

## INTEGRATION BY PARTS FORMULA AND LOGARITHMIC SOBOLEV INEQUALITY ON THE PATH SPACE OVER LOOP GROUPS

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The geometric stochastic analysis on the Riemannian path space developed recently gives rise to the concept of tangent processes. Roughly speaking, it is the infinitesimal version of the Girsanov theorem. Using this concept, we shall establish a formula of integration by parts on the path space over a loop group. Following the martingale method developed in Capitaine, Hsu and Ledoux, we shall prove that the logarithmic Sobolev inequality holds on the full paths. As a particular case of our result, we obtain the Driver–Lohrenz’s heat kernel logarithmic Sobolev inequalities over loop groups. The stochastic parallel transport introduced by Driver will play a crucial role.

**Introduction.** Let  $G$  be a connected compact Lie group. We shall be concerned with the following based loop group:

$$\mathcal{L}_e(G) = \{l: [0, 1] \rightarrow G \text{ continuous}; l(0) = l(1) = e\},$$

where  $e$  is the unit element of  $G$ . Let  $\mathcal{S}$  be the Lie algebra of  $G$ . Take an Ad-invariant metric  $\langle \cdot, \cdot \rangle_{\mathcal{S}}$  on  $\mathcal{S}$ , which defines a biinvariant Laplacian operator on  $G$ . The associated Brownian motion  $g_x(t)$  on  $G$  induces a probability measure  $Q$  on the path space over  $G$ . The conditioning of  $Q$  by  $g_x(1) = e$  gives rise to the Wiener measure  $\nu$  on  $\mathcal{L}_e(G)$ , which has been extensively studied (see [23], [24], [1], [16], [17], [18], [29], [13]). The study by means of Brownian motion on loop groups was proposed by Malliavin in [26]. The law of Brownian motion at a fixed time gives rise to a heat kernel measure. In this spirit, a log.sob inequality, without an additional potential term comparing to the case in [17], with respect to heat kernel measures on  $\mathcal{L}_e(G)$ , has been obtained by Driver and Lohrenz [9]. Their method was based on the concept of  $\Gamma_2$  of Bakry, Emery and Ledoux. On the other hand, Brownian motion on  $\mathcal{L}_e(G)$  defines the Wiener measure  $\mu$  on the path space over  $\mathcal{L}_e(G)$ . Using the induction argument, an integration by parts formula with respect to  $\mu$  for constant vector fields has been established recently by Driver in [7]. Our work, which benefits very much from [7], will develop the concept of tangent processes in our infinite-dimensional setting. We shall prove that Driver’s integration by parts formula holds for all adapted vector fields. This is a necessary step in order to obtain the Clark–Ocone martingale representation formula. Now following the approach of [3], we shall obtain the Logarithmic Sobolev inequality on the

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path space over  $\mathcal{L}_e(G)$ . As a particular case, by taking one-point cylindrical functions, we obtain Driver and Lohrentz's heat kernel logarithmic Sobolev inequalities over  $\mathcal{L}_e(G)$ .

**1. Concept of tangent processes.** The recent development of stochastic analysis on Riemannian path space gives rise to the concept of tangent processes, which allow transferring the differential calculus on the path space to that on the flat Wiener space (see [4], [5], [10], [11], [14], [19], [23]). In this section, we shall develop this concept in our infinite-dimensional setting.

Let  $\mathbb{P}_0^0(\mathbb{R}^d)$  be the Wiener space of a  $d$ -dimensional Brownian bridge,

$$\mathbb{P}_0^0(\mathbb{R}^d) = \{w : [0, 1] \rightarrow \mathbb{R}^d \text{ continuous; } w(0) = w(1) = 0\}.$$

Let us recall briefly the construction of Brownian motion on  $\mathbb{P}_0^0(\mathbb{R}^d)$ . Consider the Cameron–Martin subspace,

$$H_0(\mathbb{R}^d) = \left\{ h \in \mathbb{P}_0^0(\mathbb{R}^d); |h|_{H_0}^2 = \int_0^1 |\dot{h}(\theta)|^2 d\theta < +\infty \right\}.$$

Denote by  $\mathbb{P}_0^0(\mathbb{R}^d)'$  the dual space of  $\mathbb{P}_0^0(\mathbb{R}^d)$ . An element  $l \in \mathbb{P}_0^0(\mathbb{R}^d)'$  will be identified with  $\tilde{l} \in H_0(\mathbb{R}^d)$  by relation  $\langle l, h \rangle = \langle \tilde{l}, h \rangle_{H_0}$  for all  $h \in H_0(\mathbb{R}^d)$ . Let  $c_n(\theta) = \sqrt{2} ((\sin \pi n \theta)/n\pi)$  for  $n \geq 1$  and  $\{\varepsilon_1, \dots, \varepsilon_d\}$  be the canonical basis of  $\mathbb{R}^d$ . Define  $h_{n,i}(\theta) = c_n(\theta) \varepsilon_i$ . Then  $\{h_{n,i}\}$  is an orthonormal basis of  $H_0(\mathbb{R}^d)$ .

Let  $(\Omega, \mathcal{F}_t, \mathcal{F}, P)$  be a filtered probability space satisfying the usual hypothesis. Consider a sequence of independent real Brownian motion  $x_{n,i}(\omega, t)$  defined on  $(\Omega, \mathcal{F}_t, \mathcal{F}, P)$ . It is well known that the following random series:

$$(1.1) \quad x_\omega(t, \theta) = \sum_{n,i} x_{n,i}(\omega, t) h_{n,i}(\theta)$$

converges uniformly in  $(t, \theta) \in [0, 1] \times [0, 1]$  almost surely. A Brownian motion  $x_\omega$  on  $\mathbb{P}_0^0(\mathbb{R}^d)$  with the covariance operator  $\langle \cdot, \cdot \rangle_{H_0}$  is a continuous adapted process on  $\mathbb{P}_0^0(\mathbb{R}^d)$  such that

$$\mathbb{E}(\langle l_1, x(s) \rangle \langle l_2, x(t) \rangle) = s \wedge t \langle l_1, l_2 \rangle_{H_0}$$

for all  $l_1, l_2 \in \mathbb{P}_0^0(\mathbb{R}^d)'$ . This is equivalent to saying that

$$(1.2) \quad \mathbb{E}(\exp i \langle l, x(t) - x(s) \rangle \mid \mathcal{F}_s) = \exp \left\{ -\frac{(t-s)}{2} |l|_{H_0}^2 \right\} \quad \text{for all } l \in \mathbb{P}_0^0(\mathbb{R}^d)'.$$

The process  $x_\omega(t)$  defined in (1.1) is a  $\mathbb{P}_0^0(\mathbb{R}^d)$ -valued Brownian motion with covariance operator  $\langle \cdot, \cdot \rangle_{H_0}$ .

In what follows, for simplicity we shall write  $x(t) = \sum_n x_n(\omega, t) h_n$ . Let  $K$  be a separable Hilbert space. Consider an  $H_0(\mathbb{R}^d) \otimes K$ -valued adapted process  $f_\omega(t)$  satisfying  $\mathbb{E}(\int_0^1 |f_\omega(t)|_{\text{HS}}^2 dt) < +\infty$  where  $|\cdot|_{\text{HS}}$  denotes the Hilbert–Schmidt norm. It is known that the stochastic integral  $\int_0^T \langle f_t, dx_\omega(t) \rangle$  is well defined. We have the following properties.

- PROPOSITION 1.1. (i)  $T \rightarrow M_T = \int_0^T \langle f_t, dx(t) \rangle$  is a  $K$ -valued martingale.  
 (ii)  $\mathbb{E} | \int_0^T \langle f_t, dx(t) \rangle_K |^2 = \mathbb{E} \int_0^T |f_\omega(t)|_{HS}^2 dt$ .  
 (iii) The quadratic variation  $dM_T dM_T$  of  $M_T$  is given by  $dM_T dM_T = \int_0^T |f_\omega(t)|_{HS}^2 dt$ .

For the proof, see [22].

Now take  $K = \mathbb{R}$ , for any  $H_0(\mathbb{R}^d)$ -valued adapted process  $f_\omega(t)$  such that

$$\mathbb{E} \left( \int_0^1 |f_\omega(t)|_{H_0}^2 dt \right) < +\infty,$$

the stochastic integral  $M_t = \int_0^t \langle f_s, dx(s) \rangle$  is a real valued continuous martingale.

Let  $\mathcal{B}(H_0)$  be the Banach space of bounded linear operators on  $H_0(\mathbb{R}^d)$  with the endomorphism norm.

DEFINITION 1.2. A  $\mathcal{B}(H_0)$ -valued process  $q_t(\omega)$  is said to be  $\mathcal{F}_t$ -adapted if for all  $h \in H_0(\mathbb{R}^d)$ ,  $q_t(\omega)h$  is an  $H_0(\mathbb{R}^d)$ -valued adapted process.

Now denote by  $\mathcal{O}(H_0)$  the group of unitary operator on  $H_0(\mathbb{R}^d)$ . Let  $U(t)$  be an adapted process in  $\mathcal{O}(H_0)$ . Denote by  $U^*(t)$  the adjoint operator of  $U(t)$ . We are going to define the stochastic integral  $y(T) = \int_0^T U(t) dx(t)$ . Let  $h_n \in \mathbb{P}_0^0(\mathbb{R}^d)$  be an orthonormal basis of  $H_0(\mathbb{R}^d)$  and consider the series

$$(1.3) \quad \sum_n \left( \int_0^T \langle U^*(t)h_n, dx(t) \rangle \right) h_n.$$

THEOREM 1.3. The series (1.3) converges uniformly with respect to  $T \in [0, 1]$  in  $\mathbb{P}_0^0(\mathbb{R}^d)$  almost surely and defines a Brownian motion  $y(T)$  on  $\mathbb{P}_0^0(\mathbb{R}^d)$  with covariance operator  $\langle \cdot, \cdot \rangle_{H_0}$ .

PROOF. Denote  $y_n(T) = \int_0^T \langle U^*(t)h_n, dx(t) \rangle$ . It is sufficient to see that  $\{y_n(T); n \geq 1\}$  are mutually independent real Brownian motions. By Proposition 1.1(iii), the quadratic variations between  $y_n$  and  $y_m$  are given by

$$\begin{aligned} dy_n(T)dy_m(T) &= \int_0^T \langle U^*(t)h_n, U^*(t)h_m \rangle_{H_0} dt \\ &= \int_0^T \langle h_n, h_m \rangle_{H_0} dt = T \delta_{nm}. \end{aligned}$$

Now by the Lévy characterization theorem,  $y_n$  are independent Brownian motions.  $\square$

Consider now a  $\mathcal{B}(H_0)$ -valued adapted process  $q_t$  such that for all  $h, k \in H_0(\mathbb{R}^d)$ ,

$$\langle q_t h, k \rangle_{H_0} = -\langle h, q_t k \rangle_{H_0} \quad \text{and} \quad \mathbb{E} \left( \int_0^1 |q_t h|_{H_0}^2 dt \right) < +\infty.$$

It is clear that for all  $n \geq 1$ ,  $q_t^n$  are  $\mathcal{B}(H_0)$ -valued adapted processes. Define

$$\exp(q_t) = \sum_{n \geq 0} \frac{q_t^n}{n!}.$$

Then  $e^{q_t}$  is an adapted process in  $\mathcal{O}(H_0)$ . Now, according to [4] and to [6], we shall introduce the concept of tangent process.

DEFINITION 1.4. A  $\mathcal{B}(H_0) \times H_0(\mathbb{R}^d)$ -valued process  $(q_t(\omega), z_\omega(t))$  is called a tangent process if:

- (i)  $t \rightarrow (q_t(\omega), z_\omega(t))$  is adapted;
- (ii) for  $h, k \in H_0(\mathbb{R}^d)$ ,  $\langle q_t(\omega)h, k \rangle_{H_0} = -\langle h, q_t(\omega)k \rangle_{H_0}$ ;
- (iii)  $\int_0^1 |z_\omega(t)|_{H_0}^2 dt \leq C < +\infty$  a.s.

Now for  $q_t(\omega)$  as above, we denote:  $y_q(t) = \int_0^t e^{q_s} dx(s)$ .

THEOREM 1.5 (Girsanov). Let  $(q_t(\omega), z_\omega(t))$  be a tangent process, denote

$$K_{q,z} = \exp \left\{ - \int_0^1 \langle \exp(-q_s) z_s, dx(s) \rangle - \frac{1}{2} \int_0^1 |z_s|_{H_0}^2 ds \right\},$$

then under the probability law  $dQ = K_{q,z} dP$ , the process  $y_q(t) + \int_0^t z_s ds$  is a Brownian motion, with  $\langle \cdot, \cdot \rangle_{H_0}$  as the covariance operator.

PROOF. Denote by  $\mathbb{E}_Q$  the expectation with respect to the probability measure  $Q$  and  $\tilde{y}(t) = y_q(t) + \int_0^t z_s ds$ . By characterization (1.2), it is sufficient to prove

$$\mathbb{E}_Q(\exp\{i \langle l, \tilde{y}(t) - \tilde{y}(s) \rangle\} | \mathcal{F}_s) = \exp\{-(t-s)|l|_{H_0}^2/2\} \quad \text{for all } l \in \mathbb{P}_0^0(\mathbb{R}^d)'.$$

The verification is the same as that in the finite-dimensional case (see [28]).  $\square$

In what follows, we shall denote by

$$X = \{x: [0, 1] \rightarrow \mathbb{P}_0^0(\mathbb{R}^d) \text{ continuous; } x(0) = 0\}.$$

Endow  $X$  with the probability law  $P$  induced by the Brownian motion on  $\mathbb{P}_0^0(\mathbb{R}^d)$ . Denote  $\mathcal{F}_t = \sigma\{\langle l, x(s) \rangle; s \leq t, l \in \mathbb{P}_0^0(\mathbb{R}^d)'\}$ . According to the above theorem, we shall define the derivative along a tangent process.

DEFINITION 1.6. Letting  $(q_t(\omega), z_\omega(t))$  be a tangent process and  $F: X \rightarrow \mathbb{R}$  be a measurable function, we say that  $F$  is differentiable along  $(q, z)$  if

$$(D_{q,z}F)(x) = \lim_{\varepsilon \rightarrow 0} \frac{F(y_{\varepsilon q} + \varepsilon \int_0^\bullet z_s ds) - F(x)}{\varepsilon} \quad \text{exists in } L^2(X).$$

THEOREM 1.7. We have

$$\mathbb{E}(D_{q,z}F) = \mathbb{E} \left( F \int_0^1 \langle z_s, dx(s) \rangle \right).$$

PROOF. Denote

$$K_\varepsilon = \exp \left\{ -\varepsilon \int_0^1 \langle \exp(-\varepsilon q_s) z_s, dx(s) \rangle - \frac{\varepsilon^2}{2} \int_0^1 |z_s|^2 ds \right\}.$$

Then

$$\left\{ \frac{dK_\varepsilon}{d\varepsilon} \right\}_{\varepsilon=0} = - \int_0^1 \langle z_s, dx(s) \rangle \quad \text{in } L^2(X).$$

Now by the Girsanov Theorem 1.5,  $\mathbb{E}(F(y_{\varepsilon q} + \varepsilon \int_0^1 z_s ds) K_\varepsilon) = \mathbb{E}(F)$ . Taking the derivative with respect to  $\varepsilon$ , at  $\varepsilon = 0$ , we obtain

$$\mathbb{E}(D_{q,z} F - F \int_0^1 \langle z_s, dx(s) \rangle) = 0,$$

which gives the result.  $\square$

**2. Stochastic parallel transport.** Let  $G$  be a compact Lie group and  $\mathcal{G}$  its Lie algebra. Take on  $\mathcal{G}$  an  $\text{Ad}_G$ -invariant metric  $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ . Consider

$$H_0(\mathcal{G}) = \left\{ h: [0, 1] \rightarrow \mathcal{G}; h(0) = h(1) = 0 \text{ and } \int_0^1 |\dot{h}(\theta)|_{\mathcal{G}}^2 d\theta < +\infty \right\}.$$

For  $h, k \in H_0(\mathcal{G})$ , define the Lie bracket between  $h$  and  $k$  by  $[h, k](\theta) = [h(\theta), k(\theta)]$ . Then  $H_0(\mathcal{G})$  is a Lie algebra. To  $h \in H_0(\mathcal{G})$ , we shall associate a left-invariant vector field  $\tilde{h}$  on  $\mathcal{L}_e(G)$  defined by

$$(\tilde{h}F)(l) = \left\{ \frac{d}{d\varepsilon} F(l e^{\varepsilon h}) \right\}_{\varepsilon=0}$$

for all cylindrical functions  $F: \mathcal{L}_e(G) \rightarrow \mathbb{R}$  written in the form

$$F(l) = f(l(\tau_1), \dots, l(\tau_k)) \quad \text{where } f \in \mathcal{C}^\infty(G^k).$$

Denote by  $\mathcal{F}C^\infty(\mathcal{L}_e(G))$  the class of cylindrical functions on  $\mathcal{L}_e(G)$ . Then we have the relation  $[\tilde{h}, \tilde{k}] = [\tilde{h}, \tilde{k}]$  on  $\mathcal{F}C^\infty(\mathcal{L}_e(G))$ . Therefore, the computation on the Lie group can be reduced to its Lie algebra. The Levi-Civita connection on  $H_0(\mathcal{G})$  defined by

$$\langle \nabla_h k, z \rangle_{H_0} = \frac{1}{2} \{ \langle [h, k], z \rangle_{H_0} - \langle [h, z], k \rangle_{H_0} - \langle [k, z], h \rangle_{H_0} \}$$

has the explicit expression (see [15] or [9]),

$$(2.1) \quad \widehat{\nabla_h k}(\theta) = [h(\theta), \dot{k}(\theta)] - \int_0^1 [h(\theta), \dot{k}(\theta)] d\theta.$$

It follows that for  $k \in H_0(\mathcal{G})$  given, the operator  $h \rightarrow \nabla_h k$  is Hilbert-Schmidt. Following [15], the curvature tensor is not of trace class but its two steps trace exists. More precisely, let  $e_1, \dots, e_d$  be an orthonormal basis of  $\mathcal{G}$  and  $c_n$  be an orthonormal basis of  $H_0(\mathbb{R})$ . Let  $h_{n,i} = c_n e_i$ . Then (see [15] and [9]),

$$(2.2) \quad \text{Ric } h = \sum_n \sum_i ([\nabla_h, \nabla_{h_{n,i}}] - \nabla_{[h, h_{n,i}]}) h_{n,i} \quad \text{converges in } H_0(\mathcal{G}).$$

Moreover, the Ricci is given by

$$(2.3) \quad \langle \text{Ric } h, h \rangle_{H_0} = \int_0^1 \mathcal{K}(\bar{h}(\theta), \bar{h}(\theta)) d\theta,$$

where  $\mathcal{K}(a, b) = \text{trace}(ad(a) \circ ad(b))$  is the killing form on  $\mathcal{G}$  and  $\bar{h}(\theta) = h(\theta) - \int_0^1 h(\tau) d\tau$ .

The following result, giving another approach to the Ricci tensor, will be used in the next section.

**THEOREM 2.1.** *Let  $\mathcal{B}$  be an orthonormal basis of  $H_0(\mathcal{G})$ . Then for  $h \in H_0(\mathcal{G})$ ,*

$$(2.4) \quad \sum_{k, \bar{k} \in \mathcal{B}} \langle \nabla_k h, \bar{k} \rangle_{H_0} \langle \nabla_{\bar{k}} h, k \rangle_{H_0} = -\langle \text{Ric } h, h \rangle_{H_0}.$$

For the proof, see [13], page 395.

In order to introduce the stochastic parallel transport, we shall need the following basic estimates. In the sequel, we shall fix an orthonormal basis  $\{h_n; n \geq 1\}$  of  $H_0(\mathcal{G})$ .

**PROPOSITION 2.2.** *Let  $C = \frac{1}{2} \sup_{|a|=1} |\mathcal{K}(a, a)|^{1/2}$ , then:*

- (i)  $\|\nabla z\|_{H_0 \otimes H_0} \leq C |z|_{H_0}, z \in H_0(\mathcal{G});$
- (ii)  $\Delta^1 z = \sum_n \nabla_{h_n} \nabla_{h_n} z$  converges in  $H_0(\mathcal{G});$
- (iii)  $|\Delta^1 z|_{H_0} \leq C^2 |z|_{H_0}.$

**PROOF.** See [9], pages 403 and 424. For (i), see also [13], page 395. In what follows, we shall give another proof of (ii) and (iii) using (i). Letting  $k \in H_0(\mathcal{G})$ , for  $p, q \geq 1$ , we have

$$\begin{aligned} \left| \left\langle \sum_p^q \nabla_{h_n} \nabla_{h_n} z, k \right\rangle_{H_0} \right| &= \left| \sum_p^q \langle \nabla_{h_n} z, \nabla_{h_n} k \rangle \right| \\ &\leq \left( \sum_p^q |\nabla_{h_n} z|^2 \right)^{1/2} \left( \sum_p^q |\nabla_{h_n} k|^2 \right)^{1/2} \\ &\leq \left( \sum_p^q |\nabla_{h_n} z|^2 \right)^{1/2} |\nabla k|_{H_0 \otimes H_0}. \end{aligned}$$

According to (i), we obtain, as  $p, q \rightarrow +\infty$ ,

$$\left| \sum_p^q \nabla_{h_n} \nabla_{h_n} z \right| \leq C \left( \sum_p^q |\nabla_{h_n} z|^2 \right)^{1/2} \rightarrow 0.$$

Letting  $p = 1, q \rightarrow +\infty$ , we have

$$|\Delta^1 z|_{H_0} \leq C |\nabla z|_{H_0 \otimes H_0} \leq C^2 |z|_{H_0}. \quad \square$$

Now let  $\mathbb{P}_0^0(\mathcal{S})$  be the Wiener space of the Brownian bridges over  $\mathcal{S}$ , starting from the origin. Let  $x(t, \theta) = \sum_n x_n(t)h_n(\theta)$  be a Brownian motion on  $\mathbb{P}_0^0(\mathcal{S})$ . We shall introduce the stochastic parallel transport in  $H_0(\mathcal{S})$  along the Brownian motion  $x$ , following [7].

**THEOREM 2.3 (Driver).** *There exists a unique continuous  $H_0(\mathcal{S})$ -valued process  $z_t$  satisfying the following family of s.d.e. on  $H_0(\mathcal{S})$ :*

$$(2.5) \quad d_t z(t, \theta) = - \sum_n (\nabla_{h_n} z_t)(\theta) \circ dx_n(t) \quad z_0 \text{ is given,}$$

where  $d_t$  denotes the stochastic differential with respect to  $t$ . Moreover,

$$\langle z_t, z_t \rangle_{H_0} = \langle z_0, z_0 \rangle_{H_0} \quad \text{for all } t \in [0, 1].$$

According to Proposition 2.2, the Picard iterated method does work. See [7], Section 4.

Letting  $h \in H_0(\mathcal{S})$ , define

$$(2.6) \quad U(t)h = \sum_n \langle h, h_n \rangle_{H_0} z_t(h_n),$$

where  $z_t(h_n)$  is the solution of s.d.e. (2.5) such that  $z_0 = h_n$ . It is obvious that the series (2.6) converges uniformly in  $t$ . Therefore almost surely, for all  $h \in H_0(\mathcal{S})$ ,  $t \rightarrow U(t)h$  is continuous and for all  $t$ ,  $U(t)$  is isometric. The following result was claimed in [7], but not proved. We give a proof here. Another proof may be found in [8].

**THEOREM 2.4.** *Almost surely, for all  $t$ ,  $U(t)$  is unitary.*

**PROOF.** Let  $N > 0$  and consider the finite-dimensional vector space  $V_N = \text{span}\{h_1, \dots, h_N\}$ . Define  $A_n^N: H_0 \rightarrow H_0$  by  $A_n^N = P_N \nabla_{h_n} P_N$  where  $P_N$  is the orthogonal projection onto  $V_N$ . It is clear that  $A_n^N$  are skew-symmetric on  $H_0$ .

Consider the following s.d.e.:

$$dz_t^N = - \sum_{n=1}^N A_n^N z_t^N \circ dx_n(t), \quad z_0^N = z_0.$$

Define  $U_{t,x}^N z_0 = z_t^N$ . We see that  $U_{t,x}^N z_0 = z_0$  if  $z_0 \in V_N^\perp$  and  $V_N$  is stable under  $U_{t,x}^N$ . Denote by  $\widetilde{U}_{t,x}^N: V_N \rightarrow V_N$ . Then  $\widetilde{U}_{t,x}^N$  is a unitary operator on  $V_N$ . It follows that  $U_{t,x}^N$  is a unitary operator on  $H_0$ .

Let  $T > 0$ , consider  $\hat{x}_n^T(t) = x_n(T-t) - x_n(T)$  and  $\hat{x}^T(t) = \sum_n \hat{x}_n^T(t)h_n$ . Then  $\hat{x}^T(t)$  is a Brownian motion with respect to the filtration

$$\mathcal{F}_t^T = \sigma\{\langle l, x(T-s) - x(T) \rangle; s \leq t, l \in P_0^0(\mathcal{S})'\}.$$

Consider the following s.d.e.:

$$dk_t^N = - \sum_{n=1}^N A_n^N k_t^N \circ d\hat{x}_n^T(t), \quad k_0^N = k_0.$$

Denote  $U_{t, \hat{x}^T}^N k_0 = k_t^N$ . Then we have (see [21], page 250 or [25])

$$U_{T-t, \hat{x}^T}^N = U_{t, x}^N \circ U_{T, \hat{x}^T}^N \quad \text{on } V_N.$$

It follows that  $(U_{T, x}^N)^* = U_{T, \hat{x}^T}^N$  on  $H_0$ . Now by straightforward calculation,

$$\lim_{N \rightarrow +\infty} E \left( \sup_{0 \leq t \leq 1} \|U_{t, x}^N h - U_{t, x} h\|^2 \right) = 0.$$

In the same way,  $\lim_{N \rightarrow +\infty} E(\sup_{0 \leq t \leq T} \|U_{t, \hat{x}^T}^N h - U_{t, \hat{x}^T} h\|^2) = 0$ . Therefore, for all  $h \in H_0$ , up to a subsequence, a.s.,

$$U_{T, x} h = \lim_{N \rightarrow +\infty} U_{T, x}^N h \quad \text{and} \quad U_{T, \hat{x}^T} h = \lim_{N \rightarrow +\infty} U_{T, \hat{x}^T}^N h.$$

As  $H_0$  is separable, it follows that almost surely,

$$(2.7) \quad U_{T, \hat{x}^T} = U_{T, x}^*.$$

In order to obtain our result from (2.7), we need the continuity of  $T \rightarrow U_{T, \hat{x}^T}$ . To this end, it is more convenient to use the Itô backward stochastic integrals.

Let  $h \in H_0(\mathcal{S})$ , denote  $H_T(t) = U_{T-t, \hat{x}^T} h$  for  $t \leq T$ . As  $U_{t, \hat{x}^T} h$  is  $\mathcal{F}_t^T$ -measurable, then the Itô backward stochastic integral  $\int_t^T \nabla_{dx(\tau)}^- H_T(\tau)$  can be defined as follows:

$$\int_t^T \nabla_{dx(\tau)}^- H_T(\tau) = \sum_n \int_t^T \nabla_{h_n} H_T(\tau) dx_n^-(\tau) = - \sum_n \int_0^{T-t} \nabla_{h_n} H_T(T-\tau) d\hat{x}_n^T(\tau).$$

Therefore  $H_T(t)$  satisfies the following backward s.d.e.:

$$h - H_T(t) + \int_t^T \nabla_{dx(\tau)}^- H_T(\tau) + \frac{1}{2} \int_t^T \Delta^1 H_T(\tau) d\tau = 0.$$

Let  $T_1 < T_2$  in  $]0, 1[$ . We have, for all  $t \leq T_1$ ,

$$\begin{aligned} & H_{T_1}(t) - H_{T_2}(t) \\ &= \int_t^{T_1} \nabla_{dx(\tau)}^- H_{T_1}(\tau) - \int_t^{T_2} \nabla_{dx(\tau)}^- H_{T_2}(\tau) \\ & \quad + \frac{1}{2} \left[ \int_t^{T_1} \Delta^1 H_{T_1}(\tau) d\tau - \int_t^{T_2} \Delta^1 H_{T_2}(\tau) d\tau \right] \\ (2.8) \quad &= \int_t^{T_1} \nabla_{dx(\tau)}^- H_{T_1}(\tau) - \int_t^{T_1} \nabla_{dx(\tau)}^- H_{T_2}(\tau) - \int_{T_1}^{T_2} \nabla_{dx(\tau)}^- H_{T_2}(\tau) \\ & \quad + \frac{1}{2} \left[ \int_t^{T_1} \Delta^1 (H_{T_1}(\tau) - H_{T_2}(\tau)) d\tau + \int_{T_1}^{T_2} \Delta^1 H_{T_2}(\tau) d\tau \right]. \end{aligned}$$



We have  $\mathcal{F}_t^{T_1} \subset \mathcal{F}_{T_2-T_1+t}^{T_2}$  and  $\hat{x}_{t_2}^{T_1} - \hat{x}_{t_1}^{T_1} = \hat{x}_{T_2-T_1+t_2}^{T_2} - \hat{x}_{T_2-T_1+t_1}^{T_2}$ . It follows that  $\hat{x}_t^{T_1}$  is a  $\mathcal{F}_{T_2-T_1+t}^{T_2}$  Brownian motion. Therefore,

$$\int_t^{T_1} \nabla_{dx(\tau)}^- H_{T_1}(\tau) = - \int_0^{T_1-t} \nabla_{d\hat{x}^{T_1}(\tau)} H_{T_1}(T_1 - \tau) = - \int_{T_2-T_1}^{T_2-t} \nabla_{d\hat{x}^{T_2}(\tau)} H_{T_1}(T_2 - \tau).$$

It follows that

$$\begin{aligned} & \int_t^{T_1} \nabla_{dx(\tau)}^- H_{T_1}(\tau) - \int_t^{T_2} \nabla_{dx(\tau)}^- H_{T_2}(\tau) \\ &= - \int_{T_2-T_1}^{T_2-t} [\nabla_{d\hat{x}^{T_2}(\tau)}(H_{T_1}(T_2 - \tau) - H_{T_2}(T_2 - \tau))]. \end{aligned}$$

By the Burkholder inequality, for  $p > 1$ , we have

$$\begin{aligned} & \mathbb{E} \left| \int_{T_2-T_1}^{T_2-t} [\nabla_{d\hat{x}^{T_2}(\tau)}(H_{T_1}(T_2 - \tau) - H_{T_2}(T_2 - \tau))] \right|^{2p} \\ & \leq C_p \mathbb{E} \left( \int_t^{T_1} |\nabla(H_{T_1}(\tau) - H_{T_2}(\tau))|_{H_0 \otimes H_0}^2 d\tau \right)^p. \end{aligned}$$

Now according to Proposition 2.2, using (2.8) and by the Gronwall inequality, we obtain for  $t \leq T_1$ ,

$$\mathbb{E}(|H_{T_1}(t) - H_{T_2}(t)|^{2p}) \leq C_p |T_1 - T_2|^p.$$

In particular,  $\mathbb{E}(|H_{T_1}(0) - H_{T_2}(0)|^{2p}) \leq C_p |T_1 - T_2|^p$ . By the Kolmogoroff modification theorem, almost surely,  $T \rightarrow H_T(0)$  is continuous. In other words,  $t \rightarrow U_{t, \hat{x}^t} h$  is continuous. Using the expression  $U_{t, \hat{x}^t} h = \sum_n \langle h, h_n \rangle U_{t, \hat{x}^t} h_n$ , we see that almost surely, for all  $h \in H_0(\mathcal{S})$ ,  $t \rightarrow U_{t, \hat{x}^t} h$  is continuous. Now using (2.7), almost surely, for all rational  $t \in [0, 1]$  and for all  $h, k \in H_0(\mathcal{S})$ , we have

$$\langle U_{t, x} h, k \rangle = \langle h, U_{t, \hat{x}^t} k \rangle.$$

By continuity, we obtain that  $U_{t, x}^* = U_{t, \hat{x}^t}$  for all  $t \in [0, 1]$ .  $\square$

**3. Malliavin calculus on the Brownian motion over  $\mathcal{L}_e(G)$ .** Let  $x(t)$  be the Brownian motion on  $\mathbb{P}_0^0(\mathcal{S})$ . Let  $\{e_1, \dots, e_d\}$  be an orthonormal basis of  $\mathcal{S}$ . Denote  $x^i(t, \theta) = \langle x(t, \theta), e_i \rangle_{\mathcal{S}}$ . For  $\theta \in [0, 1]$ , consider the following Stratanovich s.d.e. with parameter  $\theta$ :

$$(3.1) \quad d_t g_x(t, \theta) = \sum_{i=1}^d g_x(t, \theta) e_i \circ d_t x^i(t, \theta), \quad g_x(0, \theta) = e.$$

**THEOREM 3.1 (Malliavin).** *There exists a unique  $\mathcal{L}_e(G)$ -valued continuous adapted process  $g_x(t)$  such that for all  $\theta \in [0, 1]$ ,  $g_x(t, \theta) = g_x(t)(\theta)$  satisfies the s.d.e. (3.1). Moreover,  $(t, \theta) \rightarrow g_x(t, \theta)$  is continuous.*

For the proof, see [26], pages 19–22 and [7].

*Convention of notations.* In what follows, we shall use the prime to denote the derivative with respect to  $t$ , the parameter for paths and the dot, the derivative with respect to  $\theta$ , the parameter for loops.

Now consider an  $H_0(\mathcal{G})$ -valued adapted process  $z(t)$ . We suppose that there exists a process  $z'(t)$  satisfying

$$(3.2) \quad \sup_{t \in [0, 1]} |z'(t)|_{H_0} \leq C < +\infty \quad \text{almost surely}$$

such that  $z(t) = \int_0^t z'(s) ds$ . Let  $\varepsilon \geq 0$  and consider

$$g_{x, \varepsilon}(t, \theta) = g_x(t, \theta) \exp(\varepsilon(U_t z(t))(\theta)).$$

Denote  $k_t = U_t z(t)$ . By the Itô formula, we have

$$\begin{aligned} d_t g_{x, \varepsilon}(t, \theta) &= d_t g_x(t, \theta) \circ \exp(\varepsilon k_t(\theta)) \\ &\quad + g_x(t, \theta) \circ (\varepsilon \exp'(\varepsilon k_t(\theta)) \circ d_t k_t(\theta)) \\ &= \sum_{i=1}^d (g_x(t, \theta) e_i \exp(\varepsilon k_t(\theta))) \circ d_t x^i(t, \theta) \\ &\quad + \varepsilon (g_x(t, \theta) \exp'(\varepsilon k_t(\theta))) \circ d_t k_t(\theta) \\ &= g_{x, \varepsilon}(t, \theta) \circ [\text{Ad}(\exp(-\varepsilon k_t(\theta))) e_i \circ d_t x^i(t, \theta) \\ &\quad + \varepsilon \exp(-\varepsilon k_t(\theta)) \exp'(\varepsilon k_t(\theta)) d_t k_t(\theta)]. \end{aligned}$$

Let  $M_x(t, \theta) = \{(d/d\varepsilon)g_{x, \varepsilon}(t, \theta)\}_{\varepsilon=0}$ . Then  $M_x(t, \theta)$  satisfies the following family of s.d.e.:

$$(3.3) \quad \begin{aligned} d_t M_x(t, \theta) &= M_x(t, \theta) \circ d_t x(t, \theta) \\ &\quad + g_x(t, \theta) \circ [-ad(k_t(\theta)) \circ d_t x(t, \theta) + d_t k_t(\theta)] \end{aligned}$$

with the initial conditions  $M_x(0, \theta) = 0$ . By Theorem 4.4 in [6], we have

$$(3.4) \quad \begin{aligned} d_t k_t(\theta) &= - \sum_{n \geq 1} (\nabla_{h_n} k_t)(\theta) dx_n(t) + (U_t z'(t))(\theta) dt \\ &\quad + \frac{1}{2} \sum_{n \geq 1} (\nabla_{h_n} \nabla_{h_n} k_t)(\theta) dt. \end{aligned}$$

LEMMA 3.2. *We have*

$$(3.5) \quad \begin{aligned} -ad(k_t(\theta)) \circ d_t x(t, \theta) &= - \sum_{n \geq 1} [k_t, h_n](\theta) dx_n(t) \\ &\quad + \frac{1}{2} \sum_{n \geq 1} [(\nabla_{h_n} k_t), h_n](\theta) dt. \end{aligned}$$

PROOF. Let  $l \in \mathbb{P}_0^0(\mathcal{G})'$ . Let  $f_t$  be a real adapted process such that  $\int_0^1 |f_t|^2 dt < +\infty$ . It is easy to see that

$$(3.6) \quad \int_0^T f_t d \langle l, x_t \rangle = \int_0^T \langle f_t l, dx_t \rangle.$$

Now for  $\theta \in [0, 1]$  and  $i = 1, \dots, d$ , we define  $l_\theta^i \in \mathbb{P}_0^0(\mathcal{A})$  by  $\langle l_\theta^i, x \rangle = x^i(\theta)$  for  $x \in \mathbb{P}_0^0(\mathcal{A})$ . By (3.4), (3.6) and Proposition 1.1(iii), for all  $\xi \in \mathcal{A}$ , we have

$$(3.7) \quad d_t [k_t(\theta), \xi] d_t x(t, \theta) = - \sum_n \langle [(\nabla_{h_n} k_t)(\theta), \xi], h_n(\theta) \rangle dt.$$

On the other hand,

$$(3.8) \quad \begin{aligned} \langle \xi, -ad(k_t(\theta)) d_t x(t, \theta) \rangle &= \langle [k_t(\theta), \xi], d_t x(t, \theta) \rangle \\ &= \langle [k_t(\theta), \xi], \langle l_\theta, dx_t \rangle \rangle \\ &= \sum_{i=1}^d \langle \langle [k_t(\theta), \xi], e_i \rangle_{\mathcal{A}} l_\theta^i, dx(t) \rangle \\ &= \sum_n \sum_{i=1}^d \langle \langle [k_t(\theta), \xi], e_i \rangle_{\mathcal{A}} l_\theta^i, h_n \rangle dx_n(t) \\ &= \sum_n \langle [k_t(\theta), \xi], h_n(\theta) \rangle_{\mathcal{A}} dx_n(t). \end{aligned}$$

Then (3.5) follows from (3.7) and (3.8).  $\square$

DEFINITION 3.3. Letting  $z$  be a process satisfying condition (3.2), we define the operators  $q_z(t): H_0(\mathcal{A}) \rightarrow H_0(\mathcal{A})$  by

$$(3.9) \quad q_z(t) h = -[k_t, h] - \nabla_h k_t, \quad h \in H_0(\mathcal{A}).$$

As the torsion is free, we have  $\nabla_h k_t - \nabla_{k_t} h = [h, k_t]$ . It follows that  $q_t h = -\nabla_{k_t} h$ . Now by the antisymmetry of  $\nabla_{k_t}$ , we obtain

$$(3.10) \quad \langle q_z(t) h_1, h_2 \rangle_{H_0} = -\langle h_1, q_z(t) h_2 \rangle_{H_0}.$$

LEMMA 3.4. We have, for all  $T \in [0, 1]$ ,

$$(3.11) \quad \int_0^T \langle q_t l_\theta, dx_t \rangle = \sum_n \int_0^T ([k_t, h_n](\theta) + (\nabla_{h_n} k_t)(\theta)) dx_n(t).$$

PROOF. Remark first that  $\mathbb{E} \int_0^1 |q_t l_\theta|_{H \otimes \mathcal{A}}^2 dt < +\infty$ . The stochastic integral  $\int_0^T \langle q_t l_\theta, dx_t \rangle$  is well defined. Now it is sufficient to verify that

$$\langle q_t l_\theta, h_n \rangle = [k_t(\theta), h_n(\theta)] + (\nabla_{h_n} k_t)(\theta),$$

which follows from Definitions (3.9) and (3.10).  $\square$

LEMMA 3.5. We have

$$(3.12) \quad \sum_n \nabla_{h_n} \nabla_{h_n} k_t + \sum_n [\nabla_{h_n} k_t, h_n] = \text{Ric } k_t.$$

PROOF. By torsion free, we have

$$\nabla_{h_n} \nabla_{h_n} k_t + [\nabla_{h_n} k_t, h_n] = \nabla_{\nabla_{h_n} k_t} h_n = \sum_m \nabla_{h_m} h_n \langle \nabla_{h_n} k_t, h_m \rangle_{H_0}.$$

Therefore, for all  $h \in H_0(\mathcal{L})$ ,

$$\begin{aligned} & \left\langle h, \sum_n (\nabla_{h_n} \nabla_{h_n} k_t + [\nabla_{h_n} k_t, h_n]) \right\rangle_{H_0} \\ &= \sum_{n,m} \langle \nabla_{h_m} h_n, h \rangle_{H_0} \langle \nabla_{h_n} k_t, h_m \rangle_{H_0} \\ &= - \sum_{n,m} \langle \nabla_{h_m} h, h_n \rangle_{H_0} \langle \nabla_{h_n} k_t, h_m \rangle_{H_0} = \langle \text{Ric } k_t, h \rangle_{H_0}, \end{aligned}$$

the convergence of the above series being guaranteed by the fact that  $z \rightarrow \nabla_z h$  is a Hilbert–Schmidt operator and the last equality follows from (2.4).  $\square$

THEOREM 3.6. Here  $M_x(t, \theta)$  satisfies the following family of s.d.e.:

$$\begin{aligned} (3.13) \quad & d_t M_x(t, \theta) \\ &= M_x(t, \theta) \circ d_t x(t, \theta) \\ &+ g_x(t, \theta) \circ [-\langle q_t l_\theta, dx_t \rangle + ((U_t z'_t)(\theta) + \frac{1}{2}(\text{Ric } U_t z_t)(\theta)) dt] \end{aligned}$$

PROOF. According to (3.4), (3.5), (3.11) and (3.12), the s.d.e. (3.3) can be written in the form (3.13).  $\square$

Now consider the operator  $\exp(\varepsilon q_z(t)): H_0(\mathcal{L}) \rightarrow H_0(\mathcal{L})$ . By (4.10),  $q_z(t)$  is antisymmetric, so that  $\exp(\varepsilon q_z(t))$  is an unitary operator on  $H_0(\mathcal{L})$ . Denote

$$y^\varepsilon(t) = \int_0^t e^{\varepsilon q_s} dx(s).$$

Consider  $\tilde{g}_{x,\varepsilon}(t, \theta)$  the solution of the following s.d.e.:

$$d_t \tilde{g}_{x,\varepsilon}(t, \theta) = \tilde{g}_{x,\varepsilon}(t, \theta) \circ d_t y^\varepsilon(t, \theta) + \varepsilon \tilde{g}_{x,\varepsilon}(t, \theta) \hat{z}(t, \theta) dt, \quad \tilde{g}_{x,\varepsilon}(0, \theta) = e,$$

where  $\hat{z}(t) = U_t z'_t + \frac{1}{2} \text{Ric } U_t z_t$ . For any  $\theta \in [0, 1]$ , almost surely  $\tilde{g}_{x,0}(t, \theta) = g_x(t, \theta)$  for all  $t \in [0, 1]$ . We have  $d_t y^\varepsilon(t, \theta) = \langle \exp(-\varepsilon q_t) l_\theta, dx_t \rangle$ . As in [2], Chapter II-c, or in [14],

$$\tilde{M}_x(t, \theta) = \left\{ \frac{d}{d\varepsilon} \tilde{g}_{x,\varepsilon}(t, \theta) \right\}_{\varepsilon=0} \quad \text{exists in } L^2,$$

and  $\tilde{M}_x(t, \theta)$  satisfies the s.d.e.,

$$d_t \tilde{M}_x(t, \theta) = \tilde{M}_x(t, \theta) \circ d_t x(t, \theta) + g_x(t, \theta) \circ [-\langle q_t l_\theta, dx_t \rangle + \hat{z}(t, \theta) dt].$$

This means that for all  $\theta \in [0, 1]$ ,  $\tilde{M}_x(t, \theta)$  satisfies the s.d.e. (3.13). By unicity, for any  $\theta \in [0, 1]$ ,

$$(3.14) \quad M_x(t, \theta) = \tilde{M}_x(t, \theta) \quad \text{for all } t \in [0, 1] \text{ almost surely.}$$

Now we are going to establish the formula of integration by parts on the path space over  $\mathcal{L}_e(G)$ . Let  $\mathbf{e}$  be the constant identity loop on  $\mathcal{L}_e(G)$ :  $\mathbf{e}(t) = e$ . Denote

$$\mathbb{P}_e(\mathcal{L}_e(G)) = \{ \gamma: [0, 1] \rightarrow \mathcal{L}_e(G) \text{ continuous; } \gamma(0) = \mathbf{e} \}.$$

The Brownian motion  $g_x(t)$  over  $\mathcal{L}_e(G)$  induces a probability measure  $\mu$  on  $\mathbb{P}_e(\mathcal{L}_e(G))$ . A function  $F: \mathbb{P}_e(\mathcal{L}_e(G)) \rightarrow \mathbb{R}$  is said to be cylindrical if it is in the form

$$(3.15) \quad F(\gamma) = f(\gamma(\tau_1, \theta_1), \dots, \gamma(\tau_k, \theta_k)), \quad f \in \mathcal{C}^\infty(G^k).$$

Given an  $H_0(\mathcal{L})$ -valued adapted process  $z(t)$  satisfying the condition (3.2), we shall define the derivative of a cylindrical function along  $z$  according to [7].

DEFINITION 3.7. Letting  $F$  be a cylindrical function on  $\mathbb{P}_e(\mathcal{L}_e(G))$ , we define

$$(D_z F)(g_x) = \left\{ \frac{d}{d\varepsilon} F(g_x e^{\varepsilon U z}) \right\}_{\varepsilon=0} \quad \text{in } L^2.$$

For a cylindrical function in the form (3.15), we have

$$(3.16) \quad (D_z F)(g_x) = \sum_{j=1}^k \langle \partial_j f, g_x(\tau_j, \theta_j) (U_{\tau_j} z_{\tau_j})(\theta_j) \rangle_{T_{g_x(\tau_j, \theta_j)} G},$$

where  $\partial_j$  denotes the partial gradient with respect to the  $j$ -component.

Now, using Definitions 1.6 and 3.7, by (3.14) we obtain the following result.

THEOREM 3.8. Let  $F: \mathbb{P}_e(\mathcal{L}_e(G)) \rightarrow \mathbb{R}$  be a cylindrical function and denote  $\tilde{F}(x) = F(g_x)$ . Then we have

$$(D_z F)(g_x) = (D_{q_z, \hat{z}} \tilde{F})(x)$$

for all adapted process  $z$  verifying the condition (3.2).

In what follows, we shall compute the gradient of  $F$ . By (3.16), denoting  $g_{\tau_j}(\theta) = g_x(\tau_j, \theta)$ , we have

$$\begin{aligned} (D_z F)(g_x) &= \sum_j \left\langle g_{\tau_j}^{-1}(\theta_j) (\partial_j f), \left[ U_{\tau_j} \int_0^{\tau_j} z'(t) dt \right] (\theta_j) \right\rangle_{\mathcal{L}} \\ &= \sum_j \int_0^1 \left\langle g_{\tau_j}^{-1}(\theta_j) (\partial_j f) \mathbf{1}_{(t < \tau_j)}, (U_{\tau_j} z'(t)) (\theta_j) \right\rangle_{\mathcal{L}} dt \\ &= \sum_j \int_0^1 \left\langle g_{\tau_j}^{-1}(\theta_j) (\partial_j f) \mathbf{1}_{(t < \tau_j)}, \int_0^1 \frac{d}{d\theta} (U_{\tau_j} z'(t)) G(\theta_j, \theta) d\theta \right\rangle_{\mathcal{L}} dt \\ &= \sum_j \int_0^1 dt \int_0^1 \left\langle g_{\tau_j}^{-1}(\theta_j) (\partial_j f) G(\theta_j, \theta) \mathbf{1}_{(t < \tau_j)}, \frac{d}{d\theta} (U_{\tau_j} z'(t)) \right\rangle d\theta, \end{aligned}$$

where  $G(\theta_j, \theta) = \theta_j \wedge \theta - \theta\theta_j$  is the Green function such that

$$\int_0^1 G(\theta_j, \theta) \dot{h}(\theta) d\theta = h(\theta_j) \quad \text{for all } h \in H_0(\mathcal{L}).$$

For  $(g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_k) \in G^{k-1}$  given, define

$$f_{j, g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_k}(g) = f(g_1, \dots, g_{j-1}, g, g_{j+1}, \dots, g_k).$$

For  $l \in \mathcal{L}_e(G)$ , let

$$F_{j, g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_k}(l) = f_{j, g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_k}(l(\theta_j)),$$

which is a cylindrical function on  $\mathcal{L}_e(G)$ . Now for a cylindrical function  $\phi$  on  $\mathcal{L}_e(G)$ , written in the form  $\phi(l) = \tilde{\phi}(l(\theta_1), \dots, l(\theta_m))$ , where  $\tilde{\phi}$  is a smooth function on  $G^m$ , we define the gradient operator  $\nabla^{\mathcal{L}_e(G)}$  on  $\mathcal{L}_e(G)$  by

$$(\nabla^{\mathcal{L}_e(G)} \phi)(l)(\theta) = \sum_{j=1}^m l(\theta_j)^{-1} \partial_j \tilde{\phi} G(\theta_j, \theta).$$

We have

$$\begin{aligned} & (\nabla^{\mathcal{L}_e(G)} F_{j, g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_k})(l)(\theta) \\ &= l^{-1}(\theta_j) \nabla^G f_{j, g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_k}(l(\theta_j)) G(\theta_j, \theta), \end{aligned}$$

where  $\nabla^G$  denotes the gradient operator on the Lie group  $G$ .

DEFINITION 3.9. We define

$$(\nabla_j^{\mathcal{L}_e(G)} F)(\gamma) = \nabla^{\mathcal{L}_e(G)} F_{j, \gamma_{\tau_1}(\theta_1), \dots, \gamma_{\tau_k}(\theta_k)}(\gamma_{\tau_j}(\theta_j)),$$

where  $\gamma_{\tau_j}(\theta_j) = \gamma(\tau_j, \theta_j)$ .

Then  $D_z F$  can be written in the form

$$\begin{aligned} (D_z F)(g_x) &= \sum_j \int_0^1 \langle (\nabla_j^{\mathcal{L}_e(G)} F)(g_x) \mathbf{1}_{(t < \tau_j)}, U_{\tau_j} z'(t) \rangle_{H_0} dt \\ &= \sum_j \int_0^1 \langle U_{\tau_j}^* (\nabla_j^{\mathcal{L}_e(G)} F)(g_x) \mathbf{1}_{(t < \tau_j)}, z'(t) \rangle_{H_0} dt. \end{aligned}$$

DEFINITION 3.10. Define the gradient operator over  $\mathbb{P}_e(\mathcal{L}_e(G))$  by

$$(3.17) \quad D_t^{\mathbb{P}} F(g_x) = \sum_j U_{\tau_j}^* (\nabla_j^{\mathcal{L}_e(G)} F)(g_x) \mathbf{1}_{(t < \tau_j)}.$$

Using (3.17), we have

$$(3.18) \quad (D_z F)(g_x) = \int_0^1 \langle (D_t^{\mathbb{P}} F)(g_x), z'(t) \rangle_{H_0} dt.$$

**THEOREM 3.11.** *Let  $z(t) = \int_0^t z'(s) ds$  be an adapted  $H_0(\mathcal{G})$ -valued process such that*

$$\mathbb{E} \int_0^1 |z'(t)|_{H_0}^2 dt < +\infty.$$

*Then the following formula of integration by parts holds:*

$$(3.19) \quad \mathbb{E}(D_z F) = \mathbb{E} \left( F \int_0^1 \langle U_t z'_t + \frac{1}{2} \text{Ric } U_t z_t, dx_t \rangle \right)$$

*for all cylindrical function  $F$ .*

The proof follows from Proposition 1.7 and Theorem 3.8.

**4. Clark–Ocône representation formula.** In this section, we shall establish the Clark–Ocône formula on the path space  $\mathbb{P}_e(\mathcal{L}_e(G))$  over  $\mathcal{L}_e(G)$ . For flat case, see [30].

**LEMMA 4.1.** *Let  $F \in L^2(X, P)$ ; then there exists a unique  $H_0(\mathcal{G})$ -valued predictable process  $a_t$  such that:*

- (i)  $\mathbb{E}(\int_0^1 |a_t|_{H_0}^2 dt) < +\infty$ ;
- (ii)  $F = \mathbb{E}(F) + \int_0^1 \langle a_t, dx_t \rangle$ .

**PROOF.** Unicity follows easily from the Itô energy identity. For existence, we shall follow the proof of Itô’s classical martingale representation theorem (see [28]). Consider the space  $\mathcal{S}$  generated by those functions  $F \in L^2(X, P)$  for which there exists an  $H_0(\mathcal{G})$ -valued predictable process  $a_t$  such that (i) and (ii) hold. Consider the following simple predictable process:

$$a_t = \sum_{j, \text{ finite}} l_j \mathbf{1}_{]t_{j-1}, t_j]}, \quad l_j \in \mathbb{P}_0^0(\mathcal{G})'.$$

Then  $\int_0^1 \langle a_t, dx_t \rangle = \sum_j \langle l_j, x(t_j) - x(t_{j-1}) \rangle$ . Denote

$$\mathcal{E}_s(a) = \exp \left\{ \int_0^s \langle a_t, dx_t \rangle - \frac{1}{2} \int_0^s |a_t|_{H_0}^2 dt \right\}.$$

Then by the Itô formula,

$$\mathcal{E}_1(a) = 1 + \int_0^1 \langle \mathcal{E}_t(a) a_t, dx_t \rangle.$$

Therefore  $\mathcal{E}_1(a) \in \mathcal{S}$ . To complete the proof, we have to verify (i) that the functions  $\mathcal{E}_1(a)$  are dense in  $L^2(X, P)$  and (ii) that the space  $\mathcal{S}$  is closed in  $L^2(X, P)$ . The same argument in [28], pages 186 and 187, gives the results.  $\square$

**THEOREM 4.2.** *Let  $F$  be a cylindrical function on  $\mathbb{P}_e(\mathcal{L}_e(G))$  and define  $\tilde{F}(x) = F(g_x)$ . Then the following Clark–Ocône formula holds:*

$$(4.1) \quad \tilde{F} = \mathbb{E}(F) + \int_0^1 \langle a_t(F)(g_x), dx(t) \rangle,$$

*where  $a_t(F)$  will be defined in (4.6).*

PROOF. Consider the following Hilbert space:

$$L_a^2(\chi) = \left\{ z_t; H_0(\mathcal{L})\text{-valued adapted process such that} \right. \\ \left. \|z\|^2 = \mathbb{E} \left( \int_0^1 |z_t|_{H_0}^2 dt \right) < +\infty \right\}.$$

Define the operator  $\mathcal{H}: L_a^2(\chi) \rightarrow L_a^2(\chi)$  by

$$(4.2) \quad (\mathcal{H}z)_t = U_t z_t + \frac{1}{2} \text{Ric } U_t \int_0^t z_s ds \quad \text{for } z \in L_a^2(\chi).$$

By Lemma 4.1, there exists  $a \in L_a^2(\chi)$  such that

$$\tilde{F}(x) = \mathbb{E}(F) + \int_0^1 \langle a_t, dx_t \rangle.$$

Therefore for any  $z \in L_a^2(\chi)$ , by Itô's energy equality, we have

$$(4.3) \quad \mathbb{E} \left( \tilde{F} \int_0^1 \langle \mathcal{H}z, dx \rangle \right) = \mathbb{E} \left( \int_0^1 \langle a_t, dx(t) \rangle \int_0^1 \langle \mathcal{H}z, dx \rangle \right) \\ = \mathbb{E} \left( \int_0^1 \langle a_t, (\mathcal{H}z)_t \rangle_{H_0} dt \right) \\ = \mathbb{E} \left( \int_0^1 \langle (\mathcal{H}^* a)_t, z_t \rangle_{H_0} dt \right),$$

where  $\mathcal{H}^*$  is the dual operator of  $\mathcal{H}$  in  $L_a^2(\chi)$ . On the other hand, by Theorem 3.11, we have

$$(4.4) \quad \mathbb{E} \left( \tilde{F} \int_0^1 \langle \mathcal{H}z, dx \rangle \right) = \mathbb{E} \left( \int_0^1 \langle D_t^{\mathbb{P}} F, z_t \rangle_{H_0} dt \right).$$

LEMMA 4.3. *Let  $z_x(s, t) = \mathbb{E}^{\mathcal{F}_s}(D_t^{\mathbb{P}} F)$ , then for  $t$  fixed,*

$$\mathbb{E}^{\mathcal{F}_t}(D_t^{\mathbb{P}} F) = \lim_{s \rightarrow t} z_x(s, t) \quad \text{exists in } L^2(X, H_0(\mathcal{L})).$$

Moreover,  $|\mathbb{E}^{\mathcal{F}_t}(D_t^{\mathbb{P}} F)|_{H_0} \leq \mathbb{E}^{\mathcal{F}_t}(|D_t^{\mathbb{P}} F|_{H_0})$ .

PROOF. Let  $G$  be an  $H_0(\mathcal{L})$ -valued simple measurable function  $G = \sum_{j, \text{finite}} h_j f_j$  where  $h_j \in H_0(\mathcal{L})$  and  $f_j$  are real bounded measurable functions. We can choose  $h_j$  to be mutually orthogonal. Then  $\mathbb{E}^{\mathcal{F}_t}(G)$  is well defined. Moreover,

$$\mathbb{E}(|\mathbb{E}^{\mathcal{F}_t}(G)|_{H_0}^2) \leq \mathbb{E}(|G|_{H_0}^2).$$

By density argument, we see that  $\mathbb{E}^{\mathcal{F}_t}(G)$  is well defined for any  $G \in L^2(X, H_0(\mathcal{L}))$ . Now by Lemma 4.1, for a simple function  $G$ , there exists an  $H_0(\mathcal{L}) \otimes H_0(\mathcal{L})$ -valued predictable process  $a_t$  such that

$$(4.5) \quad G = \mathbb{E}(G) + \int_0^1 \langle a_t, dx(t) \rangle.$$



We have  $\mathbb{E}(|G|_{H_0}^2) = |\mathbb{E}(G)|_{H_0}^2 + \mathbb{E}(\int_0^1 |a_t|_{\text{HS}}^2 dt)$ . It follows, by density, that for any  $G \in L^2(X, H_0(\mathcal{I}))$  there exists an  $H_0(\mathcal{I}) \otimes H_0(\mathcal{I})$ -valued predictable process  $a_t(G)$  such that  $\mathbb{E}(\int_0^1 |a_t(G)|_{\text{HS}}^2 dt) < +\infty$  and the relation (4.5) holds. Now for  $t$  fixed, taking  $G = D_t^{\mathbb{P}} F$ , we obtain

$$\mathbb{E}^{\mathcal{F}_s}(D_t^{\mathbb{P}} F) = \mathbb{E}(D_t^{\mathbb{P}} F) + \int_0^s \langle a_\tau(F), dx(\tau) \rangle.$$

Therefore for  $s, s' \in [0, 1]$ ,

$$\mathbb{E}(|\mathbb{E}^{\mathcal{F}_s}(D_t^{\mathbb{P}} F) - \mathbb{E}^{\mathcal{F}_{s'}}(D_t^{\mathbb{P}} F)|^2) = \mathbb{E}\left(\int_{s'}^s |a_\tau(F)|_{\text{HS}}^2 d\tau\right) \rightarrow 0 \quad \text{as } s, s' \rightarrow t.$$

By the Cauchy criterion, we obtain the result.  $\square$

Now by (4.3) and (4.4), we obtain

$$\mathbb{E}\left(\int_0^1 \langle (\mathcal{H}^* a)_t, z_t \rangle_{H_0} dt\right) = \mathbb{E}\left(\int_0^1 \langle D_t^{\mathbb{P}} F, z_t \rangle_{H_0} dt\right) \quad \text{for all } z \in L_a^2(\chi).$$

It follows that  $(\mathcal{H}^* a)_t = \mathbb{E}^{\mathcal{F}_t}(D_t^{\mathbb{P}} F)$ . The operator  $\mathcal{H}^*$  is invertible [we shall give the explicit expression for  $(\mathcal{H}^*)^{-1}$  below], so we obtain

$$(4.6) \quad a_t(F) = (\mathcal{H}^*)^{-1} \mathbb{E}^{\mathcal{F}_t}(D_t^{\mathbb{P}} F). \quad \square$$

**5. Logarithmic Sobolev inequalities.** In this section, we shall deduce from (4.1) the logarithmic Sobolev inequality on the path space over  $\mathcal{L}_e(G)$ , following the ideas in [3]. To this end, we shall need the explicit expression of the operator  $(\mathcal{H}^*)^{-1}$ . In the case of Riemannian paths, it was computed for the first time in [20]. Remark first,

$$(\mathcal{H}^*)^{-1} = (\mathcal{H}^{-1})^*.$$

Now determine the operator  $\mathcal{H}^{-1}$ . Denote by  $\mathcal{B}(H_0(\mathcal{I}))$  the Banach space of bounded operators on  $H_0(\mathcal{I})$ , with the endomorphism norm. Consider the resolvent equation in  $\mathcal{B}(H_0(\mathcal{I}))$ , for  $t > s$ ,

$$(5.1) \quad \frac{d Q_{t,s}}{dt} = -\frac{1}{2} \text{Ric}_t Q_{t,s}, \quad Q_{s,s} = \text{Id}_{H_0(\mathcal{I})},$$

where  $\text{Ric}_t = U_t^* \text{Ric} U_t$ . Then  $Y_t = \int_0^t Q_{t,s} U_s^* k_s ds$  solves the following ordinary differential equation:

$$\frac{d Y_t}{dt} + \frac{1}{2} \text{Ric}_t Y_t = U_t^* k_t.$$

It follows that  $(d Y_t/dt) = U_t^* k_t - \frac{1}{2} \text{Ric}_t \int_0^t Q_{t,s} U_s^* k_s ds$ . Therefore, we have

$$(5.2) \quad (\mathcal{H}^{-1} h)_t = U_t^* h_t - \frac{1}{2} \text{Ric}_t \int_0^t Q_{t,s} U_s^* h_s ds.$$

Now let  $z \in L_a^2(\chi)$ ; then

$$\begin{aligned} & \mathbb{E} \left( \int_0^1 \langle (\mathcal{H}^{-1} h)_t, z(t) \rangle_{H_0} dt \right) \\ &= \mathbb{E} \left( \langle h_t, U_t z_t \rangle_{H_0} dt \right) - \frac{1}{2} \mathbb{E} \left( \int_0^1 \left\langle \int_0^t \mathbf{Q}_{t,s} U_s^* h_s ds, \text{Ric}_t z_t \right\rangle_{H_0} dt \right) \\ &= \mathbb{E} \left( \langle h_t, U_t z_t \rangle_{H_0} dt \right) - \frac{1}{2} \mathbb{E} \left( \int_0^1 \left\langle h_s, \int_s^1 U_s \mathbf{Q}_{t,s}^* \text{Ric}_t z_t dt \right\rangle_{H_0} ds \right). \end{aligned}$$

Then we have the following proposition.

PROPOSITION 5.1. *We have, for  $z \in L_a^2(\chi)$ ,*

$$(5.3) \quad ((\mathcal{H}^{-1})^* z)_t = U_t z_t - \frac{1}{2} \mathbb{E}^{\mathcal{F}_t} \left( \int_t^1 U_t \mathbf{Q}_{s,t}^* \text{Ric}_s z_s ds \right).$$

LEMMA 5.2. *Denote  $K = \|\text{Ric}\|_{\text{End}(H_0(\mathcal{L}))}$ ; then we have*

$$(5.4) \quad |(\mathcal{H}^{-1})^* z|_{H_0} \leq |z_t|_{H_0} + \frac{K}{2} \mathbb{E}^{\mathcal{F}_t} \left( \int_t^1 \exp(K(s-t)/2) |z_s|_{H_0} ds \right).$$

PROOF. By the resolvent equation (5.1), we obtain

$$\|\mathbf{Q}_{t,s}^*\|_{\text{End}(H_0(\mathcal{L}))} \leq \exp(K(t-s)/2) \quad \text{for } t > s.$$

Now using (5.3) and Lemma 4.3, we obtain (5.4).  $\square$

Now we shall follow [3] to deduce the logarithmic Sobolev inequality on the path space over  $\mathcal{L}_e(G)$ . Denote  $M_t = \mathbb{E}^{\mathcal{F}_t}(F)$ . By Lemma 4.1, we have

$$M_t = \mathbb{E}(F) + \int_0^t \langle a_s(F), dx(s) \rangle.$$

It follows that

$$M_1 = \tilde{F}, \quad M_0 = \mathbb{E}(F), \quad dM_t dM_t = |a_t|_{H_0}^2 dt.$$

Suppose  $F \geq \delta > 0$ ; applying Itô's formula to the function  $\phi(\xi) = \xi \log \xi$ ,

$$\mathbb{E}(M_1 \log M_1) - \mathbb{E}(M_0 \log M_0) = \frac{1}{2} \mathbb{E} \left( \int_0^1 \frac{|a_t(F)|^2}{M_t} dt \right),$$

or

$$(5.5) \quad \mathbb{E}(F \log F) - \mathbb{E}(F) \log \mathbb{E}(F) = \frac{1}{2} \mathbb{E} \left( \int_0^1 \frac{|a_t(F)|^2}{M_t} dt \right).$$

Denote  $k_t(F) = \mathbb{E}^{\mathcal{F}_t}(D_t^{\mathbb{P}} F)$ . Then  $k_t(F^2) = 2\mathbb{E}^{\mathcal{F}_t}(F D_t^{\mathbb{P}} F)$ . Let  $j_t = 2|F| |D_t^{\mathbb{P}} F|_{H_0}$ . We have  $|k_t(F^2)|_{H_0} \leq \mathbb{E}^{\mathcal{F}_t}(j_t)$ .

Now thanks to (5.4), it follows that

$$|\alpha_t(F^2)| \leq \mathbb{E}^{\mathcal{F}_t} \left( j_t + \frac{K}{2} \int_t^1 \exp(K(s-t)/2) j_s ds \right).$$

The same computation as in [3] yields the following result.

**THEOREM 5.3.** *We have*

$$(5.6) \quad \mathbb{E}(F^2 \log F^2) - \mathbb{E}(F^2) \log \mathbb{E}(F^2) \leq 2 e^K \mathbb{E} \left( \int_0^1 |D_t^{\mathbb{P}} F|_{H_0}^2 dt \right)$$

for all cylindrical functions  $F$  on  $\mathbb{P}_e(\mathcal{L}_e(G))$ .

Now let  $T > 0$ . Denote by  $\nu_T$  the law of the Brownian motion  $g_x(t)$  on  $\mathcal{L}_e(G)$  at the time  $T$  and  $\mathbb{E}_T$  the expectation with respect to  $\nu_T$ . Then we have the following theorem.

**THEOREM 5.4 (Driver-Lohrenz).** *It holds that*

$$(5.7) \quad \mathbb{E}_T(f^2 \log f^2) - \mathbb{E}_T(f^2) \log \mathbb{E}_T(f^2) \leq 2Te^{KT} \mathbb{E}_T(|\nabla^{\mathcal{L}_e(G)} f|_{H_0}^2)$$

for all cylindrical function  $f$  on  $\mathcal{L}_e(G)$ .

**PROOF.** Consider the space of continuous paths from  $[0, T]$  into  $\mathcal{L}_e(G)$  and  $F(\gamma) = f(\gamma_T)$ . By (5.6), we have

$$(5.8) \quad \mathbb{E}_T(f^2 \log f^2) - \mathbb{E}_T(f^2) \log \mathbb{E}_T(f^2) \leq 2e^{KT} \mathbb{E} \left( \int_0^T |D_t^{\mathbb{P}} F|_{H_0}^2 dt \right).$$

Now by (3.17),  $\int_0^T |D_t^{\mathbb{P}} F|_{H_0}^2 dt \leq T |\nabla^{\mathcal{L}_e(G)} f|_{H_0}(\gamma_T)|_{H_0}$ . Therefore, combining with (5.8), we obtain (5.7).  $\square$

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