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RATIO LIMIT THEOREMS FOR A BROWNIAN MOTION KILLED AT THE BOUNDARY OF A BENEDICKS DOMAIN

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We consider a Brownian motion in a Benedicks domain with absorption at the boundary. We show ratio limit theorems for the associated heat kernel. When the hole is compact, therefore the Martin boundary is two dimensional; we obtain sharp estimates on the lifetime probabilities and we identify, in probabilistic terms, the various constants appearing in the theory.

1. Introduction. Let $n \geq 2$ and consider E a closed, proper subset of $\mathbf{R}^{n-1}$. We denote by $\mathscr{O} = \mathbf{R}^{n-1} \setminus E$ and we identify it with $\mathscr{O} \times \{0\}$, in what follows. We assume that each point of $E \times \{0\}$ is regular for the Dirichlet's problem associated to the operator $\mathscr{L} = \frac{1}{2}\Delta$ in $\mathscr{D} = \mathbf{R}^n \setminus E \times \{0\}$.

Let $\mathscr{P}_E$ be the cone of positive harmonic functions in $\mathscr{D}$ vanishing at E. A result of Benedicks [4] (see also [1], [2]) states that:

B1. Either all functions in $\mathscr{P}_E$ are proportional or
B2. $\mathscr{P}_E$ is generated by two linearly independent, minimal positive harmonic functions.

We refer to Theorem 4 in [4] for an integral test characterizing both cases. We remark that in both cases there is a unique, up to a multiplicative constant, positive harmonic and symmetric function, denoted below by v_s, which satisfies $v_s(z) = v_s(\bar{z})$ where $z = (x, y) \in \mathscr{D}$ and $\bar{z} = (x, -y)$, $x \in \mathbf{R}^{n-1}$, $y \in \mathbf{R}$. We fix a point in $\mathscr{O}$ denoted by $\mathbf{0}$ and we also fix v_s by imposing $v_s(\mathbf{0}) = 1$. We recall that from Herglotz' theorem v_s has the following representation for $y > 0$:

$$v_s(z) = \chi y + y\mathscr{C}_n \int_{\mathscr{O}} \frac{v_s(\xi, 0)\, d\xi}{\left(|x - \xi|^2 + y^2\right)^{n/2}}, \tag{1.1}$$

where

$$\mathscr{C}_n = \int_0^\infty \frac{s^{-(n+2)/2} \exp(-1/2s)}{(2\pi)^{n/2}}\, ds,$$

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and

$$\chi = \lim_{y \to \infty} \frac{v_s(x, y)}{y}.$$

Theorem 3 in [4] ensures that $\chi > 0$ if and only if B2 holds. We point out that a representation such as (1.1) can be written for any nontrivial positive harmonic function, and it is easy to check that all of them are nonintegrable. Notice that for the domains we consider, since the inner radius is infinite, the largest point of the spectrum for $\mathscr{L}$ in $L^2(\mathscr{D}, dz)$, is $\underline{\lambda} = 0$. Let (W_t) denote the standard n-dimensional Brownian motion. We consider $\mathscr{T}$ the hitting time of $\partial\mathscr{D}$ and for $z, z' \in \mathscr{D}$ we denote by $p_t(z, z')\, dz' := \mathbb{P}_z(W_t \in dz', \mathscr{T} > t)$ the heat kernel on $\mathscr{D}$ with Dirichlet boundary conditions.

THEOREM 1.1.

(i) *For any Benedicks' domain* $\mathscr{D}$ *and for any* $z, z' \in \mathscr{D}$, $s \geq 0$, *the following ratio limit exists*:

$$\lim_{t \to \infty} \frac{p_{t+s}(z, z')}{p_t(z, z')} = 1.$$

(ii) *For any Benedicks' domain* $\mathscr{D}$, *the quasi-limiting distribution is zero*; *namely*, *for any* $z, z' \in \mathscr{D}$,

$$\lim_{t \to \infty} \mathbb{P}_z\big(W_t \in dz' | \mathscr{T} > t\big) = 0.$$

(iii) *If* $B1$ *holds then the following ratio limit exists for* $z, z' \in \mathscr{D}$:

$$\lim_{t \to \infty} \frac{p_t(z, z')}{p_t(z, z)} = \frac{v_s(z')}{v_s(z)},$$

uniformly on compact subsets of $\mathscr{D}$.

(iv) *If* $B2$ *holds*, *then the following ratio limit exists for* $z \in \mathscr{Q}$, $z' \in \mathscr{D}$:

$$\lim_{t \to \infty} \frac{p_t(z, z')}{p_t(z, z)} = \frac{v_s(z')}{v_s(z)},$$

uniformly on (z, z') *on compact subsets of* $\mathscr{Q} \times \mathscr{D}$.

We get sharper results on the case $\mathscr{Q}$ is bounded, which constitute the main part of this work. We point out that Corollary 2 of [4] ensures that in this case, the Martin boundary has two extremal points.

THEOREM 1.2. *Assume* $\mathscr{Q}$ *is bounded. Then the following limits exist and satisfy*:

(i) $\lim_{t \to \infty} p_t(\mathbf{0}, \mathbf{0}) t^{(n+2)/2} = \chi^{-2}(2\pi)^{-n/2}$.

(ii) *For all* $z, z' \in \mathscr{D}$,

$$\lim_{t \to \infty} \frac{p_t(z, z')}{p_t(\mathbf{0}, \mathbf{0})} = v_s(z) v_s(z') + \chi^2 y y'.$$

Moreover, the function

$$w_\infty(z) = \int_0^\infty ds \int_{\mathscr{Q}} p_s((\xi, 0), z)\, d\xi$$

is finite on $\mathbf{R}^n$ *and satisfies*

(iii) $$\lim_{|y| \to \infty} \frac{w_\infty(x, y)}{|y|} = 0.$$

(iv) $v_s(z) = \chi(|y| + w_\infty(z))$ *for all* $z \in \mathbf{R}^n$.

(v) $$w_\infty(\mathbf{0}) = \chi^{-1} = \int_0^\infty ds \int_{\mathscr{Q}} p_s((\xi, 0), \mathbf{0})\, d\xi.$$

Let $\mathscr{T}$ *denote the exit time of* $\mathscr{D}$. *The survival probability* $\mathbb{P}_z(\mathscr{T} > t)$ *satisfies*

(vi) $$\mathbb{P}_z(\mathscr{T} > t) = \sqrt{\frac{2}{\pi t}}\, [|y| + w_\infty(z)] + o\left(\frac{1}{\sqrt{t}}\right),$$

in particular for all $z, z' \in \mathscr{D}$,

(vii) $$\lim_{t \to \infty} \frac{\mathbb{P}_z(\mathscr{T} > t)}{\mathbb{P}_{z'}(\mathscr{T} > t)} = \frac{v_s(z)}{v_s(z')}.$$

THEOREM 1.3. *For any* $s > 0$ *and for any* $A \in \mathscr{F}_s$,

$$\lim_{t \to \infty} \mathbb{P}_z(W \in A | \mathscr{T} > t) = \Pi_z(A),$$

where Π_z *is the distribution of a diffusion with transition probability densities given by*

$$\pi_t(z, z')\, dz' = \frac{v_s(z')}{v_s(z)} \mathbb{P}_z(W_t \in dz', \mathscr{T} > t).$$

We mention here that the technique we use to prove Theorem 1.1 can be generalized to contain other domains and semigroups. A basic assumption we use is that the semigroup is similar to a self-adjoint one; that is, $p_t(z, z') = e^{S(z)} q_t(z, z') e^{-S(z')}$, where $q_t(z, z')$ is self-adjoint in $L^2(\mathscr{D}, dz)$. In this generality, part (i) always holds, but since $\underline{\lambda}$ is not 0 the limit should be read as $e^{-\underline{\lambda} s}$. More specifically, in proving (ii) we use that any ground state of the infinitesimal generator is nonintegrable. As for part (iii), the crucial hypothesis is the uniqueness of the ground state.

Ratio limits theorems are well understood for general diffusions on compact domains (see [10], [11] and references therein). For half lines in the case of general one-dimensional diffusions see [5]. For some planar domains see [3]. The large time asymptotics of the diffusion kernel, under some integrability conditions on the ground state, is given in [9], even in nonsymmetric cases.

2. Proof of basic results and Theorem 1.1. In the following lemmas, we shall establish monotone properties as well as convergence of ratios for the heat kernel. A basic tool we use is the parabolic Harnack's inequality (see [7] or [12]), which allows us to compare the kernel at different points of the domain. We state it in the following form: for any compact set $K \subset \mathscr{D}$ there is a constant C_K such that for any $\delta \leq d(K, \partial\mathscr{D})/2$, for any points z, z', z'' in K such that $d(z, z') < \delta$ and $d(z', z'') < \delta$ and for any $t > \delta^2$ it is verified that

$$p_t(z, z') \leq C_K p_{t+\delta^2}(z', z') \leq C_K^3 p_{t+3\delta^2}(z, z''). \tag{2.1}$$

LEMMA 2.1. *If φ is a nonnegative nonzero function belonging to $C_0^\infty(\mathscr{D})$, then the function $\langle \varphi, e^{t\mathscr{L}}\varphi \rangle$ is log-convex.*

PROOF. From the spectral theorem for bounded self-adjoint semigroups, there is a positive finite measure μ such that

$$\langle \varphi, e^{t\mathscr{L}}\varphi \rangle = \int_{-\infty}^0 e^{\lambda t}\, d\mu(\lambda).$$

Therefore,

$$\frac{\partial_t \langle \varphi, e^{t\mathscr{L}}\varphi \rangle}{\langle \varphi, e^{t\mathscr{L}}\varphi \rangle} = \frac{\int_{-\infty}^0 e^{\lambda t}\lambda\, d\mu(\lambda)}{\int_{-\infty}^0 e^{\lambda t}\, d\mu(\lambda)} \quad \text{and} \quad \frac{\partial_t^2 \langle \varphi, e^{t\mathscr{L}}\varphi \rangle}{\langle \varphi, e^{t\mathscr{L}}\varphi \rangle} = \frac{\int_{-\infty}^0 e^{\lambda t}\lambda^2\, d\mu(\lambda)}{\int_{-\infty}^0 e^{\lambda t}\, d\mu(\lambda)}$$

and using Schwarz's inequality, it follows easily that

$$\partial_t^2 \log(\langle \varphi, e^{t\mathscr{L}}\varphi \rangle) \geq 0. \qquad \square$$

COROLLARY 2.2. *Let φ be a nonnegative nonzero function belonging to $C_0^\infty(\mathscr{D})$, and s a fixed number. For $t > \max\{-s, 0\}$ the positive function $\langle \varphi, \exp(t+s)\mathscr{L})\varphi \rangle / \langle \varphi, e^{t\mathscr{L}}\varphi \rangle$ is nondecreasing when $s \geq 0$ (nonincreasing when $s \leq 0$) and bounded above (respectively, below) by 1. Similarly, for any $z \in \mathscr{D}$ and for $t > \max\{-s, 0\}$ the function $(p_{t+s}(z, z))/(p_t(z, z))$ is nondecreasing for $s \geq 0$ (nonincreasing for $s \leq 0$), bounded above (respectively, below) by 1.*

PROOF. We have

$$\partial_t \log\left(\frac{\langle \varphi \exp((t+s)\mathscr{L})\varphi \rangle}{\langle \varphi, e^{t\mathscr{L}}\varphi \rangle} \right) = \int_t^{t+s} \partial_\tau^2 \log(\langle \varphi, e^{\tau\mathscr{L}}\varphi \rangle)\, d\tau,$$

then the monotonic property follows immediately from Lemma 2.1. Boundedness follows directly from the bound on the spectrum of $\mathscr{L}$. The corresponding properties for the kernel are deduced from the continuity of $p_t(\cdot, \cdot)$ by letting φ converge to a Dirac measure. $\square$

LEMMA 2.3. *There is a finite number λ_* such that for any nonnegative nonzero function $\varphi \in C_0^\infty(\mathscr{D})$ and for any $s \geq 0$,*

$$\lim_{t\to\infty} \frac{\langle \varphi, \exp((t+s)\mathscr{L})\varphi\rangle}{\langle \varphi, e^{t\mathscr{L}}\varphi\rangle} = \exp(-\lambda_* s). \tag{2.2}$$

Moreover, for any $z \in \mathscr{D}$ and $s \geq 0$, we have

$$\lim_{t\to\infty} \frac{p_{t+s}(z,z)}{p_t(z,z)} = \exp(-\lambda_* s).$$

PROOF. It follows at once from Corollary 2.2 that the limit in (2.2) exists. Let us denote this limit by $a_\varphi(s)$. It is easy to verify that if s_1 and s_2 are positive numbers, we have $a_\varphi(s_1+s_2) = a_\varphi(s_1)a_\varphi(s_2)$. Since by Lemma 2.2 and Corollary 2.2, $0 < a_\varphi(s) \leq 1$ we have $a_\varphi(s) = \exp(-\lambda_\varphi s)$ for some finite nonnegative number λ_φ and it remains to prove that this number is independent of φ.

Let K denote a compact set contained in $\mathscr{D}$. Let 3η denotes the distance of K to $\partial\mathscr{D}$. We can find a finite array of points $\mathscr{N}$ such that any point in K is at a distance less than η of some point, in $\mathscr{N}$. We choose once and for all a point y in $\mathscr{N}$. Using (2.1) several times, we conclude that there is a constant $C_1 > 1$ and a positive number $T_K \leq |\mathscr{N}|\eta$ such that for any $t > \eta^2$ we have for any $z, z', z'' \in K$,

$$p_t(z', z'') \leq C_1 p_{t+T(z',z'')}(z,z),$$

where $0 \leq T(z', z'') \leq 2T_K$. Using Corollary 2.2, we obtain

$$p_t(z', z'') \leq C_1 p_t(z,z). \tag{2.3}$$

On the other hand, using again (2.1), there is a constant $C_2 > 1$ such that for any $u > \eta^2$,

$$p_u(z,z) \leq C_2 p_{u+T(z',z'')}(z', z'').$$

Taking $u = t - T(z', z'')$ (for $t > 2T_K + 1$) it follows that

$$p_{t-T(z',z'')}(z,z) \leq C_2 p_t(z', z'').$$

From Corollary 2.2 for $t > 2T_K + 1$, we have

$$\frac{p_{t-T(z',z'')}(z,z)}{p_t(z,z)} \geq 1.$$

This implies that there is a constant $C_3 > 1$ such that for $t > 2T_K + 1$,

$$p_t(z,z) \leq C_3 p_t(z', z''). \tag{2.4}$$

Let $\varphi \in C_0^\infty(\mathscr{D})$ be nonnegative nonzero and with support in K, multiplying inequalities (2.3) and (2.4) by $\varphi(x)\varphi(z)$ and integrating over z' and z'', we obtain that

$$\left(C_1\|\varphi\|_1^2\right)^{-1}\langle \varphi, e^{t\mathscr{L}}\varphi\rangle \leq p_t(z,z) \leq C_3\|\varphi\|_1^{-2}\langle \varphi, e^{t\mathscr{L}}\varphi\rangle.$$

From this equation we deduce that for any s and for any t large enough, we have

$$\left(C_3 C_1 \|\varphi\|_1^4\right)^{-1} \frac{\langle \varphi, \exp((t+s)\mathscr{L})\varphi\rangle}{\langle \varphi, e^{t\mathscr{L}}\varphi\rangle} \le \frac{p_{t+s}(z,z)}{p_t(z,z)} \le C_3 C_1 \|\varphi\|_1^4 \frac{\langle \varphi, \exp((t+s)\mathscr{L})\varphi\rangle}{\langle \varphi, e^{t\mathscr{L}}\varphi\rangle}.$$

It follows as above that the function $p_{t+s}(z,z)/p_t(z,z)$ converges when t goes to infinity, and that the limit is $\exp(-\lambda(z)s)$. Therefore the limit in (2.2) does not depend on φ and $\lambda_\varphi = \lambda(z) = \lambda_K$. It follows easily that λ_K does not depend on K. We refer to [8] for a related result. □

LEMMA 2.4. $\lambda_* = 0$.

PROOF. From Corollary 2.2 and Lemma 2.3, we deduce that $-\lambda_* \le 0$. Denote by $(E_\lambda)_\lambda$ the spectral family of projections associated to $\mathscr{L}$. Since 0 is the maximum of the spectrum of $\mathscr{L}$, and since $C_0^\infty(\mathscr{D})$ is dense in the space $L^2(\mathscr{D})$, for every $\varepsilon > 0$, we can find a $\varphi \in C_0^\infty(\mathscr{D})$, nonnegative such that

$$\|E_{[-\varepsilon,0]}\varphi\|_{L^2(\mathscr{D})} > \frac{\|\varphi\|_{L^2(\mathscr{D})}}{\sqrt{2}};$$

otherwise the spectral projection $E_{[-\varepsilon,0]} := E_0 - E_{-\varepsilon}$, would have a norm smaller than 1.

For this function φ we have

$$\langle \varphi, e^{t\mathscr{L}}\varphi\rangle \ge \int_{-\varepsilon}^0 e^{\lambda t} d\langle \varphi, E_\lambda \varphi\rangle \ge e^{-\varepsilon t}\int_{-\varepsilon}^0 d\langle \varphi, E_\lambda\varphi\rangle \ge e^{-\varepsilon t}\|\varphi\|_{L^2(\mathscr{D})}^2/2.$$

It follows from Corollary 2.2 and Lemma 2.3 that for any integer $n \ge 1$,

$$\langle \varphi, e^{n\mathscr{L}}\varphi\rangle \le \langle \varphi, e^{\mathscr{L}}\varphi\rangle \exp(-(n-1)\lambda_*).$$

From these estimates the result follows. □

LEMMA 2.5. *For any compact set K contained in $\mathscr{D}$, and for any point z in the interior of $\mathscr{D}$, there is a number $\delta_{K,z} > 0$ and a constant $C_{K,z} > 0$ such that if z' and z'' belong to K we have for any $t > 3$,*

$$\left|\frac{p_t(z,z')}{p_t(z,z)} - \frac{p_t(z,z'')}{p_t(z,z)}\right| \le C_{K,z}|z' - z''|^{\delta_{K,z}}.$$

PROOF. For a given $t > 3$, we will denote by m the integer part of t, and consider on the time interval $\tau \in [m-2, m+1]$ the function

$$u(z',\tau) = \frac{p_\tau(z,z')}{p_m(z,z)}.$$

Using (2.1) and Corollary 2.2, we conclude that this function is bounded on compact sets (in z') uniformly in τ (in $[m-2, m+1]$) and uniformly in $m \geq 3$. We can apply the results of [12] or [1] to conclude that it is Hölder continuous on K (in z'), uniformly in τ and m. The same result also holds for the function

$$\frac{p_t(z, z')}{p_t(z, z)} = u(z', t) \frac{p_m(z, z)}{p_t(z, z)},$$

since by Corollary 2.2 the number $p_m(z, z)/p_t(z, z)$ is bounded by

$$\sup_{s \in [0,1]} \frac{p_3(z, z)}{p_{3+s}(z, z)} < \infty. \qquad \square$$

LEMMA 2.6. *For every fixed $z \in \mathscr{D}$ there exists a finite constant $C = C(z) > 0$ such that for all $t \geq 1$ and all $z' \in \mathscr{D}$,*

$$p_t(z, z') \leq C^{1+\|z'\|} p_t(z, z).$$

PROOF. We first assume that $z = \mathbf{0}$. Consider $z' = (x, y) \in \mathscr{D}$ where $y \geq 0$ and define $z^* = \mathbf{0} + (0, -1)$. By Harnack's inequality we have for all $t \geq 1$, $p_t(z', \mathbf{0}) \leq A p_{t+u}(z', z^*)$, where A and u are finite positive constants. Using a reflection argument with respect to the hyperplane $L = \{(x, y) \in \mathbf{R}^n / y = -1/2\}$ and Harnack's inequality,

$$p_{t+u}(z', z^*) \leq p_{t+u}((x, -1-y), z^*) \leq C_1^{1+\|z'\|} p_{t+u+u'}(z^*, z^*),$$

where C_1 is a finite positive constant and u' is also positive and finite but depends on z'. We now use Corollary 2.2 to get

$$p_t(z', \mathbf{0}) \leq A C_1^{1+\|z'\|} p_t(z^*, z^*).$$

The result for $z = \mathbf{0}$ follows by an application of Harnack's inequality in conjunction with Corollary 2.2. The case $y < 0$ is similar. For a general z, we use again Harnack's inequality and Corollary 2.2 to get

$$p_t(z', z) \leq C_2 p_{t+s}(z', \mathbf{0}) \leq C_2 C_1^{1+\|z'\|} p_t(\mathbf{0}, \mathbf{0}) \leq C_2 C_3 C_1^{1+\|z'\|} p_t(z, z),$$

where C_2, C_3, s only depend on z, from which the result follows. $\square$

We are now going to derive some properties of the function $p_t(z, z')/p_t(z, z)$ as a function of z', where z is a fixed point in the interior of $\mathscr{D}$. More precisely, we will need later on the Hölder continuity of this function uniformly in t, on compact subsets of $\mathscr{D}$.

PROOF OF THEOREM 1.1. Let us prove part (i). The case $z = z'$ is shown already in Lemma 2.3, and we will from now on assume that $z \neq z'$. We introduce the notation

$$I_t(z, z') = p_t(z, z) + p_t(z', z') + 2 p_t(z, z').$$

Consider a fixed compact set $K \subset \mathscr{D}$ such that $z, z' \in \mathring{K}$. Using (2.3) and (2.4), we deduce that for some constants $C_1 = C_1(K)$, $C_3 = C_3(K)$ and $t \geq t_0(K)$,

$$\sup_{z_1, z_2 \in K} p_t(z_1, z_2) \leq C_1 I_t(z, z') \tag{2.5}$$

and

$$I_t(z, z') \leq C_3 \inf_{z_1, z_2 \in K} p_t(z_1, z_2). \tag{2.6}$$

Now take φ and ψ two nonnegative functions in $C_0^\infty(K)$ such that

$$z \in \overbrace{\text{Supp}\,\varphi}^{\circ},\ z' \in \overbrace{\text{Supp}\,v}^{\circ},\ Supp\,\varphi \cap Supp\,\psi = \varnothing,$$

and

$$\|\varphi\|_1 = \|\psi\|_1 = 1$$

(here Supp φ denotes the support of φ). Therefore from (2.5) and (2.6) we deduce that there exists a constant $C_4 = C_4(K)$ such that for all $s > 0$ and $t \geq t_0(K)$,

$$\begin{aligned} &C_4^{-1} \frac{\langle \varphi + \psi, \exp((t+s)\mathscr{L})(\varphi + \psi)\rangle}{\langle \varphi + \psi, e^{t\mathscr{L}}(\varphi + \psi)\rangle} \\ &\quad \leq \frac{I_{t+s}(z, z')}{I_t(z, z')} \leq C_4 \frac{\langle \varphi + \psi, \exp((t+s)\mathscr{L})(\varphi + \psi)\rangle}{\langle \varphi + \psi, e^{t\mathscr{L}}(\varphi + \psi)\rangle}. \end{aligned} \tag{2.7}$$

On the other hand, letting φ and ψ converge to the Dirac measures of z and z', respectively, we obtain that for all $t > 0$ and all $s > 0$ the function of t,

$$\frac{I_{t+s}(z, z')}{I_t(z, z')}$$

is monotone nondecreasing and bounded above by 1. Once again,

$$\begin{aligned} &\lim_{t \to \infty} \frac{I_{t+s}(z, z')}{I_t(z, z')} \\ &\quad = \lim_{t \to \infty} \frac{p_{t+s}(z, z) + p_{t+s}(z', z') + 2p_{t+s}(z, z')}{p_t(z, z) + p_t(z', z') + 2p_t(z, z')} = \exp(-\lambda(z, z')s) \end{aligned}$$

for some $0 \leq \lambda(z, z')$. From (2.7) and the previous lemmas, we get $\lambda(z, z') = 0$. In particular we have

$$\lim_{t \to \infty} \frac{1 + (2p_{t+s}(z, z'))/(p_{t+s}(z, z) + p_{t+s}(z', z'))}{1 + (2p_t(z, z'))/(p_t(z, z) + p_t(z', z'))} = 1. \tag{2.8}$$

From (2.3) and (2.4) we get that for large τ

$$\frac{2p_\tau(z, z')}{p_\tau(z, z) + p_\tau(z', z')}$$

is bounded above and below away from zero. Therefore it follows from (2.8) that

$$\lim_{t\to\infty} \frac{p_{t+s}(z,z')}{p_{t+s}(z,z)+p_{t+s}(z',z')} \frac{p_t(z,z)+p_t(z',z')}{p_t(z,z')} = 1,$$

from which (i) follows.

We now turn to the proof of (ii). Let fix a point $z_0 \in \mathscr{D}$. By Lemma 2.5, on any fixed compact set, the family of functions of z,

$$v_t(z) = \frac{p_t(z_0,z)}{p_t(z_0,z_0)}$$

is equicontinuous; therefore we can extract a convergent subsequence. Using a covering argument we conclude that there is a sequence (t_n) such that v_{t_n} converges to a function v_* uniformly on compact subsets of $\mathscr{D}$. This function is moreover continuous in $\mathscr{D}$. Note that this function is nontrivial since we have $v_*(z_0)=1$, and by Lemma 2.6 is bounded by $g(z) := C^{1+\|z\|}$.

From the semigroup property, we have for any $s>0$,

$$\begin{aligned} &\frac{p_{t_n+s}(z_0,z)}{p_{t_n}(z_0,z)} \frac{p_{t_n}(z_0,z)}{p_{t_n}(z_0,z_0)} \frac{p_{t_n}(z_0,z_0)}{p_{t_n+s}(z_0,z_0)} \\ &\qquad = \frac{p_{t_n+s}(z_0,z)}{p_{t_n+s}(z_0,z_0)} = \frac{p_{t_n}(z_0,z_0)}{p_{t_n+s}(z_0,z_0)} \int_{\mathscr{D}} p_s(z,z')v_{t_n}(z')\,dz'. \end{aligned}$$

Using (i) and the dominated convergence theorem, we conclude that for any $s>0$ small enough,

$$v_*(z) = \int_{\mathscr{D}} p_s(z,z')v_*(z')\,dz'.$$

It follows from Lemma 2.6 that

$$v_*(z) \le \mathbb{E}_z(g(W_s),\mathscr{T}>s) \le \sup_{\substack{|z'-z|\le 1\\ z'\in\mathscr{D}}} \left[\mathbb{E}_{z'}\left(g(W_s)^2,\mathscr{T}>s\right)\right]^{1/2} \left(\mathbb{P}_z(\mathscr{T}>s)\right)^{1/2}.$$

Since every point of the boundary of $\mathscr{D}$ is assumed to be regular, we get that the last term converges to zero as z converges to the boundary of $\mathscr{D}$. This means that any accumulation point, as t tends to ∞, for the family of functions $(p_t(z_0,\cdot)/p_t(z_0,z_0))$ is a nontrivial positive harmonic function. From Herglotz's theorem each one of them is nonintegrable on $\mathscr{D}$.

We now prove part (ii). By definition we have

$$\mathbb{P}_{z_0}(W_t\in dz|\mathscr{T}>t) = \frac{p_t(z_0,z)\,dy}{\int_{\mathscr{D}} p_t(z_0,z')\,dz'}.$$

Let (K_m) denote an increasing sequence of compact sets converging to $\mathscr{D}$. Take any sequence (t_n) such that

$$\limsup_{t\to\infty} \frac{p_t(z_0,z)}{\int_D p_t(z_0,z')\,dz'} = \lim_{n\to\infty} \frac{p_{t_n}(z_0,z)}{\int_D p_{t_n}(z_0,z')\,dz'}.$$

By taking a further subsequence, if necessary, we can assume that the family of functions $(p_{t_n}(z_0, \cdot)/p_{t_n}(z_0, z_0))$ converges to some positive harmonic function v_*, uniformly on each K_m. Since

$$\limsup_{t\to\infty} \frac{p_t(z_0, z)}{\int_{\mathscr{D}} p_t(z_0, z')\,dz'} = \lim_{n\to\infty} \frac{p_{t_n}(z_0, z)}{\int_D p_{t_n}(z_0, z')\,dz'}$$

$$\leq \lim_{n\to\infty} \frac{p_{t_n}(z_0, z)}{\int_{K_m} p_{t_n}(z_0, z')\,dz'} = \frac{v_*(z)}{\int_{K_m} v_*(z')\,dz'}.$$

and the nonintegrability of v_* we conclude the proof of part (i).

Now we proceed with the proof of part (iii). It follows from B1 that any accumulation point v_* is unique up to a multiplicative constant and therefore $v_*(z) = v_s(z)/v_s(z_0)$, from which (iii) follows immediately.

The proof of (iv) is entirely analogous and is based on the symmetry of the considered ratio on z'. □

From part (iv) of Theorem 1.1 we deduce the following result.

COROLLARY 2.7. *Let $z_1, z_2 \in \mathscr{Q}$, $z, z' \in \mathscr{D}$; then the following limits exist*:

$$\lim_{t\to\infty} \frac{p_t(z_2, z)}{p_t(z_1, z_1)} = \frac{v_s(z_2)v_s(z)}{(v_s(z_1))^2},$$

$$\lim_{t\to\infty} \frac{p_t(z, z') + p_t(z, \bar{z}')}{2p_t(z_1, z_1)} = \frac{v_s(z)v_s(z')}{(v_s(z_1))^2}$$

and they are uniform on compact sets in each variable.

3. Proof of Theorems 1.2 and 1.3. In what follows a main role is played by the heat kernel $p_t^+(\cdot,\cdot)$ on the upper half space $H = \mathbf{R}^{n-1} \times \mathbf{R}_+^*$ with Dirichlet boundary conditions on $\partial H = \mathbf{R}^{n-1} \times \{0\}$. If we denote by $z = (x, y)$ a point in $\mathbf{R}^n$ with $x \in \mathbf{R}^{n-1}$, $y \in \mathbf{R}$, then from the independence of the coordinates it follows easily that for $y, y' > 0$,

$$p_t^+(z, z') = \frac{\exp\left(-(x-x')^2/2t\right)}{(2\pi t)^{(n-1)/2}} \frac{\exp\left(-(y-y')^2/2t\right)}{(2\pi t)^{1/2}} 2\sinh\frac{yy'}{t}.$$

We observe that

$$p_t^+(z, z') = \frac{2yy'}{(2\pi)^{n/2}} t^{-(n+2)/2} + o(t^{-(n+2)/2}) \tag{3.1}$$

uniformly on compact subsets of H.

The basic relation between p_t and p_t^+ is given by

$$p_t(z, z) = p_t(z, \bar{z}) + p_t^+(z, z). \tag{3.2}$$

This follows at once by considering separately the trajectories which do not cross $\mathscr{Q}$ plus a reflection argument for those which cross it.

LEMMA 3.1. *For $z, z' \in D$ with $y > 0$ and $y' < 0$ (or analogously $y < 0$ and $y' > 0$, we have*

$$p_t(z, z') = \frac{y}{(2\pi)^{n/2}} \int_0^t s^{-(n+2)/2} \exp\left(-\frac{y^2}{2s}\right) ds \int_{\mathcal{Q}} \exp\left(-\frac{(x-\xi)^2}{2s}\right) \times p_{t-s}((\xi, 0), z')\, d\xi \tag{3.3}$$

PROOF. Let $\mathcal{T}_{\mathcal{Q}}$ be the hitting time of $\mathcal{Q}$ and $\mathcal{T}_{\partial H}$ be the exit time from $\mathbf{R}^{n-1} \times \mathbf{R}_+^*$. Notice that the trajectories from z to z' must cross $\mathcal{Q}$. From the Markov property we get

$$\begin{aligned} p_t(z, z')\, dz' &= \mathbb{P}_z(W_t \in dz', \mathcal{T} > t) = \int_0^t \mathbb{P}_z(W_t \in dz', \mathcal{T}_{\partial H} \in ds, \mathcal{T} > t) \\ &= \int_0^t \mathbb{P}_z\big(\mathcal{T}_{\mathcal{Q}} \in ds, \mathcal{T} > s, \mathbb{E}_{W_s}(W_{t-s} \in dz', \mathcal{T} > t - s)\big) \\ &= \int_0^t \int_{\mathcal{Q}} \mathbb{P}_z(\mathcal{T}_{\mathcal{Q}} \in ds, \mathcal{T} > s, W_s \in d(\xi, 0)) p_{t-s}((\xi, 0), z')\, dz' \\ &= \int_0^t \int_{\mathcal{Q}} \mathbb{P}_z(\mathcal{T}_{\partial H} \in ds, W_s \in d(\xi, 0)) p_{t-s}((\xi, 0), z')\, dz'. \end{aligned}$$

Standard formulas (for instance, see [6], page 197) and independence of the coordinates for a Brownian motion give.

$$\mathbb{P}_z\big(\mathcal{T}_{\partial H} \in ds, W_s \in d(\xi, 0)\big) = \frac{y \exp(-y^2/2s)}{\sqrt{2\pi s^3}} \frac{\exp\big(-(x-\xi)^2/2s\big)}{(2\pi s)^{((n-1)/2)}}\, ds\, d\xi.$$

Therefore the result follows. □

The main technical lemma is the following one.

LEMMA 3.2. *If $\mathcal{Q}$ is bounded then for any z, z' in the upper half space H there exists a constant $C(z, z') < \infty$ such that*

$$p_t^+(z, z') \le p_t(z, z') \le C(z, z') p_t^+(z, z') \quad \textit{for } t \ge 1.$$

PROOF. The left inequality is obvious. Let us show the second inequality. Observe that from (3.1) and $t \ge 1$ we have

$$p_t^+(z, z') \ge A t^{-(n+2)/2} \quad \text{for } t \ge 1,$$

where $A > 0$ is a constant depending on z, z'. On the other hand, if $z_0 = (0, 1)$, we obtain from Harnack's inequality and Corollary 2.2 that $p_t(z, z') \le C_1 p_t(z_0, z_0)$, for some constant C_1 depending on z, z'. Therefore the result will follow if we can prove

$$p_t(z_0, z_0) \le C t^{-(n+2)/2} \quad \text{for } t \ge 1.$$

Hence for $t \geq 1$, using (3.1) and (3.2) we get

$$p_t(z_0, z_0) \leq p_t(z_0, \bar{z}_0) + C_2 t^{-(n+2)/2}, \tag{3.4}$$

where C_2 is a finite constant.

We now introduce the function $w_t(z) = \int_0^t ds \int_{\mathscr{Q}} p_s((\xi, 0), z)\, d\xi$, which is well defined for $z \notin \mathscr{Q}$.

In what follows we first consider the case $n \geq 3$. Using the Gaussian bound,

$$p_s((\xi,0), z) \leq \frac{1}{(2\pi s)^{n/2}} \exp\left(-\frac{(\xi - x)^2}{2s} - \frac{y^2}{2s} \right),$$

we obtain that $w_\infty(z) < \infty$.

The next step is to split the integral over $[0, t]$ in (3.3) into two pieces to get $p_t(z_0, \bar{z}_0) = I_t + J_t$ where

$$I_t = \frac{1}{(2\pi)^{n/2}} \int_0^{t/2} s^{-(n+2)/2} \exp(-1/2s)\, ds \int_{\mathscr{Q}} \exp(-\xi^2/2s) p_{t-s}((\xi,0), \bar{z}_0)\, d\xi,$$

$$J_t = \frac{1}{(2\pi)^{n/2}} \int_{t/2}^{t} s^{-(n+2)/s} \exp(-1/2s)\, ds \int_{\mathscr{Q}} \exp(-\xi^2/2s) p_{t-s}((\xi,0), \bar{z}_0)\, d\xi.$$

For the second term we have

$$\begin{aligned} J_t &\leq \frac{1}{(2\pi)^{n/2}} \left(\frac{2}{t} \right)^{(n+2)/2} \int_{t/2}^{t} ds \int_{\mathscr{Q}} p_{t-s}((\xi,0), \bar{z}_0)\, d\xi \\ &\leq \frac{1}{(2\pi)^{n/2}} \left(\frac{2}{t} \right)^{(n+2)/2} w_\infty(\bar{z}_0). \end{aligned}$$

Using a reflection argument with respect to the hyperplane $L = \{(x,y) \in \mathbf{R}^n / y = -1/2\}$ (see Figure 1), we obtain for all $u \geq 0$ and for all $\xi \in \mathscr{Q}$,

$$p_u((\xi,0), \bar{z}_0) \leq p_u((\xi,-1), \bar{z}_0).$$

Since $\mathscr{Q}$ is bounded we get from Harnack's inequality and Corollary 2.2,

$$p_u((\xi,-1), \bar{z}_0) \leq C_3^{1+d(\mathscr{Q})} p_u(\bar{z}_0, \bar{z}_0) = C_3^{1+d(\mathscr{Q})} p_u(z_0, z_0),$$

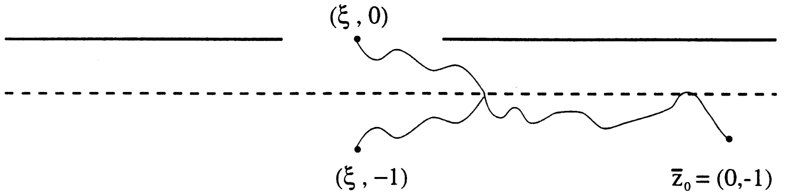

FIG. 1. *Reflection construction to move away from the boundary.*

where C_3 is a finite constant depending only on n, and

$$d(\mathscr{O}) = \sup_{v \in \mathscr{O}} \|v\|.$$

Hence we get from (3.4),

$$\begin{aligned} p_t(z_0, z_0) \le & \left[\frac{2}{\pi^{n/2}} w_\infty(\bar{z}_0) + C_2 \right] t^{-(n+2)/2} \\ & + \frac{C_3^{1+d(\mathscr{O})}}{(2\pi)^{n/2}} |\mathscr{O}| \int_0^{t/2} s^{-(n+2)/2} \exp(-1/2s) p_{t-s}(z_0, z_0)\, ds, \end{aligned} \tag{3.5}$$

where $|\mathscr{O}|$ is the $(n-1)$ dimensional Lebesgue measure of $\mathscr{O}$.

If $\lim_{t \to \infty} p_t(z_0, z_0) t^{(n+2)/2} = \infty$, since $p_t(z_0, z_0) \le t^{-n/2}$ for all $t > 0$ we obtain that for any fixed number $R > 0$ there exists $T > 0$ such that $p_t(z_0, z_0) < R t^{-(n+2)/2}$ for $t < T$ and $p_T(z_0, z_0) = R T^{-(n+2)/2}$. Therefore we get from (3.5) that

$$\begin{aligned} R T^{-(n+2)/2} \le & \left[\frac{2}{\pi^{n/2}} w_\infty(\bar{z}_0) + C_2 \right] T^{-(n+2)/2} \\ & + \frac{C_3^{1+d(\mathscr{O})} |\mathscr{O}|}{(2\pi)^{n/2}} \int_0^{T/2} s^{-(n+2)/2} \exp\left(-\frac{1}{2s} \right) R (T-s)^{-(n+2)/2}\, ds \\ \le & \left[\frac{2}{\pi^{n/2}} w_\infty(\bar{z}_0) + C_2 \right] T^{-(n+2)/2} \\ & + \frac{C_3^{1+d(\mathscr{O})} |\mathscr{O}|}{(2\pi)^{n/2}} R \left(\frac{2}{T} \right)^{(n+2)/2} \int_0^{\infty} s^{-(n+2)/2} \exp\left(-\frac{1}{2s} \right) ds. \end{aligned}$$

Therefore,

$$R \le \left[\frac{2}{\pi^{n/2}} w_\infty(\bar{z}_0) + C_2 \right] + C_3^{1+d(\mathscr{O})} |\mathscr{O}| K_n R,$$

where

$$K_n = \frac{2}{\pi^{n/2}} \int_0^\infty s^{-(n+2)/2} \exp\left(-\frac{1}{2s} \right) ds.$$

If $C_3^{1+d(\mathscr{O})} |\mathscr{O}| < K_n^{-1}$ we get a contradiction. Thus we have proven the result for small $\mathscr{O}$.

When $\mathscr{O}$ is not small we consider $\tilde{p}_t(z, z') = p_{t/b^2}(z/b, z'/b)$. It follows immediately that $\tilde{p}$ is the heat kernel on $\tilde{D} = \mathbf{R}^n \setminus \tilde{E} \times \{0\}$ where $\tilde{E} = \mathbf{R}^{n-1} \setminus \tilde{\mathscr{O}}$ and $\tilde{\mathscr{O}} = b\mathscr{O}$. Obviously, we have $d(\tilde{\mathscr{O}}) = b d(\mathscr{O})$ and $|\tilde{\mathscr{O}}| = b^{n-1} |\mathscr{O}|$. Therefore if we choose $b > 0$ small enough we get $C_3^{1+d(\tilde{\mathscr{O}})} |\tilde{\mathscr{O}}| < K_n^{-1}$. Hence, there exists a constant $\tilde{C}$ such that for all $t \ge 1$,

$$\tilde{p}_t(z_0, z_0) \le \tilde{C} t^{-(n+2)/2},$$

from which by Harnack's inequality the result follows for $n \ge 3$.

We now consider the case $n = 2$. In this situation the Gaussian bound is not good enough to prove that $w_\infty(z) < \infty$. Nevertheless, it gives us the following a priori estimate for $t \geq 1$:

$$w_t(z_0) \leq \int_0^t ds \int_{\mathscr{Q}} \frac{\exp(-(v^2+1)/2s)}{2\pi s}\, dv$$
$$\leq \frac{|\mathscr{Q}|}{2\pi}\int_0^t \frac{\exp(-1/2s)}{s}\, ds \leq \frac{|\mathscr{Q}|}{2\pi}(1+\log(t)).$$

We obtain by a similar argument an upper bound analogous to (3.5),

$$(3.6)\quad p_t(z_0, z_0) \leq \frac{C_2}{t^2} + \frac{|\mathscr{Q}|}{\pi^2}\frac{(1+\log(t))}{t^2} + \frac{C_3^{1+d(\mathscr{Q})}|\mathscr{Q}|}{2\pi}\int_0^{t/2} s^{-2}\exp\left(-\frac{1}{2s}\right) p_{t-s}(z_0, z_0)\, ds.$$

Again, if $\limsup_{t\to\infty} p_t(z_0, z_0)(t^2/1+\log(t)) = \infty$, since $p_t(z_0, z_0) \leq t^{-1/2}$ for all $t > 0$, we obtain that for any fixed number $R > 0$ there exists $T > 0$ such that $p_t(z_0, z_0) < R((1+\log(t))/t^2)$ for $0 < t < T$ and $p_T(z_0, z_0) = R((1+\log(T))/T^2)$. Hence from (3.6),

$$R\left(\frac{1+\log T}{T^2}\right) \leq \frac{C_2}{T^2} + \frac{|\mathscr{Q}|}{\pi^2}\left(\frac{1+\log(T)}{T^2}\right) + \frac{C_3^{1+d(\mathscr{Q})}|\mathscr{Q}|R}{2\pi}\int_0^{T/2} s^{-2}\exp\left(\frac{-1}{2s}\right)\frac{1+\log(T-s)}{(T-s)^2}\, ds$$
$$\leq \frac{C_2}{T^2} + \frac{|\mathscr{Q}|}{\pi^2}\frac{1+\log(T)}{T^2} + \frac{C_3^{1+d(\mathscr{Q})}|\mathscr{Q}|R}{2\pi}\left(\frac{1+\log(T/2)}{(T/2)^2}\right)\int_0^\infty s^{-2}\exp\left(-\frac{1}{2s}\right) ds$$
$$\leq \frac{C_2}{T_2} + \frac{|\mathscr{Q}|}{\pi^2}\frac{1+\log(T)}{T^2} + C_3^{1+d(\mathscr{Q})}|\mathscr{Q}|\frac{1+\log(T)}{T^2}K_2.$$

Thus $R \leq C_2 + |\mathscr{Q}|/\pi^2 + C_3^{1+d(\mathscr{Q})}|\mathscr{Q}|K_2 R$. Similarly to the case $n \geq 3$, we get a contradiction. Hence, there exists a constant C_4 such that for all $t > 1$, $p_t(z_0, z_0) \leq C_4((1+\log(t))/t^2)$. Using a reflection argument and Harnack's inequality as above, we obtain that for some finite constant C_5,

$$w_t(z_0) \leq C_5\left(1 + \int_1^t \frac{1+\log(s)}{s^2}\, ds\right) \leq C_5\left(1 + \int_1^\infty \frac{1+\log(s)}{s^2}\, ds\right) < \infty.$$

Finally we proceed as in the case $n \geq 3$ to conclude that

$$\limsup_{t\to\infty} p_t(z_0, z_0)t^2 < \infty. \qquad \square$$

Notice that from the previous proof we find that

$$w_\infty(z) = \int_0^\infty ds \int_{\mathscr{Q}} p_s((v,0),z)\,dv < \infty \qquad \text{for } z \notin \mathscr{Q},$$

independently of the dimension n.

Using Harnack's inequality and Theorem 1.1(i), we obtain the following corollary.

COROLLARY 3.3. *Let $K_1 \subset H$ and $K_2 \subset \mathscr{D}$ be two compact sets. Consider z_1, z_2 in K_1 and z_3, z_4 in K_2; then there exist two finite and positive constants A, B, depending on K_1, K_2 such that $\forall\ t \geq 1$,*

$$Ap_t^+(z_1, z_2) \leq p_t(z_3, z_4) \leq Bp_t^+(z_1, z_2). \qquad \square$$

PROOF OF THEOREM 1.2. We fix $\varepsilon > 0$ and let $T = T(\varepsilon) > 0$ to be determined later on. Consider $z \in H$ and $z' \in -H$. For $t > T$, we split the time integral on (3.3) into three pieces,

$$p_t(z, z') = L_{t,T} + M_{t,T} + U_{t,T},$$

corresponding, respectively, to the integration over the s variable in the intervals $[0, T]$, $[T, t - T]$ and $[t - T, t]$. Let us start with $L_{t,T}$ given by

$$L_{t,T} = \frac{y}{(2\pi)^{n/2}} \int_0^T s^{-(n+2)/2} \exp\left(-\frac{y^2}{2s}\right) ds \times \int_{\mathscr{Q}} \exp\left(-\frac{(\xi - x)^2}{2s}\right) p_{t-s}((\xi, 0), z')\, d\xi.$$

From Corollary 2.7 and Theorem 1.1(i), we get for all $s \in [0, T]$,

$$\lim_{t\to\infty} \frac{p_{t-s}((\xi,0),z')}{p_t(\mathbf{0},\mathbf{0})} = v_s((\xi,0))\,v_s(z').$$

Using again a reflection argument, Harnack's inequality, Lemma 2.3 and the fact that $\mathscr{Q}$ is bounded, we obtain

$$p_{t-s}((\xi,0),z') \leq Cp_t(\mathbf{0},\mathbf{0}),$$

where C depends on z' and T. Therefore by the dominated convergence theorem we get

$$\lim_{t\to\infty} \frac{L_{t,T}}{p_t(\mathbf{0},\mathbf{0})} = \frac{v_s(z')\,y}{(2\pi)^{n/2}} \int_0^T s^{-(n+2)/2} \exp\left(-\frac{y^2}{2s}\right) ds \int_{\mathscr{Q}} \exp\left(-\frac{(\xi - x)^2}{2s}\right) v_s(\xi, 0)\, d\xi$$

and

$$\lim_{T\to\infty} \underset{t\to\infty}{\mathrm{im}} \frac{L_{t,T}}{p_t(\mathbf{0},\mathbf{0})}$$
$$= \frac{v_s(z')y}{(2\pi)^{n/2}} \int_0^\infty s^{-(n+2)/2} \exp\left(-\frac{y^2}{2s}\right) ds \int_{\mathscr{Q}} \exp\left(-\frac{(\xi-x)^2}{2s}\right) v_s(\xi,0)\, d\xi.$$

Using the change of variable $u = s/((\xi - x)^2 + y^2)$, we get

$$\lim_{T\to\infty} \lim_{t\to\infty} \frac{L_{t,T}}{p_t(\mathbf{0},\mathbf{0})} = \mathscr{C}_n v_s(z') y \int_{\mathscr{Q}} \frac{v_s(\xi,0)}{\left((\xi-x)^2+y^2\right)^{n/2}}\, d\xi.$$

Recall that

$$v_s(z) = \chi y + y\mathscr{C}_n \int_{\mathscr{Q}} \frac{v_s(\xi,0)}{\left((\xi-x)^2+y^2\right)^{n/2}}\, d\xi.$$

Hence there exists $T_0 = T_0(\varepsilon)$, such that for all $T \geq T_0$ there exists $t_0 = t_0(T)$ such that for all $t > t_0$

$$\left|\frac{L_{t,T}}{p_t(\mathbf{0},\mathbf{0})} - v_s(z')\left(v_s(z) - \chi y\right)\right| \leq \varepsilon. \tag{3.7}$$

We now consider

$$U_{t,T} = \frac{y}{(2\pi)^{n/2}} \int_{t-T}^{t} \exp\left(-\frac{y^2}{2s}\right) s^{-(n+2)/2}\, ds$$
$$\times \int_{\mathscr{Q}} \exp\left(-\frac{(\xi-x)^2}{2s}\right) p_{t-s}((\xi,0),z')\, d\xi.$$

Therefore we obtain

$$t^{(n+2)/2} U_{t,T} = \frac{y}{(2\pi)^{n/2}} \int_0^T \left(\frac{t}{t-u}\right)^{(n+2)/2} \exp\left(-\frac{y^2}{2(t-u)}\right) du$$
$$\times \int_{\mathscr{Q}} \exp\left(-\frac{(\xi-x)^2}{2(t-u)}\right) p_u((\xi,0),z')\, d\xi.$$

By the dominated convergence theorem we get

$$\lim_{t\to\infty} t^{(n+2)/2} U_{t,T} = \frac{y}{(2\pi)^{n/2}} w_T(z'),$$

from which follows

$$\lim_{T\to\infty} \lim_{t\to\infty} t^{(n+2)/2} U_{t,T} = \frac{y}{(2\pi)^{n/2}} w_\infty(z') < \infty.$$

Hence there exists $T_1 = T_1(\varepsilon)$ such that for all $T \geq T_1$, there exists $t_1 = t_1(T)$ such that for all $t \geq t_1$,

$$\left| t^{(n+2)/2} U_{t,T} - \frac{y}{(2\pi)^{n/2}} w_\infty(z') \right| \leq \varepsilon. \tag{3.8}$$

Using a reflection argument as in Figure 1, Harnack's inequality and Lemma 3.2, we obtain for every $t - s \geq 1$,

$$p_{t-s}((\xi, 0), z') \leq C_1 p_{t-s}(z', z') \leq C_2 (t-s)^{-(n+2)/2},$$

where C_1, C_2 are finite constants depending on z'. Thus we get the following upper bound for $M_{t,T}$:

$$M_{t,T} \leq C_3 \int_T^{t-T} s^{-(n+2)/2} (t-s)^{-(n+2)/2}\, ds \leq C_4 T^{-n/2} t^{-(n+2)/2},$$

where C_3, C_4 are finite constants, which depend on z, z', and the last inequality holds for $t \geq t_2(T)$. Hence

$$\lim_{T \to \infty} \lim_{t \to \infty} M_{t,T} t^{(n+2)/2} = 0 \tag{3.9}$$

Let a be any accumulation point of the set $\{p_t(\mathbf{0}, \mathbf{0}) t^{(n+2)/2},\ t \geq 1\}$, which by Corollary 3.3 must satisfy $0 < a < \infty$. Also we consider a subsequence $t_k \nearrow \infty$ such that

$$a = \lim_{k \to \infty} p_{t_k}(\mathbf{0}, \mathbf{0}) t_k^{(n+2)/2}.$$

From the estimates (3.7), (3.8) and (3.9) we get

$$\lim_k \frac{p_{t_k}(z, z')}{p_{t_k}(\mathbf{0}, \mathbf{0})} = v_s(z')(v_s(z) - \chi y) + \frac{y}{a(2\pi)^{n/2}} w_\infty(z'). \tag{3.10}$$

On the other hand, for all $t > 0$,

$$\frac{p_t(z, z')}{p_t(\mathbf{0}, \mathbf{0})} = \frac{p_t(z, z') + p_t(z, \bar{z}')}{2 p_t(\mathbf{0}, \mathbf{0})} - \frac{p_t^+(z, \bar{z}')}{2 p_t(\mathbf{0}, \mathbf{0})}.$$

from which by Corollary 2.7 and (3.1) we get

$$\lim_k \frac{p_{t_k}(z, z')}{p_{t_k}(\mathbf{0}, \mathbf{0})} = v_s(z) v_s(z') + \frac{y y'}{a(2\pi)^{n/2}}. \tag{3.11}$$

From (3.10) and (3.11) we find the following relation:

$$w_\infty(z') - v_s(z') a (2\pi)^{n/2} \chi = y', \tag{3.12}$$

which implies that a is uniquely determined by v_s and w_∞. Therefore the following limits exist:

$$a = \lim_{t \to \infty} p_t(\mathbf{0}, \mathbf{0}) t^{(n+2)/2} \tag{3.13}$$

and

$$\lim_{t\to\infty}\frac{p_t(z,z')}{p_t(\mathbf{0},\mathbf{0})} = v_s(z)v_s(z') + \frac{yy'}{(2\pi)^{n/2}a}. \tag{3.14}$$

We now turn to the proof of Theorem 1.2(iii),

$$\lim_{|y|\to\infty}\frac{w_\infty(x,y)}{|y|} = 0.$$

We first consider the case $n \geq 3$. Using the Gaussian bound,

$$p_t((\xi,0),z) \leq \frac{1}{(2\pi t)^{n/2}}\exp\left(-\frac{(\xi-x)^2}{2t}\right)\exp\left(-\frac{y^2}{2t}\right),$$

we obtain for $|y| \geq 1$,

$$w_\infty(z) \leq |\mathscr{O}|\int_0^\infty \frac{\exp(-1/2t)}{t^{n/2}}\,dt < \infty,$$

from which the result follows for $n \geq 3$.

We now assume $n = 2$. Consider the heat kernel $\tilde{p}$ on $\mathbf{R}^2$ associated to $\tilde{D} = \mathbf{R}^2 \setminus (\mathbf{R} \setminus (-\rho,\rho)) \times \{0\}$ where $\rho > 0$ is chosen such that $\mathscr{O} \subseteq (-\rho,\rho) = \tilde{\mathscr{O}}$. Obviously we have $p_t(z,z') \leq \tilde{p}_t(z,z')$ for all $t > 0$, $z, z' \in \mathbf{R}^2$. Therefore it suffices to prove the result for $\tilde{w}_\infty(z) = \int_0^\infty ds \int_{\tilde{\mathscr{O}}} \tilde{p}_s((\xi,0),z)\,d\xi$.

We consider the auxiliary function Λ defined for $z \in \tilde{D}$,

$$\Lambda(z) = \int_0^\infty ds\,\tilde{p}_s((0,0),z),$$

which is finite because, from Corollary 3.3, we have the a priori estimate $\tilde{p}_s((0,0),z) \leq Cs^{-2}$ for $s \geq 1$ and $\tilde{p}_s((0,0),z) \leq C$ for $s \leq 1$, where C depends on z. Now for all $s > 0$, $x \in \mathbf{R}$, $\tilde{p}_s((0,0),(x,y)) \leq \tilde{p}_s((0,0),(0,y))$, follows by reflecting the trajectories on their first visit to the line $x' = x/2$.

Therefore, $\Lambda(x,y) \leq \Lambda(0,y)$. Assume that $0 < y' < y$ and consider $L = \{(\eta,\xi) \in \mathbf{R}^2/\xi = y'\}$. Then the strong Markov property and the symmetry of the Brownian motion implies

$$\tilde{p}((0,0),(0,y)) = \int_0^s \int_L \tilde{p}_{s-u}((0,0),(\eta,y'))\mathbb{P}_{(0,y)}(\mathscr{T}_L \in du,\, W_u \in (d\eta,y')),$$

where $\mathscr{T}_L$ is the hitting time of L. Hence, by monotonicity on $|\eta|$,

$$\begin{aligned}\tilde{p}_s((0,0),(0,y)) &\leq \int_0^s \int_L \tilde{p}_{s-u}((0,0),(0,y')\mathbb{P}_{(0,y)}(\mathscr{T}_L \in du,\, W_u \in (d\eta,y'))\\ &= \int_0^s \tilde{p}_{s-u}((0,0),(0,y'))\mathbb{P}_{(0,y)}(\mathscr{T}_L \in du).\end{aligned}$$

Integrating the last inequality on s yields

$$\Lambda((0,y)) \leq \int_0^\infty \int_0^s \tilde{p}_{s-u}((0,0),(0,y'))\mathbb{P}_{(0,y)}(\mathscr{T}_L \in du)\,ds = \Lambda(0,y'),$$

which implies for all $x \in \mathbf{R}$ all $y > y' > 0$,

$$\Lambda(x, y) \le \Lambda(0, y) \le \Lambda(0, y') \text{ and then } \lim_{|y|\to\infty} \frac{\Lambda(x, y)}{|y|} = 0.$$

The next step is to control $\tilde{p}_s((\xi, 0), z)$ for $\xi \in \bar{\mathscr{O}}$.

We assume $z = (x, y)$ where $y > 1$; then a typical reflection argument shows that for all $s > 0$,

$$\tilde{p}_s((\xi, 0), z) \le \tilde{p}_s((\xi, 1), z)$$

and then by Harnack's inequality for all $s > 1$ we obtain the existence of two finite and positive constants C, δ such that

$$\tilde{p}_s((\xi, 0), z) \le C\tilde{p}_{s+\delta}((0, 0), z).$$

Thus if $y > 1$,

$$\begin{aligned}
\tilde{w}_\infty(z) &\le \int_0^1 ds \int_{\tilde{\mathscr{O}}} \tilde{p}_s((\xi, 0), z)\, d\xi + |\tilde{\mathscr{O}}| C\Lambda(z) \\
&\le \int_0^1 ds \int_{\tilde{\mathscr{O}}} \frac{\exp\big(-(\xi - x)^2/2s\big)\exp(-1/2s)}{2\pi s}\, d\xi + |\tilde{\mathscr{O}}| C\Lambda(z) \\
&\le \int_0^1 \frac{\exp(-1/2s)}{\sqrt{2\pi s}}\, ds + |\tilde{\mathscr{O}}| C\Lambda(z)
\end{aligned}$$

and (iii) follows.

Since $\lim_{|y|\to\infty} v_s(y)/|y| = \chi$ we obtain immediately from (iii) and (3.12) that

$$a = \frac{1}{\chi^2 (2\pi)^{n/2}}. \tag{3.15}$$

From (3.13) relation (i) follows.

Plugging the value of a obtained in (3.15) in (3.12) and (3.14) gives, respectively, (iv) and (ii). Now we turn to the proof of (v):

$$w_\infty(\mathbf{0}) = \frac{1}{\chi} = \int_0^\infty ds \int_{\mathscr{O}} p_s((\xi, 0), \mathbf{0})\, d\xi.$$

From (iv) we obtain $\lim_{z\to\mathbf{0},\ y\ne 0} w_\infty(z) = 1/\chi$, hence we must show

$$\lim_{\substack{z\to\mathbf{0}\\ y\ne 0}} \int_0^\infty ds \int_{\mathscr{O}} p_s((\xi, 0), z)\, ds = \int_0^\infty ds \int_{\mathscr{O}} p_s((\xi, 0), \mathbf{0})\, d\xi.$$

By the continuity of the heat kernel for $s > 0$ we obtain

$$\lim_{\substack{z\to\mathbf{0}\\ y\ne 0}} p_s((\xi, 0), z) = p_s((\xi, 0), \mathbf{0}) \qquad \forall\, \xi \in \mathscr{O}.$$

Fix $\varepsilon > 0$. Then by Harnack's inequality we obtain for all $s > \varepsilon$,

$$p_s((\xi, 0), z) \le C p_{s+\delta}((\xi, 0), (0, 1)),$$

where C and δ depend on ε. Hence by the dominated convergence theorem,

$$\lim_{\substack{z \to \mathbf{0} \\ y \neq 0}} \int_{\varepsilon}^{\infty} ds \int_{\mathcal{Q}} p_s((\xi, 0), z)\, d\xi = \int_{\varepsilon}^{\infty} ds \int_{\mathcal{Q}} p_s((\xi, 0), \mathbf{0})\, d\xi.$$

Finally,

$$\begin{aligned}
\int_0^{\varepsilon} ds \int_{\mathcal{Q}} p_s((\xi, 0), z)\, d\xi &\leq \int_0^{\varepsilon} ds \int_{\mathcal{Q}} \frac{1}{(2\pi s)^{n/2}} \exp\left(-\frac{(x-\xi)^2}{2s}\right) \exp\left(-\frac{y^2}{2s}\right) d\xi \\
&\leq \int_0^{\varepsilon} ds \frac{\exp(-y^2/2s)}{(2\pi s)^{1/2}} \leq |y| \int_0^{\varepsilon/y^2} \frac{\exp(-1/2v^2)}{(2\pi v)^{1/2}}\, dv \\
&\leq |y| \left(\int_0^1 \frac{\exp(-1/2v^2)}{(2\pi v)^{1/2}}\, dv + \sqrt{\frac{\varepsilon}{2\pi y^2}} \right).
\end{aligned}$$

Hence (v) follows from an application of the monotone convergence theorem.

A similar argument shows that for all $z_0 \in \mathbf{R}^n$,

$$\int ds \int_{\mathcal{Q}} p((\xi, 0), z)\, d\xi \to_{z \to z_0} \int ds \int_{\mathcal{Q}} p((\xi, 0), z_0)\, d\xi.$$

Finally we prove (vi). Let us first show that for $z \in -H$,

$$\mathbb{P}_z(W_t \in H, \mathcal{T} > t) = \frac{w_\infty(z)}{\sqrt{2\pi t}} + o\left(\frac{1}{\sqrt{t}}\right). \tag{3.16}$$

Integrating (3.3) over $H = \mathbf{R}^{n-1} \times \mathbf{R}_+^*$ we obtain

$$\begin{aligned}
\mathbb{P}_z(W_t \in H, \mathcal{T} > t) &= \int_H p_t(z', z)\, dz' \\
&= \int_H \frac{y'}{(2\pi)^{n/2}} \int_0^t s^{-(n+2)/2} \exp\left(-\frac{y'^2}{2s}\right) ds \\
&\qquad \times \int_{\mathcal{Q}} \exp\left(-\frac{(x'-\xi)^2}{2s}\right) p_{t-s}((\xi, 0), z)\, d\xi\, dz' \\
&= \int_0^t \frac{ds}{\sqrt{2\pi(t-s)}} \int_{\mathcal{Q}} p_s((\xi, 0), z)\, d\xi.
\end{aligned}$$

Using that $\mathcal{Q}$ is bounded, together with a reflection as in Figure 1 and the upper bound coming from Corollary 3.3, we can prove that for $s \geq 1$,

$$\theta(s) \doteq \int_{\mathcal{Q}} p_s((\xi, 0), z)\, d\xi \leq C s^{-(n+2)/2},$$

where C depends on z.

Let $0 < \eta < 1$, we get for $t \geq 1$,

$$\int_{t^\eta}^t \frac{\theta(s)}{\sqrt{t-s}} \leq C \int_{t^\eta}^t \frac{s^{-(n+2)/2}}{\sqrt{t-s}}\, ds = o\left(\frac{1}{\sqrt{t}}\right).$$

On the other hand the dominated convergence theorem allows us to conclude that

$$\sqrt{t} \int_0^{t^\eta} \frac{\theta(s)}{\sqrt{t-s}}\, ds \to_{t\to\infty} \int_0^\infty \theta(s)\, ds = w_\infty(z),$$

from which (3.16) follows.

The reflection principle gives us

$$\mathbb{P}_z(W_t \in -H, \mathscr{T} > t) = \mathbb{P}_z(W_t \in -H, \mathscr{T}_{\partial H} > t) + \mathbb{P}_z(W_t \in H, \mathscr{T} > t).$$

Noticing that (see [6], page 96)

$$\mathbb{P}_z(W_t \in -H, \mathscr{T}_{\partial H} > t) = \mathbb{P}_z(\mathscr{T}_{\partial H} > t) = \frac{2|y|}{\sqrt{2\pi t}} + o\left(\frac{1}{\sqrt{t}}\right).$$

Therefore we have the estimate in (vi),

$$\mathbb{P}_z(\mathscr{T} > t) = \sqrt{\frac{2}{\pi t}}\,(|y| + w_\infty(z)) + o\left(\frac{1}{\sqrt{t}}\right). \tag{3.17}$$

Observe that (vii) is an immediate consequence of (iv) and (vi). This ends the proof of Theorem 1.2. □

PROOF OF THEOREM 1.3. Using the strong Markov property (vi) and (vii) of Theorem 1.2, we find

$$\begin{aligned}\mathbb{P}_z(W_u \in dz' | \mathscr{T} > t) &= \mathbb{E}_z\left(\frac{\mathbb{P}_{z'}(\mathscr{T} > t-u)}{\mathbb{P}_z(\mathscr{T} > t-u)}, W_u \in dz', \mathscr{T} > u\right) \frac{\mathbb{P}_z(\mathscr{T} > t-u)}{\mathbb{P}_z(\mathscr{T} > t)} \\ &\to_{t\to\infty} \mathbb{E}_z\left(\frac{v_s(z')}{v_s(z)}, W_u \in dz, \mathscr{T} > u\right).\end{aligned}$$

Since v_s is harmonic, Itô's formula and a localization argument shows that

$$\mathbb{E}_z(v_s(W_u), \mathscr{T} > u) = v_s(z).$$

Therefore by Scheffe's lemma, we conclude that

$$\lim_{t\to\infty} \mathbb{P}_z(W_u \in A | \mathscr{T} > t) = \int_A \pi_u(z, z')\, dz',$$

where

$$\pi_u(z, z')\, dz' = \frac{v_s(z')}{v_s(z)} \mathbb{P}_z(W_u \in dz', \mathscr{T} > u).$$

Similarly as before, using the appropriate convergence on compact subsets of $\mathscr{D}$, we have for any $A \in \mathscr{F}_s$, $K \subset \mathscr{D}$ a compact set,

$$\lim_{t\to\infty} \mathbb{P}_z(W \in A,\ W_u \in K | \mathscr{T} > t) = \mathbb{E}_z\left(\frac{v_s(W_u)}{v_s(z)},\ W \in A,\ W_u \in K,\ \mathscr{T} > u\right).$$

Finally, the result follows since

$$\lim_{t\to\infty} \mathbb{P}_z(W_u \notin K | \mathscr{T} > t) = \mathbb{E}_z\left(\frac{v_s(W_u)}{v_s(z)},\ W_u \notin K,\ \mathscr{T} > u\right)$$

can be made small as desired by taking K large enough. □

As in Theorem 1.1, the quasi-limiting distribution is equal to zero; nevertheless, from (3.16) and (3.17) the next result follows.

COROLLARY 3.4. *For any $z \in \mathscr{D}$ with $y \geq 0$ we have*

$$\lim_{t\to\infty} \mathbb{P}_z(W_t \in H | \mathscr{T} > t) = \frac{w_\infty(z)}{2(y + w_\infty(z))}.$$

REMARK. We point out that when the Martin boundary has dimension one, we get at once from (3.2) and Theorem 1.1 that for all $z \in H$,

$$\lim_{t\to\infty} \frac{p_t^+(z, z)}{p_t(z, z)} = 0.$$

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