof of Theorem 2.1 we used only the convexity of $v$ in the assumptions of $v$ in (W2). Accordingly, Theorem 2.1 still holds, if we replace (W2) by the convexity of $v$ which may take the value $\infty$ and if $Z_{T,\xi}$ in (2.6) is finite (this is necessary to define $\mu^{\varphi,w}_{T,\xi}$).

(ii) Suppose $\varphi$ is time inhomogeneous; $\varphi = \varphi(t,x)$. Suppose for each $t \in I$ there exist right- and left-dominators $\psi^{\pm}(t,\cdot)$ of $\varphi(t,\cdot)$ symmetric around $\pm m$ and $\{\psi^{\pm}\}''(t,x) \geq 2a$. If we can take $m$ and $a$ are independent of $t \in I$, then the same conclusion of Theorem 2.1 holds.

**4. Proof of Theorem 2.2.** In this section we will prove Theorem 2.2 in a slightly general framework.

THEOREM 4.1. *Assume* (A1), (W1), (W3) *and $v$ is convex such that $v = \infty$ for $|x| \geq a$* $(0 < a \leq \infty)$ *and*

$$|v(x)| \leq C_{4.1} \exp(p_1|x|) \quad \textit{for all } |x| < a \textit{ for some } C_{4.1},\ p_1 > 0. \tag{4.1}$$

*Then there exists a Gibbs measure for* $(\varphi, w)$.

REMARK 4.1. (i) Since (2.2) implies (4.1), Theorem 2.2 follows from Theorem 4.1.

(ii) Let $\varphi(x) = x^2$, $v(x) = 0$ for $|x| < 1$ and $v(x) = \infty$ for $|x| \geq 1$, $\rho(t) = 1$ for $|t| \leq 1$ and $\rho(t) = 0$ for $|t| \geq 1$. Then the assumptions of Theorem 4.1 are satisfied.

Throughout this section we fix $\varphi$ and $w$ and often suppress them from the notation. Let $\mu^{\varphi,w}_{T,0,\circ}$ be the probability measure on $C([-T,T];\mathbb{R})$ given by (2.9) obeying free boundary conditions with $\xi \equiv 0$. Let $\mu_T$ be its extention to $C(\mathbb{R};\mathbb{R})$ given by

$$\mu_T = \mu^{\varphi,w}_{T,0,\circ} \quad \text{on } \sigma[X_t; |t| \leq T],$$
$$\mu_T(X_u = 0 \text{ for all } |u| \geq T) = 1.$$

Let $\mathbf{A} = \{\mathbf{a} = (a_j)_{j\in\mathbb{N}};\ a_j > 0\}$. For $\mathbf{a}, \mathbf{b} \in \mathbf{A}$ we write $\mathbf{a} \leq \mathbf{b}$ if $a_j \leq b_j$ for all $j \in \mathbb{N}$. For $k \in \mathbb{N}$ and $\mathbf{a} = (a_j) \in \mathbf{A}$ we set

$$\Xi(k,\mathbf{a}) = \left\{ \xi \in C(\mathbb{R};\mathbb{R});\ \int_{\mathbb{R}} \rho_0(u) \exp(p_1|\xi_u|)\,du \leq k,\ |\xi_{\pm j}| \leq a_j \text{ for all } j \in \mathbb{N} \right\}.$$

By Theorem 2.1 and (W3) there exist increasing sequences $k^n$ and $\mathbf{a}^n \in \mathbf{A}$ such that

$$\liminf_{T\to\infty} \mu_T(\Xi(k^n,\mathbf{a}^n)) \geq 1 - \frac{1}{n}. \tag{4.2}$$

We set $\Xi_n = \Xi(k^n,\mathbf{a}^n)$. Note that by construction $\Xi_n \subset \Xi_{n+1}$. Let

$$\mu^n_T(\cdot) = \mu_T(\cdot \cap \Xi_n).$$

Note that $\mu_T^n$ are measures on $C(\mathbb{R};\mathbb{R})$ whose total volumes are less than or equal to 1, and $\{\mu_T^n\}$ is increasing in $n$ and compatible in the sense that

$$\mu_T^{n+1}(A) = \mu_T^n(A) \quad \text{for } A \subset \Xi_n. \tag{4.3}$$

For $S \le T \in \mathbb{N}$, we set

$$\mathbf{w}_{S,T}(t, x, \xi) = \int_{S \le |u| \le T} w(t-u, x-\xi_u)\, du.$$

LEMMA 4.2. *There exists a constant $C_{4.2}$ such that*

$$\mathbf{w}_{S,T}(t, x, \xi) \le C_{4.2} \exp(p_1|x|) \tag{4.4}$$

*for all $\xi \in \Xi_n$ such that $\mathbf{w}_{S,T}(t, x, \xi) < \infty$.*

PROOF. Let $C_{4.1}$ be the constant given in (4.1). Then

$$\begin{aligned}
\mathbf{w}_{S,T}(t, x, \xi) &\le \int_{S \le |u| \le T} \rho(t-u) C_{4.1} \exp(p_1|x-\xi_u|)\, du \\
&\le C_{4.1} \exp(p_1|x|) \int_{S \le |u|} \rho(t-u) \exp(p_1|\xi_u|)\, du \\
&\le C_{4.1} \exp(p_1|x|) C_{4.3} k^n.
\end{aligned}$$

Here $C_{4.3}$ is a constant such that $\sup_{|t| \le S} \rho(u-t) \le C_{4.3}\rho_0(u)$ for all $|u| \ge S$. We remark by (W2) $C_{4.3} < \infty$. Taking $C_{4.2} = C_{4.1}C_{4.3}k^n$ completes the proof. $\square$

Let $\mathscr{W}_{S,x}$ denote the Brownian bridge conditioned $x = (x_-, x_+) \in \mathbb{R}^2$ at $\pm S$ and $E^{\mathscr{W}}_{S,x}$ its expectation. Let $Z: \mathbb{R}^2 \to \mathbb{R}$ such that

$$Z(x) = E^{\mathscr{W}}_{S,x}\left[\exp\left(-H_S(X) - C_{4.2} \int_{|t| \le S} \exp[p_1|X_t|]\, dt\right)\right].$$

Let

$$\begin{aligned}
W_{S,T}(X, \xi) &= \int_{|t| \le S} \mathbf{w}_{S,T}(t, X_t, \xi)\, dt, \\
Z_{S,T,\xi} &= E^{\mathscr{W}}_{S,\xi}[\exp(-H_S(X) - W_{S,T})].
\end{aligned}$$

LEMMA 4.3. *Let $C_{4.4} = v(0) \int_{|t| \le S \le |u|} \rho(t-u)\, dt\, du$. Then*

$$W_{S,T} \ge C_{4.4}, \tag{4.5}$$

$$Z_{S,T,\xi} \ge Z(\xi_{-S}, \xi_S) \quad \textit{for all } \xi \in \Xi_n. \tag{4.6}$$

PROOF. Since $w(t,x) \ge \rho(t)v(0)$, (4.5) is clear; (4.6) follows from (4.4) and the definition of $Z(x)$ immediately. $\square$

LEMMA 4.4. *There exists a constant $C_{4.5} = C_{4.5}(n, S)$ such that*

$$\sup_{S \le T} E_T\big[|X_s - X_t|^4; \Xi_n\big] \le C_{4.5}|s-t|^2 \tag{4.7}$$

*for all $|s|, |t| \le S$, where $E_T$ is the expectation with respect to $\mu_T$.*

PROOF. For $x = (x_-, x_+) \in \mathbb{R}^2$ let $f(x) = E^{\mathscr{W}}_{S,x}[|X_s - X_t|^4 \exp(-H_S(X))]$. Since $H_S(X) \ge -C_{4.6}$ for some $C_{4.6} = C_{4.6}(S) \ge 0$, there exists a constant $C_{4.7} = C_{4.7}(n, S)$ such that

$$f(x) \le C_{4.7}|t-s|^2 \quad \text{for all } |s|, |t| \le S,\ |x_\pm| \le a^n_S. \tag{4.8}$$

Here $\mathbf{a}^n = (a^n_k)_{k\in\mathbb{N}}$. Let $A(\xi) = \{X \in C(\mathbb{R};\mathbb{R});\ \widetilde{X} \in \Xi_n\}$, where $\widetilde{X}(t) = X(t)$ if $|t| \le S$ and $\widetilde{X}(t) = \xi(t)$ if $|t| \ge S$. Taking the conditional expectation with respect to $\sigma[X_t; |t| \ge S]$, we see

$$\begin{aligned}
&E_T[|X_s - X_t|^4; \Xi_n] \\
&\quad = E_T\big[(Z_{S,T,\xi})^{-1} E^{\mathscr{W}}_{S,\xi}\big[|X_s - X_t|^4 \exp(-H_S(X) - W_{S,T}(X,\xi));\ A(\xi)\big]\big] \\
&\quad \le E_T\big[Z(\xi_{-S}, \xi_S)^{-1} E^{\mathscr{W}}_{S,\xi}\big[|X_s - X_t|^4 \exp(-H_S(X));\ A(\xi)\big]\big] \exp(-C_{4.4}), \\
&\quad \le \left\{\int_{|x_\pm| \le a^n_S} Z(x)^{-1} f(x) m(x)\,dx\right\} \exp(-C_{4.4}) \\
&\qquad\qquad\qquad \text{where } m(x)\,dx \text{ is the distribution of } \mu_T \circ (X_{-S}, X_S)^{-1}, \\
&\quad \le C_{4.7}|s-t|^2 \left\{\int_{|x_\pm| \le a^n_S} Z(x)^{-1} m(x)\,dx\right\} \exp(-C_{4.4}).
\end{aligned}$$

Here we used Lemma 4.3 for the third line and (4.8) for the fifth line. Hence $C_{4.5} := C_{4.7}\{\sup_{|x_\pm| \le a^n_S} Z(x)^{-1}\}\{\exp(-C_{4.4})\}$ satisfies (4.7). $\square$

LEMMA 4.5. *There exists a probability measure $\mu$ on $C(\mathbb{R};\mathbb{R})$ and a subsequence $\{\mu_{T'}\}$ of $\{\mu_T\}_{T\in\mathbb{N}}$ such that $\lim_{T'\to\infty} \mu_{T'} = \mu$ weakly in $C(\mathbb{R};\mathbb{R})$.*

PROOF. By Lemma 4.4 for each $n \in N$, we see that $\mu^n_T$ is relatively compact as measures on $C(\mathbb{R};\mathbb{R})$ equipped with the vague topology. So by the diagonal method, we can choose a subsequence $\mu^n_{T'}$ such that $\mu^n_{T'}$ converge in $C(\mathbb{R};\mathbb{R})$ in vague topology for all $n \in \mathbb{N}$. Recall that $\mu^n_T$ are compatible in $n$ and increasing as we see in (4.3). Hence its increasing limit $\mu_{T'} = \lim_{n\to\infty} \mu^n_{T'}$ is also a convergent sequence of measures on $C(\mathbb{R};\mathbb{R})$ in vague topology. So let $\mu$ denote its limit. Then $\mu$ is a measure on $C(\mathbb{R};\mathbb{R})$ whose total volume is less than or equal to 1.

Since $\Xi_n$ are closed sets in $C(\mathbb{R};\mathbb{R})$, we have

$$\mu(\Xi_n) \ge \limsup_{T'} \mu_{T'}(\Xi_n) \ge 1 - \frac{1}{n}.$$

We thus see that $\mu$ is a probability measure on $C(\mathbb{R};\mathbb{R})$ such that

$$\mu\Big(\bigcup_n \Xi_n\Big) = 1. \tag{4.9}$$

Since $\mu$ is a probability measure, $\mu_{T'}$ converge to $\mu$ not only in the vague topology but also weakly in $C(\mathbb{R};\mathbb{R})$. This completes the proof. □

PROOF OF THEOREM 4.1. Let $\Upsilon_n = \{\xi \in C(\mathbb{R};\mathbb{R});\ \int_{|t|\geq T} \exp(p_1|\xi_t|)\rho_0(t)\,dt \leq n\}$. Then from (4.9) and (W2) we easily see $\mu(\bigcup_n \Upsilon_n) = 1$. Let $\{\mu_U\}_U$ be a sequence of probability measures converging to $\mu$ obtained by Lemma 4.5. (We change the notation here from $\mu_{T'}$ to $\mu_U$). Let $E^U$ and $E^\mu$ denote the expectation with respect to $\mu_U$ and $\mu$, respectively.

Let $\mathscr{F}_T = \sigma[X_t;\ |t| \leq T]$ and $\mathscr{F}_T^* = \sigma[X_t;\ |t| \geq T]$. Let $f$ be a bounded, continuous $\mathscr{F}_T$-measurable function. We set

$$F(\xi) = Z_{T,\xi}^{-1} E_{T,\xi}^{\mathscr{W}}[f(X)\exp(-H_T(X,\xi))],$$
$$F_U(\xi) = Z_{T,U,\xi}^{-1} E_{T,\xi}^{\mathscr{W}}[f(X)\exp(-H_{T,U}(X,\xi))],$$

where $H_{T,U}(X,\xi) = H_T(X) + W_{T,U}(X,\xi)$ and $Z_{T,U,\xi}$ is the normalizing constant of Gibbs measure $\mu^U$. Note that

$$F_U(\xi) = E^U\big[f(X)|\mathscr{F}_T^*\big](\xi). \tag{4.10}$$

By (A2) and the definition of $\Upsilon_n$ we easily see that $F$ and $F_U$ are bounded and continuous in $\xi$ on $\Upsilon_n$. Moreover

$$\lim_{U\to\infty} \sup_{\xi\in\Upsilon_n} |F(\xi) - F_U(\xi)| = 0. \tag{4.11}$$

Let $g$ be a bounded, continuous $\mathscr{F}_T^*$-measurable function whose support is contained in $\Upsilon_n$. Then $fg$ and $Fg$ are bounded continuous functions supported on $\Upsilon_n$. Since $\mu_U$ converge to $\mu$ weakly in $C(\mathbb{R};\mathbb{R})$, we have

$$\begin{aligned}
E^\mu[f(\xi)g(\xi)] &= \lim_{U\to\infty} E^U[f(\xi)g(\xi)] \\
&= \lim_{U\to\infty} E^U[E^U[f(\xi)|\mathscr{F}_T^*]g(\xi)] \\
&= \lim_{U\to\infty} E^U[F_U(\xi)g(\xi)] \qquad \text{by (4.10)} \\
&= \lim_{U\to\infty} E^U[F(\xi)g(\xi)] \qquad \text{by (4.11)} \\
&= E^\mu[F(\xi)g(\xi)].
\end{aligned}$$

Here for the second line we used the assumption that $g$ is $\mathscr{F}_T^*$-measurable. We thus obtain

$$E^\mu[f(\xi)g(\xi)] = E^\mu[Z_{T,\xi}^{-1} E_{T,\xi}^{\mathscr{W}}[f(X)\exp(-H_T(X,\xi))]g(\xi)]$$

for all bounded, continuous $\mathscr{F}_T$-measurable $f$, and bounded, continuous $\mathscr{F}_T^*$-measurable $g$ with support contained in *big* $\cup_n \Upsilon_n$. Since $\mu(\bigcup_n \Upsilon_n) = 1$, this implies (2.5). □

**5. Uniqueness of Gibbs measure.** In this section we prove the uniqueness of Gibbs measure. We first recall Papangelou's uniqueness result in one-time parameter dimension. This result is originally for discrete time parameter; we modify it for continuous time case suitable for our purpose. In the later half of this section we will apply this to prove Theorem 2.3.

Let $\mathsf{S}$ be a Polish space. We regard $\mathsf{S}$ as the space of spins. We set

$$\mathsf{C}^- = C((-\infty, 0]; \mathsf{S}), \qquad \mathsf{C}^+ = C([0, \infty); \mathsf{S}), \qquad \mathsf{C} = \mathsf{C}^- \times \mathsf{C}^+.$$

We endow these spaces with the compact uniform topology. These spaces are Polish spaces with this topology.

Let $\mathscr{I} = \{[a, b];\ a < b \in \mathbb{Z}\}$. For $I = [a, b]$, $J = [c, d] \in \mathscr{I}$ with $I \subset J$, we set $J \backslash I = [c, a] \cup [b, d]$. For $\eta \in C(J \backslash I; \mathsf{S})$ and $\xi \in \mathsf{C}$, we set $\eta \bullet \xi = (\xi^-, \xi^+) \in \mathsf{C}$ by

$$\eta \bullet \xi_t = \begin{cases} \xi^-_{t-(c-a)}, & \text{if } t < c - a, \\ \eta_{t+a}, & \text{if } c - a \le t \le 0, \\ \eta_{t+b}, & \text{if } 0 \le t \le d - b, \\ \xi^+_{t-(d-b)}, & \text{if } d - b < t. \end{cases}$$

Hereafter, whenever we write $\eta \bullet \xi$, both $\eta_c = \xi^-_0$ and $\eta_d = \xi^+_0$ are implicitly assumed. For a subset $A \subset \mathbb{R}$, we denote by $\pi_A$ the projection $\pi_A$: $C(\mathbb{R}; \mathsf{S}) \to C(A; \mathsf{S})$ given by $\pi_A(X) = X|_A$.

DEFINITION 5.1. Let $\mathsf{C}_0 \subset \mathsf{C}$ be a Borel subset. Let $\mathbf{d}$ be a metric on $\mathsf{C}_0$. We assume $(\mathsf{C}_0, \mathbf{d})$ is a Polish space, and the topology induced by $\mathbf{d}$ is stronger than the relative topology as a subset of $\mathsf{C}$, but the $\sigma$-fields generated by the two topology in $\mathsf{C}_0$ are the same. Let $Q = \{Q_I(\cdot, \xi)\}_{I \in \mathscr{I}, \xi \in \mathsf{C}_0}$, where $Q_I(\cdot, \xi)$ is a probability measures on $C(I; \mathsf{S})$ for each $\xi \in \mathsf{C}_0$ and for each $A \in \mathscr{B}(C(I; \mathsf{S}))$, $Q_I(A, \cdot)$ is $\mathscr{B}(\mathsf{C}_0)$-measurable. The triplicate $(Q, \mathsf{C}_0, \mathbf{d})$ is called a specification if it satisfies the compatibility condition; for each $I \subset J \in \mathscr{I}$ and $A \in \mathscr{B}(C(I; \mathsf{S}))$, $\xi \in \mathsf{C}_0$,

$$\int_{C(J \backslash I; \mathsf{S})} Q_I(A, \eta \bullet \xi) Q_J(\pi_{J \backslash I} \in d\eta) = Q_J(\pi_I \in A, \xi). \tag{5.1}$$

For $I \in \mathscr{I}$ we define $\pi^*_I$: $C(\mathbb{R}; \mathsf{S}) \to \mathsf{C}$ by

$$\pi^*_I(X) = \begin{cases} X_{t-b}, & \text{if } t \ge b, \\ X_{t-a}, & \text{if } t \le a. \end{cases} \tag{5.2}$$

For $I \in \mathscr{I}$, we define $P^*_I(\cdot) = P(\pi^*_I \in \cdot)$.

DEFINITION 5.2. We say a probability measure $P$ on $C(\mathbb{R}; \mathsf{S})$ is admitted by the specification $(Q, \mathsf{C}_0, \mathbf{d})$ if

$$P^*_I(\mathsf{C}_0) = 1 \quad \text{for all } I \in \mathscr{I}, \tag{5.3}$$

$$P(\pi_I \in \cdot | \pi^*_J = \xi) = Q_{I,J}(\cdot, \xi) \quad \text{for } P^*_J\text{-a.s. } \xi. \tag{5.4}$$

Here we set $Q_{I,J}(\cdot, \xi) = Q_J(\pi_I \in \cdot, \xi)$ for $I \subset J$.

Let $\|\cdot\|_{\mathrm{total}}$ denote the total variation of signed measure. For $\xi = (\xi^-, \xi^+) \in \mathsf{C}$, we set $\xi_0 = (\xi_0^-, \xi_0^+) \in \mathsf{S}^2$. We assume $(Q, \mathsf{C}_0, \mathbf{d})$ satisfies the following:

(U1) $Q$ is translation invariant; $Q_I = Q_{I+a}$ for each $I \in \mathscr{I}$, $a \in \mathbb{Z}$.

(U2) $\lim_{n\to\infty} \|Q_I(\cdot, \xi_n) - Q_I(\cdot, \xi)\|_{\mathrm{total}} = 0$ if $\lim_{n\to\infty} \mathbf{d}(\xi_n, \xi) = 0$ and $\xi_{n,0} = \xi_0$.

(U3) For each $\xi, \eta \in \mathsf{C}_0$ such that $\xi_0 = \eta_0$, $Q_I(\cdot, \xi)$ and $Q_I(\cdot, \eta)$ are mutually absolutely continuous.

(U4) There exists a family of Borel subsets $\{M_I(m)\}_{I\in\mathscr{I}, m\in\mathbb{N}}$ in $C(I; \mathsf{S})$ such that

$$M_I(m) \subset M_I(n) \quad \text{if } m \le n,$$

and satisfying the following: for each $\varepsilon > 0$, $\mathbf{d}$-compact set $\mathsf{K} \subset \mathsf{C}_0$, $m \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that

$$Q_{I,J}(M_I(n), \eta \bullet \xi) > 1 - \varepsilon$$

for all $I, J, K \in \mathscr{I}$ such that $I \subset J \subset K$, $\eta \in M_{K\setminus J}(m)$ and $\xi \in \mathsf{K}$.
Here $M_{K\setminus J}(m) = M_{[c,a]}(m) \times M_{[b,d]}(m)$ for $J = [a, b]$ and $K = [c, d]$.

(U5) For each $\varepsilon > 0$, any $\mathbf{d}$-compact set $\mathsf{K}$, any $m, n \in \mathbb{N}$ there exist $k \in \mathbb{N}$ such that

$$\|Q_I^n(\cdot, \zeta \bullet \xi) - Q_I^n(\cdot, \zeta \bullet \eta)\|_{\mathrm{total}} < \varepsilon \tag{5.5}$$

for all $I \subset J \in \mathscr{I}$ with $|J\setminus I|_* \ge k$, $\zeta \in M_{J\setminus I}(m)$ and $\xi, \eta \in \mathsf{K}$ with $\xi_0 = \eta_0$.
Here $|J\setminus I|_* = \min\{a - c, d - b\}$ for $I = [a, b]$ and $J = [c, d]$, and $Q_I^n$ is given by

$$Q_I^n(\cdot, \xi) = \begin{cases} \dfrac{Q_I(\cdot \cap M_I(n), \xi)}{Q_I(M_I(n), \xi)}, & \text{if } Q_I(M_I(n), \xi) > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5.6}$$

We remark, if $\xi_0 \ne \eta_0$, then $Q_I(\cdot, \xi)$ and $Q_I(\cdot, \eta)$ are always mutually singular because $Q$ is a specification of a probability measure supported on continuous paths.

DEFINITION 5.3. A probability measure $P$ on $C(\mathbb{R}; \mathsf{S})$ admitted by the specification $(Q, \mathsf{C}_0, \mathbf{d})$ is said to be $\mathbf{d}$-tame if for every $\varepsilon > 0$ there exists a $\mathbf{d}$-compact set $\mathsf{K} \subset \mathsf{C}_0$ such that

$$P_I^*(\mathsf{K}) \ge 1 - \varepsilon \quad \text{for all } I \in \mathscr{I}. \tag{5.7}$$

REMARK 5.1. Suppose that $P$ is $\mathbb{Z}$-translation invariant (i.e., $P$ is invariant under the translation $X_t \to X_{t+z}$ for all $z \in \mathbb{Z}$) and that $P_I^*(\mathsf{C}_0) = 1$ for each $I$. Then $P$ is $\mathbf{d}$-tame because $\mathsf{C}_0$ is a Polish space.

THEOREM 5.1. *Any specification* $(Q, \mathsf{C}_0, \mathbf{d})$ *satisfying* (U1)–(U5) *admits at most one tame probability measure on* $(C(\mathbb{R}; \mathsf{S}), \mathscr{B}(C(\mathbb{R}; \mathsf{S})))$. *Such a probability measure*, *if it exists*, *is translation invariant*.

This theorem was proved by Papangelou [8] when the time parameter is discrete. We modify Hypotheses (H1)–(H5) in Papangelou to (U1)–(U5) in order to fit our problem. The proof of Theorem 5.1 is similar to Theorem 5.1 in Papangelou. So we omit it.

In the rest of this section we will prove Theorem 2.3 by using Theorem 5.1. So, we find out a suitable specification. Let $\mathsf{S} = \mathbb{R}$ and for $\xi = (\xi^+, \xi^-) \in \mathsf{C}_0$ we set $\mathbf{d}(\xi) = \mathbf{d}_0(\xi) + \mathbf{d}_1(\xi)$, where $\mathbf{d}_i(\xi) = d_i(\xi^+) + d_i(\xi^-_{\cdot})$ $(i = 0, 1)$ and for $\xi \in C([0, \infty); \mathbb{R})$, $d_i(\xi)$ $(i = 0, 1)$ are defined by

$$d_0(\xi) = \left\{ \sum_{i=0}^{\infty} 2^{-i} \min\left\{ \sup_{i \le t < i+1} |\xi_t|, 1 \right\} \right\},$$

$$d_1(\xi) = \left\{ \int_0^{\infty} \rho_0(t) |\xi_t|^{p_2} \, dt \right\}^{1/p_2}.$$

Here $p_2 > p_0 + 1$. We write the metric $\mathbf{d}(\xi, \eta) = \mathbf{d}(\xi - \eta)$ by the same symbol $\mathbf{d}$. Note that $\mathbf{d}_0$ corresponds to the compact uniform topology; accordingly, the topology induced by $\mathbf{d}$ is stronger than the compact uniform topology. Let

$$\mathsf{C}_0 = \{\xi; \mathbf{d}_1(\xi) < \infty\}.$$

For $I \in \mathscr{I}$ and $\xi \in \mathsf{C}_0$, we set

$$\begin{aligned} H_I(X, \xi) = {} & \int_I \varphi(X_t) \, dt + \tfrac{1}{2} \int_{I \times I} w(t - u, X_t - X_u) \, dt \, du \\ & + \int_{I \times (\mathbb{R} \setminus I)} w(t - u, X_t - \xi_u^I) \, dt \, du \end{aligned} \tag{5.8}$$

Here for $\xi = (\xi^-, \xi^+) \in \mathsf{C}_0$ and $I = [a, b] \in \mathscr{I}$, we define

$$\xi^I \in C((-\infty, a], [b, \infty); \mathbb{R})$$

by

$$\xi_t^I = \begin{cases} \xi^-_{t-a}, & \text{for } t \le a, \\ \xi^+_{t-b}, & \text{for } t \ge b. \end{cases} \tag{5.9}$$

Similarly to (2.6), for $I = [a, b] \in \mathscr{I}$ and $\xi \in \mathsf{C}_0$ we define $\mu_{I, \xi}^{\varphi, w}$ by

$$\mu_{I, \xi}^{\varphi, w} = (Z_{I, \xi})^{-1} \exp[-H_I(X, \xi)] \, d\mathscr{W}_{I, \xi}. \tag{5.10}$$

Here $Z_{I, \xi}$ is the normalization and $\mathscr{W}_{I, \xi}$ is the Brownian bridge on $C(I; \mathbb{R})$ conditioned $\xi_0 = (\xi_0^-, \xi_0^+)$ on the boundary. In the case of the free boundary condition, we write $\mu_{I, \xi, \circ}^{\varphi, w}$.

Let $Q_I(\cdot, \xi) = \mu_{I, \xi}^{\varphi, w}$. Then it is clear that $(Q, \mathsf{C}_0, \mathbf{d})$ is a specification. We want to check $(Q, \mathsf{C}_0, \mathbf{d})$ satisfies (U1)–(U5). We take for $I = [a, b] \in \mathscr{I}$ and $m \in \mathbb{N}$,

$$M_I(m) = \{X \in C(I; \mathbb{R});\ |X_a|, |X_b|, A(X) \le m\}, \tag{5.11}$$

where

$$A(X) = \sup_{0<t<b-a} \left\{ \frac{1}{t} \int_{[a,\, a+t] \cup [b-t,\, b]} |X_t|^{p_2}\, dt \right\}.$$

For $\xi \in \mathsf{C}_0$ and $I \in \mathscr{I}$ we set

$$\varphi_{I,\,\xi}(t, x) = \varphi(x) + \int_{u \notin I} w(t-u, x-\xi_u^I)\, du. \tag{5.12}$$

Then $\varphi_{I,\,\xi}(t, x)$ is a time inhomogeneous free potential on $I$.

THEOREM 5.2. *Suppose* (A1), (W1), (W3), *and* (2.10). *Suppose that $v$ is convex. Assume for each $I = [a, b]$ and $\xi \in \mathsf{C}_0$,*

(5.13) *right- and left-dominators $\psi^{\pm}$ of $\varphi_{I,\,\xi}(t, x)$ symmetric around $\pm m$ with $\{\psi^{\pm}\}'' \geq 2\hat{a} > 0$ such that $m$ and $\hat{a}$ depend only on $\mathbf{d}_1(\xi)$ and $|b-a|$ exist.*

*Let $(Q, \mathsf{C}_0, \mathbf{d})$ and $M_I(m)$ be as above. Then* (U1)–(U5) *are satisfied. In particular, the associated $\mathbf{d}$-tame Gibbs measure is unique.*

PROOF. (U1) and (U3) are clear. We next prove (U2): Since $\xi_{n,0} = \xi_0$ for all $n$, we have $\mathscr{W}_{I,\,\xi_n} = \mathscr{W}_{I,\,\xi}$ by definition. So for (U2) it is enough to prove

$$\lim_{n\to\infty} \int_{C(I;\mathbb{R})} |\exp(-H_I(X, \xi)) - \exp(-H_I(X, \xi_n))|\, d\mathscr{W}_{I,\,\xi} = 0. \tag{5.14}$$

By $\lim_{n\to\infty} \mathbf{d}(\xi_n, \xi) = 0$, (A2) and $p_2 > p_0 + 1$, we have

$$\lim_{n\to\infty} H_I(X, \xi_n) = H_I(X, \xi) \quad \text{for each } X.$$

By assumption $H_I(X, \xi_n)$ and $H_I(X, \xi)$ are bounded from below. Thus, applying the Lebesgue convergence theorem, we have (5.14).

We next prove (U4). Let $\mathsf{K}$ be a $\mathbf{d}$-compact set and $C_{5.1} = \max\{\mathbf{d}_1(\xi); \xi \in \mathsf{K}\}$. Then $C_{5.1} < \infty$.

For $\xi \in \mathsf{K}$ and $\eta \in M_{K\setminus J}(m)$ we regard $\varphi_{J,\eta\bullet\xi}(t, x)$ as the time inhomogeneous free potential on $J$ and consider the associated Gibbs measure,

$$\mu_J^{\eta\bullet\xi} := \mu_{J,\,(\eta\bullet\xi)^J,\,\circ}^{\varphi_{J,\,\eta\bullet\xi},\, w},$$

with interaction potential $w$ on $C(J;\mathbb{R})$. Let $E_J^{\eta\bullet\xi}$ denote the expectation with respect to $\mu_J^{\eta\bullet\xi}$.

By Remark 3.1 and (5.13) there exists a constant $C_{5.2}$, depending only on $C_{5.1}$ and $m$, such that

$$E_J^{\eta\bullet\xi}[e^{|X_t|}] \leq C_{5.2} \quad \text{for all } t \in J. \tag{5.15}$$

By (5.15) for each $\varepsilon$ there exists an $n_1$ such that

$$\mu_J^{\eta\bullet\xi}(\{X;\pi_{J\setminus I}(X)\in\{\zeta;\mathbf{d}_1(\zeta\bullet\eta\bullet\xi)\le n_1\}\})\ge 1-\varepsilon/4. \tag{5.16}$$

For $\xi \in \mathsf{K}$, $\eta \in M_{K\setminus J}(m)$ and $\zeta \in \{\zeta;\mathbf{d}_1(\zeta\bullet\eta\bullet\xi)\le n_1\}$ we consider the time inhomogeneous free potential $\varphi_{I,\,\zeta\bullet\eta\bullet\xi}(t,x)$ on $I$. Then by (5.13) we see there exist right-dominators $\psi$ depending only on $n_1$, $|b-a|$ and $(\xi,\eta,\zeta)$. Let $X^+ = X_t^+ = \max\{0, X_t\}$ and set

$$F_I(X)=\max\{X_a^+, X_b^+, A(X^+)\}.$$

Then $F$ is increasing in the sense of Lemma 3.1. So by Lemma 3.1 we have

$$E_J^{\eta\bullet\xi}[F_I(X)]\le E^{\psi,\,w}_{J,\,\eta\bullet\xi^J,\,\circ}[F_I(X)]. \tag{5.17}$$

By (5.16), (5.17) and Chebyshev's inequality, there exists an $n_2$ such that

$$Q_{I,\,J}(M^+(n_2),\eta\bullet\xi)>1-\varepsilon/2.$$

Here $M^+(n)=\{F_I(X)\le n\}$. By using the left-dominator, we see there exists an $n_3$ such that $Q_{I,\,J}(M^-(n_3),\eta\bullet\xi)>1-\varepsilon/2$. Here $M^-(n)=\{F_I(-X)\le n\}$. Since $M_I(n)=M^+(n)\cap M^-(n)$, we conclude (U4).

We finally prove (U5). For this it is enough to show

$$\lim_{k\to\infty}\sup_{\mathsf{I}_k}\left\{\int_{M_I(n)}\big|\exp(-H_I(X,\zeta\bullet\xi)) - \exp(-H_I(X,\zeta\bullet\eta))\big|\,d\mathscr{W}_{I,\,\xi}\right\}=0. \tag{5.18}$$

Here $\mathsf{I}_k = \{(\xi,\eta,\zeta,I,J);\ \xi,\eta\in\mathsf{K}$ with $\xi_0=\eta_0,\ \zeta\in M_{J\setminus I}(m),\ I,J\in\mathscr{I}$ such that $|J\setminus I|_*\ge k\}$. Since $H_I(X,\zeta\bullet\xi)$ is bounded from below, (5.18) is reduced to

$$\lim_{k\to\infty}\sup_{X\in M_I(n),\,\mathsf{I}_k}|H_I(X,\zeta\bullet\xi)-H_I(X,\zeta\bullet\eta)|=0. \tag{5.19}$$

Here we set $H_I(X,\zeta\bullet\xi)-H_I(X,\zeta\bullet\eta)=0$ in the case $H_I(X,\zeta\bullet\xi)=H_I(X,\zeta\bullet\eta)=\infty$. Note that $H_I(X,\zeta\bullet\xi)=\infty$ [or $H_I(X,\zeta\bullet\eta)=\infty$] if and only if $\int_{t\in I}\varphi(X_t)\,dt=\infty$; so $H_I(X,\zeta\bullet\xi)=\infty$ if and only if $H_I(X,\zeta\bullet\eta)=\infty$. We see that

$$\begin{aligned}
&\sup_{X\in M_I(n),\,\mathsf{I}_k}|H_I(X,\zeta\bullet\xi)-H_I(X,\zeta\bullet\eta)|\\
&\quad=\sup_{X\in M_I(n),\,\mathsf{I}_k}\left|\int_{t\in I,\,u\notin J} w(t-u,X_t-\xi_u^J)-w(t-u,X_t-\eta_u^J)\,dt\,du\right|\\
&\quad\le 2\sup_{X\in M_I(n),\,\mathsf{I}'_k}\left\{\int_{t\in I,\,u\notin J}\big|w(t-u,X_t-\xi_u^J)\big|\,dt\,du\right\},
\end{aligned}$$

where $\mathsf{I}'_k = \{(\xi, I, J);\ \xi \in \mathsf{K},\ I, J \in \mathscr{I}$ such that $|J\backslash I|_* \geq k\}$. Here for the third line we use $(\zeta \bullet \xi)^I_u = \xi^J_u$ for $u \notin J$ and $(\zeta \bullet \xi)^I_u = \zeta_u$ for $u \in I$. By (2.10) we have

$$\lim_{k\to\infty} \sup_{X\in M_I(n), \mathsf{I}'_k} \left\{ \int_{t\in I,\, u\notin J} \left|w(t-u, X_t - \xi^J_u)\right| dt\, du \right\} = 0.$$

This implies (5.19), which completes the proof of (U5).

The second statement is clear from Theorem 5.1. □

LEMMA 5.3. (A1) *and* (A2) *imply* (5.13).

PROOF. Let $w_{I,\xi}(t, x) = \int_{u\notin I} w(t-u, x-\xi^I_u)\, du$. For (5.13) it is enough to show $w_{I,\xi}(t, x)$ satisfies (5.20); for each $I = [a, b]$,

(5.20) right- and left-dominators of $w_{I,\xi}(t, x)$ are symmetric around $\pm m_1$ such that $m_1$ depends only on $b-a$ and $\mathbf{d}_1(\xi^I)$ exist.

Indeed, if $d_1$ and $d_2$ are right-dominators of $\varphi$, and $w_{I,\xi}$ is symmetric around $m$ and $m_1$, respectively, then $R^+_{m_2}(d_1+d_2)$, where $m_2 = \max\{|m|, |m_1|\}$ is such a dominator. Indeed, $R^+_{m_2}(d_1+d_2)$ is symmetric around $m_2$ and

$$R^+_{m_2}(d_1+d_2)'' \geq R^+_{m_2}(d_1)'' \geq d''_1.$$

Accordingly, $\hat{a}$ is taken to be that of $d_1$. We can construct the left-dominator similarly.

Let $\bar{\rho}_t = \int_{u\notin I} \rho(t-u)\, du$. Let $\lambda_t$ be the probability measure on $\mathbb{R}$ given by

$$\lambda_t(A) = \bar{\rho}_t^{-1} \int_{u\notin I} \rho(t-u) 1_A(\xi_u)\, du.$$

Let $\bar{v}(t, x) = \int_{\mathbb{R}} (v(x-y) - v(x)) \lambda_t(dy)$. Then $w_{I,\xi}(t, x) = \bar{\rho}_t v(x) + \bar{\rho}_t \bar{v}(t, x)$. We see

$$\begin{aligned} w'_{I,\xi}(t, x) &= \bar{\rho}_t(v'(x) + \bar{v}'(t, x)) \\ &\geq \bar{\rho}_t\left(v'(x) - u(x)\left\{\sup_{t\in I} \int_{\mathbb{R}} \{1+|y|^{p_0}\}\lambda_t(dy)\right\}\right); \end{aligned} \tag{5.21}$$

by using (2.2) with $\varepsilon = 1/\{\sup_{t\in I} \int_{\mathbb{R}} \{1+|y|^{p_0}\}\lambda_t(dy)\}$, we have

$$\begin{aligned} &\geq \bar{\rho}_t(v'(x) - (v'(x) - v'(x-b))) \\ &= \bar{\rho}_t v'(x-b). \end{aligned}$$

Then $\bar{\rho}_t v(x-b)$ is the right-dominator satisfying (5.13). We can construct the left-dominator similarly. □

THEOREM 5.4. *Suppose* (A1), (A2) *and* (2.10). *Then the* $\mathbf{d}$-*tame Gibbs measure is unique*.

For the proof, the statement follows from Theorem 5.2 and Lemma 5.3.

PROOF OF THEOREM 2.3. Since any translation invariant measures satisfying (2.11) are **d**-tame, we have the first statement from Theorem 5.4. By Theorem 2.1, any limit points $\mu$ of $\{\mu_{T,\xi,\circ}^{\varphi,w}\}_T$ or $\{\mu_{T,\xi}^{\varphi,w}\}$ for $\xi$ with $\|\xi\|_\infty < \infty$ are **d**-tame. Hence by Theorem 5.4 again we obtain the second claim. □

**6. An example of nonuniqueness.** In this section we establish by example that in our general setup there can be several Gibbs measures for a given potential. For the sake of concreteness we choose

$$\varphi(x) = \beta(x^4 - x^2), \qquad \beta > 0 \tag{6.1}$$

and

$$w(t,x) = \alpha\rho(t)\tfrac{1}{2}x^2, \qquad \alpha > 0,\ \rho(t) = (1+|t|)^{-\gamma}. \tag{6.2}$$

By (2.10) we must require $\gamma > 1$. Furthermore, we need

$$1 < \gamma \le 2. \tag{6.3}$$

Let $\langle\cdot\rangle_b(T)$ be the expectation of the finite volume Gibbs measure with outside path $\xi_t = b > 0$ for $|t| \ge T$ and let $\langle\cdot\rangle_b$ be the limit measure.

THEOREM 6.1. *Let $\phi$ and $w$ be given by* (6.1), (6.2). *Then there exist $\alpha$, $\beta$, $b > 0$ such that*

$$\langle X_0\rangle_b \ge m^* > 0. \tag{6.4}$$

REMARK 6.1. Clearly $\langle X_0\rangle_{-b} = -\langle X_0\rangle_b$. Thus (6.4) establishes that there are at least two Gibbs measures for the potential (6.1), (6.2).

PROOF. Working out the square

$$\begin{aligned}
&\tfrac{1}{2}\int_{|t|,\,|u|\le T}\rho(t-u)(X_t-X_u)^2\,dt\,du\\
&\qquad = -\int_{|t|,\,|u|\le T}\rho(t-u)X_tX_u\,dt\,du\\
&\qquad\quad + \int_{|t|\le T}X_t^2\left(\int_{|u|\le T}\rho(t-u)\,du\right)dt,\\
&\tfrac{1}{2}\int_{|t|\ge T}\left(\int_{|u|\le T}\rho(t-u)(b-X_u)^2\,du\right)dt\\
&\qquad = \int_{|u|\le T}\bar\rho(u)\left(\tfrac{1}{2}X_u^2 - bX_u + \tfrac{1}{2}b^2\right)du,
\end{aligned} \tag{6.5}$$

where $\bar{\rho}(u) = \int_{|t+u|\geq T} \rho(t)\,dt$. By Griffiths II [11], $\langle X_t\rangle_b(T)$ is decreasing in the strength of the quadratic part of the potential. Therefore,

$$
\begin{aligned}
\langle X_\tau\rangle_b(T) \geq\ & Z^{-1}\int \mathscr{W}_{T,b}(dX)X_\tau \\
& \times \exp\biggl[-\int_{-T}^{T}(\varphi(X_t)+\alpha' X_t^2)\,dt \\
& \qquad + \frac{\alpha}{2}\int_{-T}^{T}\int_{-T}^{T}\rho(t-u)X_t X_u\,dt\,du \\
& \qquad + b\alpha\int_{-T}^{T}\bar{\rho}(u)X_u\,du\biggr] \\
=:\ & \langle X_\tau\rangle_b'(T)
\end{aligned}
\tag{6.6}
$$

with $\alpha' = 4\alpha\int_0^\infty \rho(t)\,dt$. Here $\mathscr{W}_{T,b}$ is the Brownian bridge on $C([-T,T];\mathbb{R})$ conditioned to $X_{\pm T} = b$.

Let $2T = (2N+1)\delta$ with integer $N$. We introduce the block variables

$$
M_n = \frac{1}{\delta}\int_{(n-1/2)\delta}^{(n+1/2)\delta} dt X_t \quad \text{with } n = -N,\dots,N \tag{6.7}
$$

and use again Griffiths II to decrease the expectation $\langle X_\tau\rangle_b'(T)$. First, we replace $\mathscr{W}_{T,b}$ by $\prod_{n=-N}^{N}\mathscr{W}_{\delta[n-1/2,\,n+1/2],\,0}$, where $\mathscr{W}_{[\cdot,\,\cdot],\,0}$ is the Brownian bridge on the interval $[\cdot,\cdot]$ with 0 boundary condition. Second, we replace $\rho(t-u)$ by $\rho_\delta(t,u)$, where $\rho_\delta$ is the largest function on $[-T,T]^2$ such that $\rho_\delta \leq \rho$, $\rho_\delta$ is constant on each square $\delta([n-\frac{1}{2},n+\frac{1}{2})\times[m-\frac{1}{2},m+\frac{1}{2}))$, $(m\neq n)$, and $\rho_\delta = 0$ on $\delta([m-\frac{1}{2},m+\frac{1}{2})\times[m-\frac{1}{2},m+\frac{1}{2}))$ $m=-N,\dots,N$. Third, we replace $\bar{\rho}$ by $\bar{\rho}_\delta$, where $\bar{\rho}_\delta$ is the largest function such that $\bar{\rho}_\delta \leq \bar{\rho}$, and constant on each interval $[m-\frac{1}{2},m+\frac{1}{2})$. We set $\rho_\delta(n-m) = \rho_\delta(t,u)$ for $t,u\in\delta([n-\frac{1}{2},n+\frac{1}{2})\times[m-\frac{1}{2},m+\frac{1}{2}))$ and $\bar{\rho}_\delta(m) = \bar{\rho}_\delta(t)$ for $t\in\delta[m-\frac{1}{2},m+\frac{1}{2})$. Then

$$
\langle X_0\rangle_b'(T) \geq \langle M_0\rangle_b'(T) \geq \langle M_0\rangle_{b,N}. \tag{6.8}
$$

Here $\langle\,\cdot\,\rangle_{b,N}$ is the expectation with respect to the measure on $\mathbb{R}^{2N+1}$ defined by

$$
\begin{aligned}
Z^{-1}\prod_{n=-N}^{N}\mu(dM_n)\exp\biggl[&\frac{\alpha\delta^2}{2}\sum_{m,\,n=-N,\,m\neq n}^{N}\rho_\delta(n-m)M_n M_m \\
&+\alpha b\delta\sum_{n=-N}^{N}\bar{\rho}_\delta(n)M_n\biggr].
\end{aligned}
\tag{6.9}
$$

We note that by construction $\rho_\delta(n) > 0$ and $\rho_\delta(n) \cong n^{-\gamma}$ for large $n$. Also $\bar{\rho}_\delta(n) = \sum_{m=N-n}^{\infty}\rho_\delta(m) + \sum_{m=N+n}^{\infty}\rho_\delta(m)$. The single site measure $\mu$ is the distribution of $M_0$ under the measure

$$
Z^{-1}\mathscr{W}_{\delta[-1/2,\,1/2],\,0}\exp\biggl[-\int_{-\delta/2}^{\delta/2}(\varphi(X_t)+\alpha' X_t^2)\,dt\biggr]. \tag{6.10}
$$

Let us choose $a > 0$ such that

$$\mu([0, a]) = 2\mu([\sqrt{2}a, \infty)). \tag{6.11}$$

Since $\mu$ is even and $\mu([a, \infty))$ strictly decreasing, there is a unique $a = a(\alpha, \beta)$ satisfying (6.11). We set $b = \delta a$. By the inequality of Wells [2], we have

$$\langle M_0 \rangle_{b, N} \geq \langle \sigma_0 \rangle_{+, N}, \tag{6.12}$$

where $\langle \cdot \rangle_{+, N}$ is the Ising measure with Hamiltonian

$$H_I = -\frac{\alpha \delta^2 a^2}{2} \sum_{m \neq n} \rho_\delta(m - n)\sigma_m \sigma_n \tag{6.13}$$

on the interval $[-N, \ldots, N]$ with $+$ boundary conditions on $\mathbb{Z}\backslash[-N, \ldots, N]$, $\sigma_m = \pm 1, m = -N, \ldots, N$.

Thus altogether we have shown, for $b = \delta a$,

$$\langle X_0 \rangle_b(T) \geq \langle \sigma_0 \rangle_{+, N}. \tag{6.14}$$

For the Ising model (6.13) it is proved that $\langle \sigma_0 \rangle_{+, N} \geq m^* > 0$ uniformly in $N$ provided the coupling is sufficiently large [3, 4]. To check that all conditions can be satified simultaneously, we set $\delta = 1$ and let $\alpha_c$ be the corresponding critical coupling of the Ising model. We note that $a(\alpha, \beta) \to 1$ as $\beta \to \infty$ for fixed $\alpha$. Thus we choose $\alpha = 4\alpha_c$ and $\beta$ so large that $\alpha a(\alpha, \beta) \geq 2\alpha_c$. Then $\langle \sigma_0 \rangle_{+, N} \geq m^* > 0$ uniformly in $N$. □

With some extra effort, one could discuss the phase diagram in the $(\alpha, \beta)$ plane. Since we just wanted to provide an example, we do not enter into details. We should mention, however, that one feature turns out to be qualitatively wrong, which teaches us that we have not yet found the "right" proof. Let us fix $\beta$ sufficiently large and the blocking interval $\delta$. ($\delta$ could be optimized, too, without changing the conclusions.) For small $\alpha$ we have $\langle X_0 \rangle_b = 0$ for any choice of $b$. According to Theorem 1 the bound

$$\langle X_0 \rangle_b \geq m^* > 0 \quad \text{with } b = a\delta$$

holds for sufficiently larger $\alpha$. As we further increase $\alpha$, $\varphi(x) + \alpha' x^2$ obtains a single minimum. Then $a$ will decrease and we can keep the effective coupling, $\alpha\delta^2/a^2$, no longer above criticality. Thus our argument fails for sufficiently large coupling $\alpha$. The reason is that the ferromagnetic inequalities refer only to the off-diagonal piece in (6.5). No monotonicity seems to be available in $\rho(t - u)$ for the full action $\int\int \rho(t - u)(X_t - X_u)^2 \, dt \, du$.

## APPENDIX

LEMMA A.1. *Let* $a: \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ *be a convex function such that* $a(t) = a(-t)$.

(i) *Then for all* $s, t, u, v \geq 0$ *such that* $s + t \leq u + v$, *and* $s, t \leq u$,

$$a(s) + a(t) \leq a(u) + a(v). \tag{A.1}$$

(ii) *Let* $x \wedge y = \min\{x, y\}$ *and* $x \vee y = \max\{x, y\}$. *Then for* $x_1, x_2, y_1, y_2 \in \mathbb{R}$

$$a(x_1 - x_2) + a(y_1 - y_2) - a(x_1 \wedge y_1 - x_2 \wedge y_2) - a(x_1 \vee y_1 - x_2 \vee y_2) \geq 0.$$

PROOF. If $v \geq \max\{s, t\}$, then the claim is clear. Hence, without loss of generality, we can assume $v \leq t$. Then

$$\begin{aligned} a(u) - a(t) &= \int_0^{u-t} a'(t+\theta)\, d\theta \geq \int_0^{u-t} a'(t + \theta - (t - v)) d\theta \\ &= a(u - t + v) - a(v). \end{aligned}$$

Since $u - t + v \geq s$, we complete the proof of (i).

Let $I$ denote the left-hand side of the inequality in (ii). If $x_1 \leq y_1$, $x_2 \leq y_2$, or $x_1 > y_1$, $x_2 > y_2$, then $I = 0$. If $x_1 \leq y_1$, $y_2 \leq x_2$, then we divide the case into six cases:

1. $y_1 \leq y_2$,
2. $x_1 \leq y_2 \leq y_1 \leq x_2$,
3. $x_1 \leq y_2 \leq x_2 \leq y_1$,
4. $y_2 \leq x_1 \leq y_1 \leq x_2$,
5. $y_2 \leq x_1 \leq x_2 \leq y_1$,
6. $y_2 \leq x_2 \leq x_1 \leq y_1$.

In Case 1, $I \geq 0$ follows from (A.1). Indeed, we can take $s = y_2 - x_1$, $t = x_2 - y_1$, $u = x_2 - x_1$ and $v = y_2 - y_1$. Case 2 also follows from (A.1) by taking $s = y_2 - x_1$, $t = x_2 - y_1$, $u = x_2 - x_1$ and $v = y_1 - y_2$. The proof of Cases 3–6 is similar, so we omit it. The case of $x_2 \leq y_2$, $y_1 \leq x_1$ is reduced to the case of $x_1 \leq y_1$, $y_2 \leq x_2$ by interchanging $x_1$ and $y_1$ by $x_2$ and $y_2$, respectively. □

LEMMA A.2. *Let* $a_i, b_i, c_{ij}$ $(1 \leq i, j)$ *be measurable functions on* $\mathbb{R}$ *such that* $a_i - b_i$ *are nondecreasing for all* $i$. *Assume* $c_{ij}$ *are convex and* $c_{ij}(t) = c_{ji}(t) = c_{ij}(-t)$ *for all* $i, j$, *and* $c_{ii} = 0$. *Let* $\mu_i$ $(i = 1, \ldots, n)$ *be Radon measures on* $\mathbb{R}$. *Set*

$$f(x) = \sum_{i=1}^n a_i(x_i) + \sum_{i,j=1}^n c_{ij}(x_i - x_j), \qquad g(x) = \sum_{i=1}^n b_i(x_i) + \sum_{i,j=1}^n c_{ij}(x_i - x_j)$$

*and*

$$A = \int_{\mathbb{R}^n} e^{-f} \prod_{i=1}^n \mu_i(dx_i), \qquad B = \int_{\mathbb{R}^n} e^{-g} \prod_{i=1}^n \mu_i(dx_i).$$

*Assume* $0 < A, B < \infty$. *Set* $F(x) = A^{-1}e^{-f(x)}$ *and* $G(x) = B^{-1}e^{-g(x)}$. *Then*

$$F(x)G(y) \leq F(x \wedge y)G(x \vee y).$$

*Here* $x \wedge y = (\min\{x_i, y_i\})_{i=1,\dots,n}$ *and* $x \vee y = (\max\{x_i, y_i\})_{i=1,\dots,n}$.

PROOF. Let $\Phi = f(x) + g(y) - f(x \wedge y) - g(x \vee y)$. Then the claim is equivalent to $\Phi \geq 0$. We write $\Phi = \sum_{i=1}^n I_i + \sum_{i,\, j=1}^n I_{ij}$, where

$$\begin{aligned} I_i &= a_i(x_i) + b_i(y_i) - a_i(x_i \wedge y_i) - b_i(x_i \vee y_i), \\ I_{ij} &= c_{ij}(x_i - x_j) + c_{ij}(y_i - y_j) \\ &\qquad - c_{ij}(x_i \wedge y_i - x_j \wedge y_j) - c_{ij}(x_i \vee y_i - x_j \vee y_j). \end{aligned}$$

Since $a_i - b_i$ is nondecreasing, we have $I_i \geq 0$. Then $I_{ij} \geq 0$ follows from Lemma A.1(ii). Collecting these completes the proof. □

We next give a sufficient condition of the existence of symmetric dominators:

LEMMA A.3. *Suppose* $\varphi(x) < \infty$ *for all* $x \in \mathbb{R}$. *Suppose there exists a convex function* $\varphi_c$ *such that* $\varphi_c''(x) \geq 2a$ *a.e.* $x$ *for some* $a > 0$, *symmetric around* $m \in \mathbb{R}$ *and* $\varphi - \varphi_c$ *is absolutely continuous and there exist* $n, b$ *satisfying* $0 \leq b \leq n - |m|$ *and*

$$\varphi'(x) \leq \varphi_c'(x + b) \quad \textit{for a.e. } x \leq -n, \tag{A.2}$$

$$\varphi'(x) \leq \varphi_c'(x + 2n - b) \quad \textit{for a.e. } x \geq -n, \tag{A.3}$$

$$\varphi'(x) \geq \varphi_c'(x - b) \quad \textit{for a.e. } x \geq n, \tag{A.4}$$

$$\varphi'(x) \geq \varphi_c'(x - 2n + b) \quad \textit{for a.e. } x \leq n. \tag{A.5}$$

*Then there exist right- and left-dominators symmetric around* $\pm n$.

PROOF. For $0 \leq b \leq n - |m|$ define

$$R_{n,\,b}^{\pm}(\varphi_c) = R_n^{\pm}(\varphi_c(\cdot \mp b)).$$

Then $R_{n,\,b}^{\pm}(\varphi_c)$ are convex and symmetric around $\pm n$. By $\varphi_c'' \geq 2a$ a.e. $x$ we see

$$R_{n,\,b}^{\pm}(\varphi_c)'' \geq 2a \quad \text{a.e. } x.$$

By (A.2)–(A.5) we see

$$R_{n,\,b}^{+}(\varphi_c)'(x) \leq \varphi'(x) \leq R_{n,\,b}^{-}(\varphi_c)'(x) \quad \text{a.e. } x. \tag{A.6}$$

Hence $R_{n,\,b}^{\pm}(\varphi_c)$ are right- and left-dominators of $\varphi$ symmetric around $\pm n$. □

REMARK A.1. We can apply Lemma A.3 to examples in Example 2.1(i) by taking $\varphi_c = |x|^q + |x|^2$ for the first one and $\varphi_c = e^{|x|}$ for the second.

## REFERENCES


[1] BRASCAMP, H. J. and LIEB, E. H. (1976). On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *J. Funct. Anal.* **22** 366–389.

[2] BRICMONT, J., LEBOWITZ, J. L. and PFISTER, C. (1981). Periodic Gibbs states of ferromagnetic spin systems. *J. Statist. Phys.* **24** 269–278.

[3] DYSON, F. J. (1971). Existence of a phase-transition in a one-dimensional Ising ferromagnet. *Comm. Math. Phys.* **12** 91–107.

[4] FRÖHLICH, J. and SPENCER, T. (1982). The phase transition in the one-dimensional Ising model with $1/r^2$ interaction energy. *Comm. Math. Phys.* **84** 87–102.

[5] GEORGII, H.-O. (1988). *Gibbs Measures and Phase Transitions*. de Gruyter, Berlin.

[6] GLIMM, J. and JAFFE, A. (1981). *Quantum Physics: A Functional Integral Point of View*. Springer, Berlin.

[7] LEBOWITZ, J. L. and PRESUTTI, E. (1976). Statistical mechanics of systems of unbounded spins. *Comm. Math. Phys.* **50** 195–218.

[8] PAPANGELOU, F. (1984). On the absence of phase transition in one-dimensional random fields I. Sufficient conditions. *Z. Wahrsch. Verw. Gebiete* **67** 255–263.

[9] PRESTON, C. J. (1974). A generalization of the FKG inequality. *Comm. Math. Phys.* **36** 223–241.

[10] RUELLE, D. (1970). Superstable interactions in classical statistical mechanics. *Comm. Math. Phys.* **18** 127–159.

[11] SIMON, B. (1979). *Functional Integration and Quantum Physics*. Academic Press, London.



GRADUATE SCHOOL OF MATHEMATICS
NAGOYA UNIVERSITY
CHIKUSA-KU, NAGOYA 464-8602
JAPAN
E-MAIL: osada@math.nagoya-u.ac.jp

DEPARTMENTS OF MATHEMATICS AND PHYSICS
TECHNISCHE UNIVERSITÄT MÜNCHEN
D-80209 MÜNCHEN
GERMANY
E-MAIL: spohn@mathematik.tu-muenchen.de