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THE LIMITS OF STOCHASTIC INTEGRALS OF DIFFERENTIAL FORMS

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This paper is concerned with the integration (of 1-forms) against the Markov stochastic process associated with a second-order elliptic differential operator in divergence form. It focuses on the limiting behavior of the integral as the process leaves a fixed point or goes to infinity. This extends previous work in the area where advantage was usually taken of the fact that the operator was self adjoint and started with the associated invariant measure. Applications are given. For example, it is a trivial consequence that the diffusion associated to a uniformly elliptic operator on a negatively curved Cartan–Hadamard manifold has an asymptotic direction (recovering and strengthening the previous arguments of Pratt, Sullivan and others). The approach can also be used to construct a Lévy area for such processes, to study the thinness of sets for the elliptic operator, and probably has wider applications.

1. Introduction. The starting point for this paper consists of a connected manifold E on which a strictly elliptic divergence form operator

$$Lu = \rho^{-1} \sum_{ij} \frac{\partial}{\partial x^i} \rho g^{ij} \frac{\partial}{\partial x^j} u + b^j \frac{\partial}{\partial x^j} u,$$

with measurable coefficients, is defined, which has been closed via its Friedrich extension and to which has been associated a diffusion process X in E (and whose finite-dimensional distributions are given by the minimal positive solutions to associated parabolic equations). An expert will realize, from the intepretation of divergence form operators via duality and the notation we use for the operator L, which puts in evidence a natural volume measure V and an energy form, that it is easy to extend some of our arguments to (new) results in other Dirichlet space settings; we have avoided such generalization in order to keep the paper accessible.

If a bounded function f is in $L^2(V)$ and has finite energy and $o \in E$ is a fixed point, then the process $f(X_t)$ can be uniquely decomposed under P^o, in the form

$$f(X_t) = f(X_s) + \tfrac{1}{2} M_t^s - \tfrac{1}{2} \overline{M}_s^t - \alpha_t^s + \beta_t^s, \qquad 0 < s \le t,$$

where for fixed s, $(M_t^s,\ t \ge s)$ is an additive functional martingale and for fixed t, $(\overline{M}_s^t, 0 < s \le t)$ is a martingale with respect to the backward filtration of the process. The processes $(\alpha_t^s, \beta_t^s, t \ge s)$, with s fixed, are additive functionals of

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finite variation. The decomposition allows one to define the stochastic integral

$$\int_s^t \omega \circ dX$$

of a 1-form ω against the process X, thus continuing the works [14], [15] and [13].

Two problems arise for such a stochastic integral: (1) the existence of the limit as $s \searrow 0$ under P^o; (2) the existence of the limit as $t \nearrow \zeta$ (the lifetime of the process). It is the aim of this paper to study these two problems. Theorem 5.2 gives rather general conditions on ω under which the stochastic integral admits P^o-almost surely finite limits as $s \searrow 0$. This result is in fact a local one. It depends on certain estimates of the logarithmic gradient of the heat kernel. For smooth enough coefficients of L, adequate estimates are classical. We give what we believe is a new estimate in the case of nonsmooth coefficients (Theorem 8.1), which turns out to be powerful enough for our purposes.

Theorem 5.4 gives general conditions ensuring that the stochastic integral has finite limits as $t \nearrow \zeta$. Explicit and more concrete results are obtained in the case of a negatively curved complete Riemannian manifold, with L the Laplace operator. Theorem 7.1 provides a precise convergence and a strong control of the rate of convergence in the spirit of [21]. A more general result is obtained in Corollary 7.6. However, the convergence given by this corollary is not very precise, being like the convergence of a semimartingale. This kind of result had been initially obtained by Prat [17] for the angular coordinate of Brownian motion on a negatively curved two-dimensional complete manifold.

Part of the results of this paper has been stated in a slightly different preliminary form in [12] without detailed proofs.

2. Preliminaries on the framework. Let E be a connected N-dimensional manifold, $N \geq 2$, and g a measurable metric such that for each point $e \in E$ there exists a chart (U, ϕ) with $e \in U$ and in local coordinates, the following conditions are satisfied:

$$\lambda^{-1}|\xi|^2 \leq g_{ij}(x)\xi^i\xi^j \leq \lambda|\xi|^2, \qquad x \in U, \ \xi \in \mathbf{R}^N, \tag{1}$$

the constant $\lambda \geq 1$ depending on the chart and the point e. Let V be a measure on E such that in local coordinates it is written with a density ρ, $V(dx) = \rho dx$, which satisfies the condition

$$\lambda^{-1} \leq \rho(x) \leq \lambda, \qquad x \in U, \tag{2}$$

with the same constant λ from above. The gradient of a function u is expressed in local coordinates as the vector with components

$$\nabla u^i = \sum_j g^{ij}\frac{\partial u}{\partial x^j},$$

where (g^{ij}) is the inverse of the matrix (g_{ij}). The energy (or perhaps more correctly "action") form is defined by integrating the pointwise scalar product of the gradients $\langle \nabla u(x), \nabla v(x) \rangle = g(\nabla u(x), \nabla v(x))$,

$$\mathscr{E}(u, v) = \int_E \langle \nabla u, \nabla v \rangle \, dV, \qquad u, v \in \mathscr{C}_c^\infty(E).$$

Let $(H, \mathscr{E})$ be the Dirichlet space obtained as closure of the space $\mathscr{C}_c^\infty(E)$ with respect to $\mathscr{E}(u, u) + \|u\|_2^2 =: \mathscr{E}_1(u, u)$ (see [6] and [16] for basic facts concerning Dirichlet spaces).

The infinitesimal generator $(\Delta, \mathscr{D}(\Delta))$ of the resolvent of this Dirichlet space is expressed in local coordinates in the variational form

$$\Delta u = \rho^{-1} \sum_{ij} \frac{\partial}{\partial x^i} \, \rho g^{ij} \, \frac{\partial}{\partial x^j} u.$$

Consider also a measurable vector field $b := b^j(\partial/\partial x^j)$, such that $|b|$ is bounded on E (the magnitude $|\ |$ of a tangent vector being measured with the matrix (g_{ij}) and using the same notation $|\ |$ for the uniform norm). The uniform or L_∞ norm of a vector field has a useful alternative characterization as the smallest value for $|b|$ so that for every $u \in L^2$, $v \in H$ the inequality

$$\int_E u b(v) \, dV \le |b| \, \|u\|_2 \sqrt{\mathscr{E}(v, v)}$$

holds. In the same way we will use $|\ |$ to indicate the magnitude of a cotangent (measured by g^{ij}) and for the unifom norm of a 1-form.

We define an operator L by setting $\mathscr{D}(L) = \mathscr{D}(\Delta)$ and

$$Lu = \Delta u + b(u), \qquad u \in \mathscr{D}(\Delta).$$

In order to obtain the semigroup generated by L, (P_t) and its adjoint P_t^* on the space $L^2(dV)$, introduce the (nonsymmetric) bilinear form

$$\overline{\mathscr{E}}(u, v) = \mathscr{E}(u, v) - (bu, v), \qquad u, v \in H.$$

For large $\gamma \in \mathbb{R}$, the form $\overline{\mathscr{E}}_\gamma = \overline{\mathscr{E}} + \gamma(\cdot, \cdot)$ becomes a coercive closed form in the sense of [16]. Therefore it generates a semigroup P_t^γ, whose infinitesimal generator turns out to be $L - \gamma$. Then the semigroup generated by L is easily obtained by the formula

$$P_t = e^{\gamma t} P_t^\gamma.$$

2.1. *Properties of the minimal semigroup generated by L.* We record in this section some standard estimates which apply to the semigroup generated by L. Our approach is to regard L as a pertubation of a self-adjoint operator Δ. Standard spectral theory applied to the self-adjoint semigroup and the perturbation estimates that come from iterating the integral form of the Campbell–Baker–Hausdorff–Dynkin equation, which goes back to Philips, quickly give all we will need.

We start with the results from spectral theory. It is standard from the theory of Dirichlet forms that the self-adjoint operator Δ defines a bounded semigroup $Q_t = \exp t\Delta$, which is a positive contraction on $L^2(dV)$ with range contained wholly in $\mathscr{D}(\Delta)$. From the inequality $x \exp -x \leq 1/e$, it is routine from the spectral theorem that

$$\|\Delta\, Q_t\, f\|_2 \leq e^{-1} t^{-1}\, \|f\|_2, \qquad f \in L^2,$$

and from $xe^{-x^2} < 1/\sqrt{2e}$ that

$$\mathscr{E}(Q_t\, f)^{1/2} \leq \frac{1}{\sqrt{2e}} t^{-1/2}\, \|f\|_2, \qquad f \in L^2$$

and $\mathscr{E}((-\Delta)^{1/2} Q_t f, f) \leq (1/\sqrt{2et})\mathscr{E}(f, f)$, which leads to

$$\|\Delta Q_t f\|_2 \leq \frac{1}{\sqrt{2e}} t^{-1/2} \mathscr{E}(f)^{1/2}, \qquad f \in H.$$

Similarly, from the inequality $x^3 e^{-x^2} \leq (3/2e)^{3/2}$, one deduces

$$\mathscr{E}(\Delta Q_t f)^{1/2} \leq \left(\frac{3}{2e}\right)^{3/2} t^{-3/2} \|f\|_2, \qquad f \in L^2$$

and, again from $x \exp -x \leq 1/e$, one obtains

$$\mathscr{E}(\Delta Q_t f)^{1/2} \leq \frac{1}{e} t^{-1} \mathscr{E}(f)^{1/2}, \qquad f \in H.$$

Finally, we have, from the estimate $e^{-x^2} \leq 1$, the bound

$$\mathscr{E}(Q_t f)^{1/2} \leq \mathscr{E}\,(f)^{1/2}, \qquad f \in H.$$

Putting the above together with our characterization of bounded vector fields, we have

$$\|b(Q_t\, f)\|_2 \leq \frac{1}{\sqrt{2e}} t^{-1/2} |b|\, \|f\|_2, \qquad f \in L^2,$$

$$\|b Q_t f\|_2 \leq |b| \mathscr{E}(f)^{1/2}, \qquad f \in H.$$

Now we consider the estimates that follow from perurbation theory. Our interest is in the semigroup (P_t) which is related to Q_t by the equality

$$P_t f = Q_t f + \int_0^t Q_{t-u} b P_u f\, du \tag{3}$$

that is easily obtained by applying the Leibnitz–Newton formula to the function $u \mapsto Q_{t-u} P_u f$. Iteration of this formula gives a sequence of asymptotic expansions that leads to the following expression for the semigroup:

$$P_t f = Q_t f + \sum_{n=1}^{\infty} \int_0^t \int_0^{u_1} \cdots \int_0^{u_{n-1}} Q_{t-u_1} b Q_{u_1-u_2} \ldots b Q_{u_{n-1}-u_n} b Q_{u_n} f\, du_n \ldots du_1.$$

In fact, we are now going to estimate the terms in the expression of $P_t f$ sufficiently carefully to see that the series converges and is adequate to obtain various other a priori estimates that will be important to us. To this end we denote by I_n the general term in the sum expressing $P_t f$,

$$I_n = \int_0^t \int_0^{u_1} \cdots \int_0^{u_{n-1}} Q_{t-u_1} b Q_{u_1-u_2} \cdots b Q_{u_{n-1}-u_n} b Q_{u_n} f \, du_n \cdots du_1$$

and observe from above that

$$\|I_n\|_2 \le \|f\|_2 \int_0^t \int_0^{u_1} \cdots \int_0^{u_{n-1}} \frac{|\, b \,|}{\sqrt{2e}\sqrt{u_1-u_2}} \cdots \frac{|\, b \,|}{\sqrt{2e}\sqrt{u_{n-1}-u_n}} \frac{|\, b \,|}{\sqrt{2e}\sqrt{u_n}} \, du_n \cdots du_1.$$

Computing the integral

$$\int_0^t \int_0^{u_1} \cdots \int_0^{u_{n-1}} \frac{du_n \cdots du_1}{\sqrt{(u_1-u_2)\cdots(u_{n-1}-u_n)u_n}} = t^{n/2}\pi^{n/2}/\Gamma(n/2+1),$$

we arrive at

$$\|I_n\|_2 \le \frac{\|f\|_2}{\Gamma(n/2+1)} \left(\frac{t\pi \,|\, b \,|^2}{2e} \right)^{n/2}$$

and using the fact that $\sum_{n=0}^{\infty} s^{n/2}/\Gamma(n/2+1) < 2e^s$, one gets the inequality

$$\|P_t f\|_2 \le 2\|f\|_2 \exp\left(\frac{t\pi \,|\, b \,|^2}{2e} \right), \qquad f \in L^2. \tag{4}$$

Similarly, one obtains the estimates

$$\mathscr{E}(I_n)^{1/2} \le \|f\|_2 \frac{|\, b \,|^n}{(\sqrt{2e})^{n+1}} \frac{t^{(n-1)/2}\pi^{(n+1)/2}}{\Gamma((n+1)/2)},$$

$$\mathscr{E}(I_n)^{1/2} \le \mathscr{E}(f)^{1/2} \frac{|\, b \,|^n}{(\sqrt{2e})^n} \frac{t^{n/2}\pi^{n/2}}{\Gamma((n+2)/2)}$$

and hence

$$\mathscr{E}(P_t f)^{1/2} \le \|f\|_2 \left(\frac{t^{-1/2}}{\sqrt{2e}} + \frac{|\, b \,|\, \pi}{e} \exp\left(\frac{t\pi \,|\, b \,|^2}{2e} \right) \right), \qquad f \in L^2, \tag{5}$$

$$\mathscr{E}(P_t f)^{1/2} \le \mathscr{E}(f)^{1/2} 2 \exp\left(\frac{t\pi \,|\, b \,|^2}{2e} \right), \qquad f \in H. \tag{6}$$

In order to differentiate with respect to time the expression of $P_t f$, one writes the general term in the form $I_n = t^n J_n$, where

$$J_n = \int_0^1 \int_0^{u_1} \cdots \int_0^{u_{n-1}} Q_{t(1-u_1)} b Q_{t(u_1-u_2)} \cdots b Q_{t(u_{n-1}-u_n)} b Q_{tu_n} f \, du_n \cdots du_1,$$

thus obtaining

$$\frac{\partial}{\partial t} I_n = n t^{n-1} J_n + t^n \frac{\partial}{\partial t} J_n,$$

which by straightforward estimates leads to

$$\left\| \frac{\partial}{\partial t} I_n \right\|_2 \leq \|f\|_2 \left(\frac{n}{t} + \frac{1}{et} + \frac{n}{2et} 3^{3/2} \right) \left(\frac{t\pi \mid b \mid^2}{2e} \right)^{n/2} \Big/ \Gamma(n/2+1),$$

$$\left\| \frac{\partial}{\partial t} I_n \right\|_2 \leq \mathscr{E}(f)^{1/2} \left(n + \frac{2}{e} + \frac{n-1}{2e} 3^{3/2} \right) \mid b \mid \left(\frac{t\pi \mid b \mid^2}{2e} \right)^{(n-1)/2} \Big/ \Gamma\left(\frac{n+1}{2} + 1 \right).$$

Finally, one deduces

$$\left\| \frac{\partial}{\partial t} P_t f \right\|_2 \leq \|f\|_2 \left[t^{-1} \frac{2}{e} \exp \frac{t\pi \mid b \mid^2}{2e} + t^{-1/2} \mid b \mid \sqrt{\frac{2}{e}} \left(1 + \frac{1}{2e} 3^{3/2} \right) + \frac{2\pi \mid b \mid^2}{e} \left(1 + \frac{1}{2e} 3^{3/2} \right) \exp \frac{t\pi \mid b \mid^2}{2e} \right], \qquad f \in L^2, \tag{7}$$

$$\left\| \frac{\partial}{\partial t} P_t f \right\|_2 \leq \mathscr{E}(f)^{1/2} \left[\frac{1}{\sqrt{2e}} t^{-1/2} + 8 \mid b \mid \exp \frac{t\pi \mid b \mid^2}{2e} \right], \qquad f \in H. \tag{8}$$

Relation (7) should be interpreted as saying that $P_t f \in \mathscr{D}(L)$ for each $f \in L^2$ and $LP_t f = (\partial/\partial t) P_t f$ satisfies the inequality.

We end this subsection by deducing some similar estimates for the adjoint semigroup. Clearly from (4) one has

$$\|P_t^* f\|_2 \leq 2\|f\|_2 \exp \frac{t\pi \mid b \mid^2}{2e}, \qquad f \in L^2. \tag{9}$$

Also, one sees easily that $P_t^* f \in \mathscr{D}(L^*)$ for each $f \in L^2$ and inequality (7) implies

$$\|L^* P_t^* f\|_2 \leq \|f\|_2 \left[t^{-1} \frac{2}{e} \exp \frac{t\pi \mid b \mid^2}{2e} + t^{-1/2} \mid b \mid \sqrt{\frac{2}{e}} \left(1 + \frac{1}{2e} 3^{3/2} \right) + \frac{2\pi \mid b \mid^2}{e} \left(1 + \frac{1}{2e} 3^{3/2} \right) \exp \frac{t\pi \mid b \mid^2}{2e} \right], \qquad f \in L^2. \tag{10}$$

From the identity

$$\mathscr{E}(P_t^* f, g) = -(f, LP_t g) + (P_t^* f, bg)$$

and making use of (8) and (9), one deduces

$$\mathscr{E}(P_t^* f)^{1/2} \leq \|f\|_2 \left(\frac{1}{2e} t^{-1/2} + 10 \mid b \mid \exp \frac{t\pi \mid b \mid^2}{2e} \right), \qquad f \in L^2. \tag{11}$$

Analogously, from the identity

$$\mathcal{E}(P_t^* f, g) = \mathcal{E}(f, P_t g) - (f, bP_t g) + (P_t^* f, bg)$$

and the inequalities (6) and (9), one gets

$$\mathcal{E}(P_t^* f)^{1/2} \leq 2 \exp \frac{t\pi \mid b \mid^2}{2e} (\mathcal{E}(f)^{1/2} + 2 \mid b \mid \|f\|_2), \qquad f \in H. \tag{12}$$

The equality

$$(L^* P_t^* f, g) = -\mathcal{E}(f, P_t g) + (f, bP_t g)$$

and the estimate (5) lead to

$$\|L^* P_t^* f\|_2 \leq (\mathcal{E}(f)^{1/2} + \mid b \mid \ \|f\|_2) \times \left(\frac{1}{\sqrt{2e}} t^{-1/2} + \frac{\mid b \mid \pi}{e} \exp \frac{t\pi \mid b \mid^2}{2e} \right), \qquad f \in H.$$

2.2. *The semigroup density.* The following theorem introduces the densities of the semigroup (P_t) and the diffusion process generated by L (whose transition function is given by the densities). The proof is rather lengthy and is postponed to Appendix A. It is based on the classical properties of parabolic equations in divergence form proved by Nash, de Giorgii, Moser and Aronson.

THEOREM 2.1. *There exists a continuous strictly positive function* $p_t(x, y)$ *defined on* $(0, \infty) \times E \times E$, *such that*

$$p_{t+s}(x, y) = \int p_t(x, z) p_s(z, y) V(dz),$$

$$P_t f(x) = \int p_t(x, y) f(y) \ V(dy),$$

$$P_t^* f(x) = \int f(y) p_t(y, x) V(dy).$$

For each fixed point y, U *a neighborhood of* y, $\varepsilon > 0$ *and* K *a compact set, the following relations hold*:

(i) $$\lim_{t \to 0} \sup_{x \in U^c,\, s \in (0,t)} p_s(x, y) = 0,$$

(ii) $$\sup_{z \in K,\, x \in E,\, s > \varepsilon} p_s(x, z) < \infty.$$

Moreover, there exists a diffusion process with transition function $p_t(x, y)\, V(dy)$.

For the purposes of this paper, it is useful that the density be a bounded function in the second variable, too; that is, a bounding relation dual to (ii) of Theorem 2.1 is of interest. More precisely, since our treatment involves a single probability measure P^o, corresponding to the process started at a fixed point, it turns out that the following is what we need throughout this work.

HYPOTHESIS. There exists a point $o \in E$ such that the function $p_t(x) = p_t(o, x)$ satisfies the following condition for each $\varepsilon > 0$:

$$\sup\{p_t(x) \mid x \in E,\ t \in [\varepsilon, \varepsilon^{-1}]\} < \infty.$$

Since $P_t 1 \le 1$, this hypothesis implies that $\| p_t\|_2 \le \sqrt{\| p_t\|_\infty} < \infty$. Therefore, $p_t = P^*_{t/2}(p_{t/2})$ belongs to $\mathscr{D}(L^*)$ and to H. Similarly, $p_t(\cdot, x)$ belongs to $\mathscr{D}(\Delta)$ for each $x \in E$. Also, if $f \in L^2$, then the random variable $f(X_t)$ is P^o integrable for each $t > 0$ and

$$E^o(| f(X_t) |) \le \|f\|_2 \sqrt{\| p_t\|_\infty}.$$

Another consequence of the hypothesis we made, which will be essentially used, is contained in the following lemma.

LEMMA 2.2. *The following inequality holds for* $0 < t < \infty$:

$$\int_s^t \left| \triangledown\, p_e\right|^2 dV\, dl \le \left|\| p_t\|_2^2 - \| p_s\|_2^2\right| + (t-s)\|b\|_\infty^2 \sup_{s\le l\le t} \| p_l\|_2.$$

PROOF. Set $F_l = \int p_l^2\, dV$ and derive

$$F_l' = 2\int p_l L^* p_l = -2\int \left| \triangledown\, p_l\right|^2 + 2\int p_l b(p_l).$$

Therefore, one deduces

$$2\int_s^t \int \left| \triangledown\, p_l\right|^2 dl = \int p_s^2 - \int p_t^2 + 2\int_s^t \int p_l b(p_l)\, dl.$$

Then use the following inequality:

$$2\left|\int p_l b(p_l)\right| \le \|b\|_\infty \left(\varepsilon \int | \triangledown\, p_l|^2 + \varepsilon^{-1} \int p_l^2\right). \qquad \square$$

The next proposition gives a sufficient condition for the fulfilment of the above hypothesis with respect of all points in E. This condition is expressed in terms of the divergence of the drift b. [In our context, the divergence of a measurable vector field X is a distribution $\operatorname{div} X$ defined by the relation

$$\operatorname{div} X(\varphi) = -\int X(\varphi)\, dV, \qquad \varphi \in \mathscr{C}_c^\infty.$$

In local coordinates one has $\operatorname{div} X = (\partial/\partial x^i)(\rho X^i)$.]

PROPOSITION 2.3. *Suppose that the divergence* $\operatorname{div} b$ *is bounded from below in the sense that there exists a constant* γ *such that* $\gamma V + \operatorname{div} b$ *is a nonnegative* *measure. Then for each* $y \in E$, *each open neighborhood* U, *each compact set* K *and* $\varepsilon > 0$, *one has*

$$\text{(i)} \qquad \lim_{t \to 0} \sup_{x \in U^c,\, s \in (0,t)} p_s(y, x) = 0,$$

$$\text{(ii)} \qquad \sup_{x \in E,\, y \in K,\, s > \varepsilon} e^{-\gamma s} p_s(y, x) < \infty.$$

PROOF. First we note that the assumption on the drift b can be rewritten as

$$\int (\gamma\varphi - b\varphi)\, dV \geq 0, \qquad \varphi \in \mathscr{C}_c^\infty(E), \qquad \varphi \geq 0.$$

This condition implies that $\overline{\mathscr{E}}_\gamma = \overline{\mathscr{E}} + \gamma(\cdot, \cdot)$ is a Dirichlet form in the sense of [16] and hence, the dual semigroup $e^{-\gamma t} P_t^*$ is sub-Markovian. The reasoning at the end of the proof of Theorem 2.1 can be applied and gives the desired estimates. □

Finally we remark that, without any assumption on the drift b, if we restrict our frame to a relatively compact open subset $E' \subset E$, then the basic hypothesis holds true for any point in E'.

3. Decomposition with forward and backward martingales. The operator L generates a diffusion process in E which, in general, is not necessarily conservative. So the infinity point Δ associated to E will, from time to time, play a (rather formal) role in what follows. We keep the traditional notation for this point and hope the distinction from the "Laplace operator" Δ, which we also use, will be clear from the context.

We denote by Ω the set of all continuous maps ω: $[0, \infty) \to E_\Delta = E \cup \{\Delta\}$ which admit lifetime $\zeta(\omega) = \inf\{t\colon \omega(t) = \Delta\}$ in the sense that $\omega(t) = \Delta$ for each $t \geq \zeta(\omega)$. As usual, the process of projections is denoted by (X_t), that is, $X_t(\omega) = \omega(t)$. We denote by $(\Im_t)$ the usual filtration, obtained by completion, and denote by $(\Im'_t)$ the "backward filtration"; that is, $\Im'_t$ is the completion of $\sigma(X_s\colon s \geq t)$. The shift operators are denoted by θ_t and the probability measures associated with the transition function of the minimal semigroup (P_t) are denoted P^x, $x \in E$. We use the common notation of Markov process theory (see [2]). For example, any function defined on E is automatically extended to $E_\triangle$ with $f(\triangle) = 0$ and so on.

If $f \in H$, then by choosing an appropriate version, we always assume that it is quasi-continuous so that $f(X_t)$ is a continuous process. This is possible according to Appendix C.

Now for each $f \in H$ we introduce the two-parameter processes αf and βf, defined by

$$\begin{aligned} \alpha f_t^s = \alpha_t^s &= \int_s^t p_u^{-1}(X_u) g(\nabla p_u, \nabla f)(X_u)\, du, \\ \beta f_s^t = \beta_s^t &= \int_s^t bf(X_u)\, du, \end{aligned} \tag{13}$$

where $0 < s < t < \infty$ and $p_t = p_t(o, \cdot)$ is the function associated to the fixed point $o \in E$. In the rest of this section we will be concerned with properties which hold under the fixed probability P^o. First we remark that the above processes have finite variation on intervals $[s,t]$ with $0 < s < t < \infty$. Indeed we have, by Lemma 2.2,

$$E^o(\operatorname{var} \alpha_t^s) = \int_s^t \int \big|g(\nabla p_u, \nabla f)\big| \, dV \, du$$
$$\leq (t-s)^{1/2} \mathscr{E}(f)^{1/2} \left(\int_s^t \int |\nabla p_u|^2 \, dV \, du \right)^{1/2} < \infty.$$

Similarly, for the process βf,

$$E^o(\operatorname{var} \beta_t^s) = \int_s^t \int |b(f)| \, p_u \, dV \, du$$
$$\leq (t-s) \mathscr{E}(f)^{1/2} \, \mid b \mid \sup_{s \leq u \leq t} \|p_u\|_\infty^{1/2} < \infty.$$

Let us now suppose that $f \in \mathscr{D}(\Delta) \cap L^\infty$ and introduce the processes Mf, $\overline{M}f$ defined by

$$Mf_t^s = M_t^s = f(X_t) - f(X_s) - \int_s^t \Delta f(X_u) \, du - \beta_t^s,$$
$$\overline{M}f_s^t = \overline{M}_s^t = f(X_s) - f(X_t) - \int_s^t \Delta f(X_u) \, du - 2\alpha_t^s + \beta_t^s.$$

The process $\int_s^t \Delta f(X_u) \, du$ has finite variation and its mean value is bounded by

$$E^o\left(\int_s^t |\Delta f|(X_u) \, du \right) \leq \|\Delta f\|_2 \int_s^t \sqrt{\|p_u\|_\infty} \, du < \infty.$$

The process Mf is meaningful when $s > 0$ is fixed and t belongs to $[s, \infty)$ as seen from the next proposition. The process $\overline{M}f$ is more interesting when t is fixed and s belongs to $(0, t]$. This is the reason for the use of the parameters s and t up or down. The processes α and β are interesting both forward and backward. All of the random variables α_t^s, β_t^s, M_t^s and $\overline{M}_s^t$ are measurable with respect to $\sigma(X_u;\ s \leq u \leq t)$ so that for fixed s, $(M_t^s\colon t \geq s)$ is adapted to $(\Im_t)$ and for fixed t, $(\overline{M}_{t-u}^t;\ 0 \leq u \leq t)$ is adapted to $(\Im'_{t-u})$.

PROPOSITION 3.1. *Under P^o the following hold.*

(i) *For each fixed $s > 0$, the process $(M_t^s, \Im_t,\ t \geq s)$ is an L^2-martingale and*

$$\langle M_\bullet^s \rangle_t = 2 \int_s^t |\nabla f|^2(X_u) \, du.$$

(ii) *For each fixed* $t > 0$, *the process* $(\overline{M}^t_{t-u}, \Im'_{t-u},\ 0 \le u < t)$ *is an* L^2-*martingale and*

$$\langle \overline{M}^t_{t-\bullet} \rangle_u = 2\int_{t-u}^t |\nabla f|^2 (X_\ell)\, d\ell.$$

(iii) *The following relations hold almost surely* $(0 < u < s < t)$:

$$\begin{aligned} f(X_t) - f(X_s) &= \tfrac{1}{2} M^s_t - \tfrac{1}{2}\overline{M}^t_s - \alpha^s_t + \beta^s_t, \\ M^s_t &= M^u_t - M^u_s, \\ \overline{M}^s_u &= \overline{M}^t_u - \overline{M}^t_s. \end{aligned}$$

(iv) *If* h *is another function in* $\mathscr{D}(\Delta) \cap L^\infty$, *then*

$$\langle M f^s_\bullet, M h^s_\bullet \rangle_t = 2\int_s^t g(\nabla f, \nabla h)(X_u)\, du,$$

$$\langle \overline{M} f^t_{t-\bullet}, \overline{M} h^t_{t-\bullet} \rangle_u = 2\int_{t-u}^t g(\nabla f, \nabla h)(X_\ell)\, d\ell.$$

PROOF. (i) The relation

$$P_{t-s} f - f = \int_0^{t-s} P_u\, L\, f\, du$$

holds in L^2 and shows that $E^x(M^s_t) = 0$, for almost evry $x \in E$. Then, by the Markov property it follows that M^s is a martingale. In order to compute its bracket, we set

$$H_t = \int_s^t L\, f(X_l)\, d\ell,$$

so that we may square the expression for M^s_t and write it in the form

$$(M^s_t)^2 = f^2(X_t) - f^2(X_s) - 2M^s_t(f(X_s) + H_t) - 2f(X_s)H_t - H^2_t.$$

On the other hand, the function f^2 belongs to L^2 and, on account of Lemma 3.2, the following equality holds in L^2:

$$P_{t-s} f^2 - f^2 = \int_0^{t-s} L P_u f^2\, du = 2\int_0^{t-s} P_u(f L f + |\nabla f|^2)\, du.$$

By the Markov property we deduce that

$$M(f^2)^s_t = f^2(X_t) - f^2(X_s) - 2\int_s^t (f\, L\, f(X_u) + |\nabla f|^2(X_u))\, du$$

is a martingale too. From these relations and using also

$$H^2_t = 2\int_s^t H_u\, dH_u,$$

$$M^s_t\, H_t = \int_s^t H_u\, dM^s_u + \int_s^t M^s_u\, dH_u,$$

one deduces

$$(M_t^s)^2 = M(f^2)_t^s + 2\int_s^t |\nabla f|^2 (X_u)\, du - 2f(X_s)M_t^s - 2\int_s^t H_u\, dM_u^s,$$

which establishes the required formula for the bracket of $M_\bullet^s$.

(ii) In order to avoid any ambiguity, in what follows we assume that the process is nonconservative and leave to the reader the simpler case when it is conservative. When the process is nonconservative, we have $P_t(x, E) < 1$, for each $x \in E$ and $t > 0$, because of ellipticity of L and connectivity of E. Then, we extend the density function $p_t(x, y)$ to E_Δ and set $\bar{p}_t(x, y) = p_t(x, y)$, for $x, y \in E$, $\bar{p}_t(x, \Delta) = 1 - P_t(x, E)$, $\bar{p}_t(\Delta, x) = 0$, for $x \in E$ and $\bar{p}_t(\Delta, \Delta) = 1$. We extend also the measure V and set $\overline{V}(dy) = 1_E(y)\, V(dy) + \epsilon_\Delta(dy)$. The semigroup (P_t) has the canonical extension $(\overline{P}_t)$ defined by

$$\overline{P}_t\, h(x) = \int_{E_\Delta} \bar{p}_t(x, y)\, h(y)\, \overline{V}(dy), \qquad x \in E_\Delta,\ h \in \mathscr{B}_b(E_\Delta).$$

The adjoint semigroup (P_t^*) is extended to E_Δ by the adjoint of $\overline{P}_t$, given for $h \in \mathscr{B}_b(E_\Delta)$ by

$$\overline{P}_t^* h(x) = P_t^*(h1_E)(x) \quad \text{if } x \in E,$$

$$\overline{P}_t^*\, h(\Delta) = h(\Delta) + \int_E (1 - P_t 1(x))\; h(x)\, V(dx).$$

The process (X_t) is Markovian under P^o, with the semigroup $(\overline{P}_t)$ on E_Δ. If we reverse the time from a fixed moment $t > 0$, the process $(X_{t-u})_{0 \le u < t}$ is Markovian, too, under P^o with the nonhomogenous transition function on E_Δ,

$$Q_{u,v}\, h(x) = \bar{p}_{t-u}(x)^{-1}\, \overline{P}_{v-u}^*(\bar{p}_{t-v}\, h)\,(x), \qquad x \in E_\Delta,\ 0 < u < v < t,$$

where $\overline{p_t}(x) = \overline{p_t}(o, x)$. (The function $\bar{p}_t$ is strictly positive on E_Δ for each $t > 0$, under the assumption of nonconservativity.) In the sequel we are going to repeat the arguments used in the above proof of (i). First we have to check that

$$E^o\big(\overline{M}_s^t \mid \Im_t'\big) = 0,$$

which clearly implies the martingale property of $\overline{M}_\bullet^t$. Because of the Markov property for the reversed process, this relation is equivalent to

$$\begin{aligned}(\bar{p}_t)^{-1}\, \overline{P}_{t-s}^*(\bar{p}_s f) = f + (\bar{p}_t)^{-1} \int_0^{t-s} & \overline{P}_u^*(\bar{p}_{t-u}(\Delta f - b(f)) \\ & + 2g(\nabla p_{t-u}, \nabla f))\, du.\end{aligned}$$

This relation should be checked on E_Δ. For x in E it becomes

$$(*)\qquad \begin{aligned} & P_{t-s}^*(p_s f)(x) - (p_t f)(x) \\ & \quad = \int_0^{t-s} P_u^*(p_{t-u}(\Delta f - b(f)) + 2g(\nabla p_{t-u}, \nabla f))\,(x)\, du\end{aligned}$$

and should be proved in $L^1(E, dV)$. Since f vanishes in Δ, for $x = \Delta$ the relation becomes

$$(**)\qquad \begin{aligned}&\int (1 - P_{t-s}1)\, p_s f\, dV \\ &\quad = \int_0^{t-s} \int (1 - P_u 1)\,(p_{t-u}(\Delta f - b(f)) + 2g(\nabla p_{t-u}, \nabla f))\, dV\, du.\end{aligned}$$

In order to check relation $(*)$ one applies the fundamental theorem of calculus to a function with values in the vector space $L^1(dV)$ to get

$$P^*_{t-s} p_s f - p_t f = \int_0^t (L^* P^*_u p_{t-u} f + P^*_u \partial_u p_{t-u} f)\, du.$$

From Appendix D we know that $p_{t-u} f \in \mathscr{D}_{L^1}(L^*)$ and have a formula for $L^*(p_{t-u} f)$. Using this and the fact that $L^* p_t = \partial_t p_t$, one can put the right-hand side of the preceding relation in the form appearing in the right part of $(*)$.

In order to prove relation $(**)$ we first write the left-hand term as

$$\int p_s f - \int P^*_{t-s}(p_s f) = \int p_t f - \int P^*_{t-s}(p_s f) + \int p_s f - \int p_t f$$

and then apply the fundamental theorem of calculus to both differences just obtained:

$$= \int_s^t \int P^*_{t-u}(\partial_u - L^*)(p_u f)\, du - \int_s^t \int \partial_u (p_u f)\, du.$$

By Lemma 3.3 we have

$$\int L^*(p_u f) = 0,$$

so that the above expression becomes

$$= \int_s^t \left[\int P^*_{t-u}(\partial_u - L^*)(p_u f) - \int (\partial_u - L^*)(p_u f) \right] du.$$

This last expression equals the right-hand term of relation $(**)$.

Now, in order to compute the bracket of $\overline{M}^t_\bullet$, one follows the calculations made for M. First write

$$\begin{aligned}\overline{M}^t_{t-u} &= f(X_{t-u}) - f(X_t) - H_u, \\ H_u &= \int_0^u \Gamma_v f(X_{t-v})\, dv, \\ \Gamma_v f &= \Delta f - b(f) + 2(p_{t-v})^{-1} g\,(\nabla p_{t-v}, \nabla f).\end{aligned}$$

By using Lemma 3.4 and the final comments of Appendix D, one obtains a similar backward martingale associated to f^2,

$$\begin{aligned}\overline{M}(f^2)^t_{t-u} &= f^2(X_{t-u}) - f^2(X_t) \\ &\qquad - 2\int_0^u f(X_{t-v})\, dH_v - 2\int_0^u |\nabla f|^2 (X_{t-v})\, dv.\end{aligned}$$

Finally one gets

$$(\overline{M}^t_{t-u})^2 = \overline{M}(f^2)^t_{t-u} - 2f(X_t)\overline{M}^t_{t-u} - 2\int_0^u H_v \, d\overline{M}^t_{t-v} + 2\int_0^u |\nabla f|^2(X_{t-v})\, dv,$$

which proves the formula for the bracket of $\overline{M}^t_{\bullet}$. The remainder of the assertions in the statement are obvious. □

LEMMA 3.2. *If* $f \in \mathscr{D}(\Delta) \cap L^\infty$ *and* $u > 0$, *then* $P_u(|\nabla f|^2) \in L^2$ *and the following relation holds*:

$$LP_u f^2 = 2P_u(fLf) + 2P_u |\nabla f|^2 .$$

PROOF. Start with the scalar product, with $\varphi \in \mathscr{C}_c^\infty$,

$$(LP_u f^2, \varphi) = (f^2, L^* P_u^* \varphi) = -\mathscr{E}(f^2, P_u^*\varphi) + (bf^2, P_u^*\varphi) = -2\int f \langle \nabla f, \nabla P_u^*\varphi\rangle + 2(fbf, P_u^*\varphi).$$

The first term from the right-hand side can be written as

$$-2\int f \langle \nabla f, \nabla P_u^*\varphi\rangle = -2\mathscr{E}(f, fP_u^*\varphi) + 2\int |\nabla f|^2 P_u^*\varphi.$$

Therefore one has

$$(LP_u f^2, \varphi) = 2(Lf, fP_u^*\varphi) + 2\int |\nabla f|^2 P_u^*\varphi. \qquad \square$$

LEMMA 3.3. *If* $u \in \mathscr{D}(L^*) \cap L^\infty$ *and* $v \in \mathscr{D}(\Delta) \cap L^\infty$, *then the following relation holds*:

$$\int L^*(uv)\, dV = 0.$$

PROOF. We apply the proposition from Appendix D and get

$$\int L^*(uv)\, dV = \int (L^*u)v + \int u(\Delta - b)v + 2\mathscr{E}(u, v) = (Lv, u) + (\Delta v - bv, u) + 2\mathscr{E}(u, v) = 0. \qquad \square$$

LEMMA 3.4. *If* $u \in \mathscr{D}(L^*) \cap L^\infty$ *and* $v \in \mathscr{D}_{L^1}(\Delta) \cap H \cap L^\infty$, *then one has* $\int L^*(uv)\, dV = 0$.

The proof is based on the same calculation as the proof of Lemma 3.3, using the comments at the end of Appendix D.

DEFINITION 3.1. Let $A = (A_{s,t}, 0 < s \le t < \infty)$ be a two-parameter process. For $o \in E$ and $s > 0$ fixed, we set

$$e^o_s(A) = \lim_{t \searrow s} E^o((A_{s,t})^2)/2(t-s),$$

provided the limit exists. We say that A has uniformly zero energy if for each $\varepsilon > 0$, the following relation holds:

$$\lim_{u\searrow 0} \sup_{\varepsilon\leq s\leq\varepsilon^{-1}} E^o((A_{s,\,s+u})^2)/u = 0.$$

It is easy to see that the sum of two processes of uniformly zero energy preserves this property.

THEOREM 3.5. *For each $f \in H\cap L^\infty$ there exist two-parameter processes $Mf = M = (M^s_t,\ 0 < s \leq t < \infty)$ and $\overline{M}f = \overline{M} = (\overline{M}^t_s,\ 0 < s \leq t < \infty)$, such that the following conditions are satisfied under the probability P^o:*

(i) *For each fixed $s > 0$, the process $(M^s_t, \Im_t,\ t \geq s)$ is a continuous L^2-martingale and its bracket is given by*

$$\langle M^s_{\bullet}\rangle_t = 2\int_s^t |\nabla f|^2(X_u)\,du.$$

(ii) *If $0 < s \leq u \leq t$, then $M^s_t - M^s_u = M^u_t$, almost surely.*

(iii) *The process $(f(X_t) - f(X_s) - M^s_t,\ 0 < s \leq t < \infty)$ has uniformly zero energy.*

(i′) *For each fixed $t > 0$, the process $(\overline{M}^t_{t-u}, \Im'_{t-u}, 0 \leq u < t)$ is a continuous L^2-martingale and its bracket is*

$$\langle \overline{M}^t_{t-\bullet}\rangle_u = 2\int_{t-u}^t |\nabla f|^2(X\ell)\,d\ell.$$

(ii′) *If $0 < s \leq u \leq t$, then $\overline{M}^t_s - \overline{M}^t_u = \overline{M}^u_s$, almost surely.*

(iii′) *The process $(f(X_t) - f(X_s) + \overline{M}^t_s, 0 < s < t \leq \infty)$ has uniformly zero energy.*

(iv) *The processes $(M^s_t + \overline{M}^t_s,\ 0 < s \leq t < \infty)$, $(\alpha^s_t,\ 0 < s \leq t < \infty)$, $(\beta^s_t,\ 0 < s \leq t < \infty)$, all have uniformly zero energy and the following relation is fullfilled:*

$$f(X_t) - f(X_s) = \tfrac{1}{2}M^s_t - \tfrac{1}{2}\overline{M}^t_s - \alpha^s_t + \beta^s_t, \qquad 0 < s \leq t.$$

PROOF. Let us first consider the case when $f \in \mathscr{D}(\Delta)\cap L^\infty$ is such that $\Delta f - bf \in L^\infty$. Then the assertions (i), (ii), (i′), and (ii′) follow from Proposition 3.1. Assertion (iii) is a consequence of Lemma 3.6. Now, let us check assertion (iii′). Because βf and $\Gamma\Delta f$ are again uniformly of zero energy by Lemma 3.6 (Γ being defined there), all we have to prove is that αf has the same property. Squaring the relation

$$-\overline{M}^t_s = f(X_t) - f(X_s) + 2\alpha^s_t + \Gamma(\Delta f - b(f))^s_t\,,$$

we obtain

$$\begin{aligned}(\overline{M}^t_s)^2 &= (f(X_t) - f(X_s))^2 + 4(\alpha^s_t)^2 + (\Gamma(\triangle f - b(f))^s_t)^2\\ &\quad + 4(f(X_t) - f(X_s))\alpha^s_t + 2(f(X_t) - f(X_s))\Gamma(\Delta f - b(f))^s_t\\ &\quad + 4\alpha^s_t\Gamma(\Delta f - b(f))^s_t.\end{aligned}$$

By Lemmas 3.7 and 3.8 we know that $e_s^o(\overline{M}) = e_s^o(f(X) - f(X)) = (p_s, |\nabla f|^2)$, with a uniform limit relation.

On the other hand, the assumption that $\Delta f - b(f)$ is bounded implies a uniform bound $\Gamma(\Delta f - b(f))_t^s \leq K(t-s)$, with K a constant independent of s and t. This implies that the mean value of the last two terms in the above expression, divided by $(t-s)$, tend to zero uniformly in s, as $t \searrow s$. If we show that

$$(*) \qquad \lim_{u \searrow 0} \sup_{\varepsilon \leq s \leq \varepsilon^{-1}} u^{-1} E^o((f(X_{s+u}) - f(X_s))\alpha_{s+u}^s) = 0,$$

then it follows, from the above relation, that αf has uniformly zero energy. Now, let us prove relation $(*)$. Because f is bounded and $f(X_{\bullet})$ is almost surely continuous, it follows that the random variables

$$Y_u = \sup_{\varepsilon \leq s \leq \varepsilon^{-1}} |f(X_{s+u}) - f(X_s)|$$

are uniformly bounded and tend almost surely to zero, as $u \searrow 0$. On the other hand, the random variables $u^{-1}\alpha_{s+u}^s$ may be expressed as

$$u^{-1}\alpha_{s+u}^s(w) = \int_0^1 h_{s,u}(w, r)\, dr,$$

where $(h_{s,u,} s \in [\varepsilon, \varepsilon^{-1}], u \in (0,1))$ are random variables on the product space $(\Omega \times (0,1), P^o \otimes dr)$, defined by $h_{s,u}(w,r) = p_{s+ru}^{-1} g(\nabla p_{s+ru}, \nabla f)(X_{s+ru})$. So relation $(*)$ would follow if we are able to show that the family of random variables on (Ω, P^o), $(p_u^{-1} g(\nabla p_u, \nabla f)(X_u) u \in [\varepsilon, \varepsilon^{-1}+1])$ is uniformly integrable. If K is a compact set and $\lambda > 0$, then we have

$$\left\{p_u^{-1}|g(\nabla p_u, \nabla f)| > \lambda\right\} \subset K^c \cup D_\lambda,$$

where $D_\lambda = \{|g(\nabla p_u, \nabla f)| > \lambda\gamma\}$ and $\gamma = \inf\{p_u(x)\colon x \in K,\, u \in [\epsilon, \epsilon^{-1}+1]\}$. Then we may write

$$\begin{aligned} &E^o(p_u^{-1}|g(\nabla p_u, \nabla f)|(X_u);\, p^{-1}|g(\nabla p_u, \nabla f)|(X_u) > \lambda) \\ &\quad \leq \int_{K^c \cup D_\lambda} |g(\nabla p_u, \nabla f)|\, dV \\ &\quad \leq \mathscr{E}(p_u)^{1/2}\left(\int_{K^c} |\nabla f|^2\right)^{1/2} + \int_{D_\lambda} |g(\nabla p_u, \nabla f)|. \end{aligned}$$

Now we choose K as large as the first term in the right-hand side is small. Then we use the fact that the family of functions

$$\left\{|g(\nabla p_u, \nabla f)|;\ u \in [\varepsilon, \varepsilon^{-1}+1]\right\}$$

is L^1-continuous, and hence uniformly integrable, so that, letting λ be large, the last term from above is small, too. This completes the proof of relation $(*)$ and of assertion (iii′).

The relation from assertion (iv) is exactly the same as in (iii) of Proposition 3.1. It may be written as

$$(**)\qquad f(X_t)-f(X_s)-M_t^s=-\tfrac{1}{2}(M_t^s+\overline{M}_s^t)-\alpha_t^s+\beta_t^s.$$

From this relation one sees that $(M_t^s+\overline{M}_s^t)$ has uniformly zero energy.

Now let us treat the case of a function of $f\in H\cap L^\infty$. We can choose a sequence $\{f_n\}\subset D(\Delta)\cap L^\infty$ such that $\Delta f_n-bf_n\in L^\infty$, $\|f_n\|_\infty\le\|f\|_\infty$ for each n, and $\lim_n f_n=f$ in H. We may do so using the resolvent (W_λ) generated by $\Delta-b$. This resolvent is sub-Markovian, so that $f_n=n\,W_nf$ is a sequence satisfying the above requirements.

For each fixed $s>0$, the martingales $(Mf_{n\bullet}^s)$ form a Cauchy sequence which has a limit denoted by $Mf_\bullet^s$. Then conditions (i) and (ii) are easily verified. To check property (iii) we write

$$\begin{aligned}f(X_t)-f(X_s)-Mf_t^s&=[f_n(X_t)-f_n(X_s)-Mf_{nt}^s]\\&\quad+[(f-f_n)(X_t)-(f-f_n)(X_s)]\\&\quad+[Mf_{nt}^s-Mf_t^s].\end{aligned}$$

The first term in the right-hand side has uniformly zero energy and the ratios measuring the energy of the other two terms are arbitrarily small as $n\to\infty$, by Lemmas 3.7 and 3.8. The assertions (i′), (ii′) and (iii′) are proved similarly. The relation from (iv) passes through the limit. Then relation $(**)$ from above and the following one,

$$f(X_t)-f(X_s)+\overline{M}_s^t=\tfrac{1}{2}(M_t^s+\overline{M}_s^t)-\alpha_t^s+\beta_t^s,$$

show that both $(M_t^s+\overline{M}_s^t)$ and $(-\alpha_t^s+\beta_t^s)$ have uniformly zero energy. Since (β_t^s) has the same property by Lemma 3.7, it follows that (α_t^s) also has uniformly zero energy. □

LEMMA 3.6. *If $f\in L^2$, then the process Γf defined by $\Gamma f_t^s=\int_s^t f(X_u)\,du$, has uniformly zero energy.*

PROOF. A direct calculation gives

$$\begin{aligned}E^o\left[\left(\int_s^{s+u}f(X_\ell)\,d\ell\right)^2\right]&=2E^o\left[\int_0^u f(X_{s+v})\int_0^{u-v}f(X_{s+v+\ell})\,d\ell\,dv\right]\\&=2\int_0^u\int_0^{u-v}P_{s+v}(fP_\ell f)(o)\,d\ell\,dv.\end{aligned}$$

On the other hand, one has

$$\left|P_{s+v}(fP_\ell f)(o)\right|=\left|(p_{s+v}f,P_\ell f)\right|\le\|f\|_2^2\sup_{\epsilon\le t\le\epsilon^{-1}+1}\|p_t\|_\infty,$$

which leads to the conclusion of the lemma. □

LEMMA 3.7. *For each* $\varepsilon > 0$, *there exists a constant* $C > 0$ *such that for any* $f \in H \cap L^\infty$, $u \in (0,1)$, *and any* $s \in [\varepsilon, \varepsilon^{-1}]$ *the following estimate holds*:

$$E^o((f(X_{s+u}) - f(X_s))^2) \le Cu\mathscr{E}(f)^{1/2}(\|f\|_\infty + \|f\|_2 + \mathscr{E}(f)^{1/2}).$$

Moreover for each fixed f, *the following limit relation acts uniformly in* $s \in [\varepsilon, \varepsilon^{-1}]$:

$$\lim_{u \searrow 0} E^o[(f(X_{s+u}) - f(X_s))^2]/2u = \int p_s |\nabla f|^2 \, dV.$$

PROOF. Write the left-hand side of the inequality as

$$\begin{aligned}(p_s, P_u f^2 - f^2) &- 2(p_s f, P_u f - f)\\ &= \int_0^u \big[-\mathscr{E}(p_s, P_\ell f^2) + (p_s, b(P_\ell f^2))\\ &\qquad + 2\mathscr{E}(p_s f, P_\ell f) - 2(p_s f, b(P_\ell f))\big]\, d\ell.\end{aligned}$$

Because $p_s = P^*_{s-\varepsilon/2} p_{\varepsilon/2}$, from estimate (11) one deduces

$$\mathscr{E}(p_s)^{1/2} \le C_\varepsilon, \ \ s \in [\varepsilon, \varepsilon^{-1}].$$

On the other hand, we have $\mathscr{E}(f^2)^{1/2} \le 2\|f\|_\infty \mathscr{E}(f)^{1/2}$ and hence

$$\mathscr{E}(P_\ell f^2)^{1/2} \le C\|f\|_\infty \mathscr{E}(f)^{1/2}.$$

Estimating term-by-term the above integral, one gets the inequality in the statement. In order to prove the limit relation, we note that the preceding expression under the integral differs from a similar one with P_ℓ replaced with $P_0 = I$, which equals $2\int p_s |\nabla f|^2$, by a quantity that is smaller than

$$C_\varepsilon[\mathscr{E}(P_\ell f^2 - f^2)^{1/2} + (\|f\|_\infty + \mathscr{E}(f)^{1/2})\mathscr{E}(P_\ell f - f)^{1/2}].$$

Now using relation (3), one deduces

$$\mathscr{E}(P_e h - Q_\ell h)^{1/2} \le C\ell^{1/2}\mathscr{E}(h)^{1/2}, \qquad h \in H.$$

Since $\lim_{\ell \to o} \mathscr{E}(Q_\ell h - h) = 0$, for each $h \in H$, these estimates imply the desired limit relation. □

LEMMA 3.8. *For each* $\varepsilon > 0$ *there exists a constant* $C > 0$ *such that*

$$E^o\left(\int_s^{s+u} |\nabla f|^2(X_\ell)\, d\ell\right) \le Cu\mathscr{E}(f),$$

for any $f \in H$, $u \in (0,1)$ *and* $s \in [\varepsilon, \varepsilon^{-1}]$. *If* $f \in \mathscr{D}(\Delta) \cap L^\infty$, *then the following limit relation holds uniformly in* $s \in [\varepsilon, \varepsilon^{-1}]$:

$$\lim_{u \searrow 0} E^o\left[\int_s^{s+u} |\nabla f|^2(X_\ell)\, d\ell\right]\Big/ u = \int p_s |\nabla f|^2.$$

PROOF. For the inequality, it suffices to write the term in the left-hand side as

$$\int_s^{s+u} \int p_\ell |\nabla f|^2 \, dV \, d\ell \le u\mathscr{E}(f) \sup_{s\le \ell \le s+u} \|p_\ell\|_\infty.$$

In order to prove the limit relation we have to estimate the difference

$$\int p_\ell |\nabla f|^2 - \int p_s |\nabla f|^2 = \int (p_\ell - p_s)|\nabla f|^2.$$

Setting $h = p_\ell - p_s$ we write

$$\int h|\nabla f|^2 = -\int hf\Delta f - \int fg(\nabla h, \nabla f),$$

so that

$$\left| \int p_\ell |\nabla f|^2 - \int p_s|\nabla f|^2 \right| \le \|f\|_\infty \|\Delta f\|_2 \|p_\ell - p_s\|_2 + \|f\|_\infty \mathscr{E}(f)^{1/2} \mathscr{E}(p_\ell - p_s)^{1/2}.$$

Since $p_\ell - p_s = P^*_{s-\varepsilon}(P^*_{\ell-s} p_\varepsilon - p_\varepsilon)$, from relation (12) we get a uniform estimate that proves the limit relation in the statement. □

REMARK 3.1. If $f, h \in H \cap L^\infty$, then from the formulas expressing the brackets of $M(f+h)$, Mf, Mh one deduces

$$\langle Mf^s_\bullet, Mh^s_\bullet \rangle_t = 2\int_s^t g(\nabla f, \nabla h)(X_\ell)\, d\ell,$$

and a similar relation for the backward martingales,

$$\langle \overline{M} f^t_{t-\bullet}, \overline{M} h^t_{t-\bullet} \rangle_u = 2\int_{t-u}^t g(\nabla f, \nabla h)(X_\ell)\, d\ell.$$

REMARK 3.2. The martingale property and properties (ii) and (iii) in Theorem 3.5 uniquely determine the process Mf. Similarly, the backward martingale property, (ii′), and (iii′) determine the process $\overline{M}f$.

Moreover, both processes Mf and $\overline{M}f$ are uniquely determined by their (forward, respectively, backward) martingale properties, properties (ii) and (ii′), the fact that their sum $Mf + \overline{M}f$ has uniformly zero energy and the relation from (iv) of Theorem 3.5.

These are consequences of the following fact: if a two parameter process $A = (A_{s,t} | 0 < s \le t < \infty)$ has uniformly zero energy and if $\{\tau_n\}$ is a sequence of partitions of an interval $[u, v] \subset (0, \infty)$, with $\delta(\tau_n) \to 0$, then

$$\lim_{n\to\infty} E^o \left(\sum_{t_i \in \tau_n} (A_{t_i, t_{i+1}})^2 \right) = 0,$$

that is, A has zero quadratic variation.

REMARK 3.3. Even if $f \in H$ is not assumed to be bounded, then the assertions (i), (ii), (i′), (ii′) of Theorem 3.5, the fact that $Mf + \overline{M}f$ has uniformly zero energy, and the relation of (iv) all still remain valid.

4. Stochastic integration of differential forms. Let $f \in H \cap L^\infty$ and $u\colon E \to \mathbf{R}$ be such that

$$\int u^2 |\nabla f|^2 \, dV < \infty. \tag{14}$$

Keeping the point $o \in E$ fixed as in the preceding section, we define under P^o the stochastic integral over an interval $[s,t]$, $0 < s < t < \infty$,

$$\begin{aligned}\int_s^t u(X_\ell) \circ df(X_\ell) =: \tfrac{1}{2}\int_s^t u(X_\ell)\, dM_\ell^s - \tfrac{1}{2}\int_0^{t-s} u(X_{t-\ell})\, d_\ell \overline{M}_{t-\ell}^t \\ - \int_s^t u(X_\ell)\, d\alpha_\ell^s + \int_s^t u(X_\ell)\, d\beta_\ell^s,\end{aligned} \tag{15}$$

where $M, \overline{M}, \alpha, \beta$ are associated to f. Note that condition (14) ensures the existence in L^2 of the stochastic integrals with respect to the forward, respectively, backward martingales $M_\bullet^s$ and $\overline{M}_{t-\bullet}^t$ as well as the finiteness of the pathwise ordinary integrals with respect to $d\alpha_\bullet^s$ and $d\beta_\bullet^s$. The next proposition shows that the above definition of stochastic integral is of the Stratonovich type.

PROPOSITION 4.1. *Let $f, u \in H \cap L^\infty$ and assume condition* (14) *is satisfied. Let $\{\tau^n\}$ be a sequence of partitions of $[s,t]$, $0 < s < t < \infty$, such that $\delta((\tau_n)) \to 0$. Then the stochastic integral is approximated in probability as follows*:

$$\int_s^t u(X_\ell) \circ df(X_\ell) = \lim_{n\to\infty} \sum_{t_k \in \tau^n} \tfrac{1}{2}[u(X_{t_k}) + u(X_{t_{k+1}})][f(X_{t_{k+1}}) - f(X_{t_k})].$$

PROOF. Each integral appearing in the right-hand side of (15) is approximated by Riemannian sums, so that their sum is approximated by

$$\begin{aligned}\tfrac{1}{2}\sum u(X_{t_k})(M_{t_{k+1}}^s - M_{t_k}^s) - \tfrac{1}{2}\sum u(X_{t_{k+1}})(\overline{M}_{t_k}^t - \overline{M}_{t_{k+1}}^t) \\ - \sum u(X_{t_{k+1}})(\alpha_{t_{k+1}}^s - \alpha_{t_k}^s) + \sum u(X_{t_{k+1}})(\beta_{t_{k+1}}^s - \beta_{t_h}^s).\end{aligned}$$

If we set $N_\ell^s = f(X_e) - f(X_l) - M_s^s$, by (iii′) of Theorem 3.5 this is uniformly of zero energy. The relation of (iv) in the same theorem tells that

$$N_\ell^s = f(X_s) - f(X_\ell) - \overline{M}_s^\ell - 2\alpha_\ell^s + 2\beta_\ell^s.$$

Therefore the above approximating sums may be written as

$$\begin{aligned}\sum \tfrac{1}{2}(u(X_{t_k}) + u(X_{t_{k+1}}))(f(X_{t_{k+1}}) - f(X_{t_k})) \\ + \tfrac{1}{2}\sum(u(X_{t_{k+1}}) - u(X_{t_k}))(N_{t_{k+1}}^s - N_{t_k}^s).\end{aligned}$$

The mean value of the modulus of the last sum is dominated by the product

$$E^o\Big(\sum(u(X_{t_{k+1}}) - u(X_{t_k}))^2\Big)^{1/2} E^o\Big(\sum(N_{t_{k+1}}^s - N_{t_k}^s)^2\Big)^{1/2},$$

The first factor of this product is bounded by Lemma 3.8. The second factor tends to zero because N has uniformly zero energy. The proposition is proved. □

The following proposition ensures the uniqueness of the forward- and backward-martingales appearing in relation (15).

PROPOSITION 4.2. *Let $f \in H \cap L^\infty$ and u be such that condition* (14) *is satisfied. Then the following process is uniformly of zero energy*:

$$\int_s^t u(x_\ell)\, dM_\ell^s + \int_0^{t-s} u(X_{t-\ell})\, d_\ell \overline{M}_{t-\ell}^t, \qquad 0 < s < t.$$

PROOF. Assume first that $f \in \mathscr{C}_c^\infty$ and $u \in \mathscr{C}_b$. We may write each term of the above expression as follows:

$$\begin{aligned}\int_s^t u(X_\ell)\, dM_\ell^s &= \int_s^t (u(X_\ell) - u(X_s))\, dM_\ell^s \\ &\qquad + \tfrac{1}{2}(u(X_s) + u(X_t))M_t^s + \tfrac{1}{2}(u(X_s) - u(X_t))M_t^s, \\ \int_0^{t-s} u(X_{t-l})\, d_\ell \overline{M}_{t-l}^t &= \int_0^{t-s} u(X_{t-l}) - u(X_t))\, d_\ell \overline{M}_{t-l}^t \\ &\qquad + \tfrac{1}{2}(u(X_s) + u(X_t))\overline{M}_s^t + \tfrac{1}{2}(u(X_t) - u(X_s))\overline{M}_s^t.\end{aligned}$$

The square of the sum of the middle terms appearing in the right-hand side of the above relations is bounded by

$$\left[\tfrac{1}{2}(u(X_s) + u(X_t))(M_t^s + \overline{M}_s^t)\right]^2 \le \|u\|_\infty^2 (M_s^t + \overline{M}_s^t)^2.$$

Then one uses (iv) of Theorem 3.5 to deduce that this sum has uniformly zero energy. Clearly, each of the other terms has uniformly zero energy, too. When f and u are in the general case, one proceeds by approximation and uses Lemma 3.8. □

In order to give a detailed definition of the stochastic integral of differential forms, we need some preparatory lemmas.

LEMMA 4.3. *Let $f^1 \cdots f^n \in H \cap L^\infty$ and $\phi \in C^1(\mathbb{R}^u)$ be such that $\phi(0) = 0$. Then $F = \phi(f^1 \cdots f^n)$ belongs to $H \cap L^\infty$ and the martingales in the decomposition of F are related to the martingales corresponding to $f^1 \cdots f^n$ by the following formulas*:

$$\begin{aligned}MF_t^s &= \sum_{i=1}^n \int_s^t \partial_i \phi(f^1 \cdots f^n)(X_\ell)\, d\, Mf_\ell^{i,s}, \\ \overline{M}F_s^t &= \sum_{i=1}^n \int_0^{t-s} \partial_i \phi(f^1 \cdots f^n)(X_{t-\ell})\, d\, \overline{M}f_{t-\ell}^{i,t}.\end{aligned}$$

Moreover, if u is such that the finiteness condition (14) *is satisfied with each f^i, $i = 1 \cdots n$, then the same condition is verified with F and the following relation holds*:

$$\int_s^t u(X_\ell) \circ dF(X_\ell) = \sum_{i=1}^n \int_s^t u\partial_i \phi(f^i \cdots f^n)(X_\ell) \circ df^i(X_\ell).$$

(The constant functions are in general not in H, so condition $\phi(0) = 0$ is needed to ensure that F is in H.)

PROOF. Just compute the bracket of the difference

$$\left\langle MF_\bullet^s - \sum_{i=1}^n \int_s^\bullet \partial_i \phi(f^1 \cdots f^n)(X_\ell)\, d\, Mf_\ell^{i,s} \right\rangle_t,$$

taking into account the formula $\nabla F = \sum \partial_i \phi(f^1 \cdots f^u) \nabla f^i$. The bracket turns out to be zero, thus proving the first formula. The proof of the second is similar. The last formula follows from the others. □

LEMMA 4.4. *If $f \in H \cap L^\infty$ is constant on an open set $U \subset E$, u vanishes outside a compact set $K \subset U$ and condition* (14) *is satisfied, then*

$$\int_s^t u(X_\ell) \circ df(X_\ell) \equiv 0.$$

PROOF. Each term appearing in definition (3.2) vanishes. For the martingale terms one sees that the brackets vanish. □

Now, let us consider an open set U and a compact set $K \subset U$. Let f be a function defined in U such that it is locally in $H \cap L^\infty$ [in the sense that $f_\bullet \varphi \in H \cap L^\infty$ for each $\phi \in C_c^\infty(U)$]. Let u be a function vanishing outside K and satisfying (14). Then we may define the integral

$$\int_s^t u(X_\ell) \circ df\,(X_\ell)$$

as follows: take a function $\varphi \in C_c^\infty(U)$ such that $\varphi = 1$ on a neighborhood of K and set $h = \varphi_\bullet f$; then define

$$\int_s^t u(X_\ell) \circ df(X_\ell) =: \int_s^t u(X_\ell) \circ dh(X_\ell).$$

By Lemma 4.4, this definition does not depend on the function φ. Also, if $f' = f + c$, with $c \in \mathbb{R}$, then

$$\int_s^t u(X_\ell) \circ df(X_\ell) = \int_s^t u(X_\ell) \circ df'(X_\ell).$$

Now let us extend the last relation of Lemma 4.3. Let $f^1 \cdots f^n$ be functions defined in U such that they are locally in $H \cap L^\infty$. Let V be an open set in $\mathbb{R}^n$ such that the vector $(f^1 \cdots f^n)$ maps U into V. Suppose that $\phi \in C_b^1(\mathbb{R}^n)$

admits an extention as a function ϕ in $C_b^1(\mathbb{R}^n)$ and set $F = \phi(f^s \cdots f^n)$. Then F is locally in $H \cap L^\infty$ and the following relation holds:

$$\int_s^t u(X_\ell) \circ dF(X_\ell) = \sum_{i=1}^n \int_s^t u\partial_i \phi(f^1 \cdots f^n)(X_\ell) \circ df^i(X_\ell),$$

for u vanishing outside K and satisfying condition (14) with respect to each function $f^1 \cdots f^n$.

Now we are able to write down the definition of the stochastic integral of a differential 1-form ω which is in L^2. So let ω be a measurable 1-form such that the pointwise norm $|\omega|(x) = |\omega|_g(x)$ is in L^2. We are going to define the integral

$$\int_s^t \omega \circ dX,$$

such that if ω is of the form $\omega = udf$, it coincides with definition (15). Let $(U_i, \phi_i)_{i\in I}$ be a covering of E with charts. Then choose a partition of the unity $1 = \sum_{i\in I} \Psi_i$, subordinated to the covering and a family $\{\sigma_i\}_{i\in I}$ of functions in $C^\infty(E)$ such that $\operatorname{supp} \sigma_i \subset U_i$ and $\sigma_i = 1$ on a neighborhood of $\operatorname{supp} \Psi_i$. If for $i \in I$, the map ϕ_i has components $\phi_i = (\phi_i^1 \cdots \phi_{i\bullet}^N)$, we set $f_i^j = \sigma_i \phi_i^j$ and if ω is written on U_i as $\omega = \sum_j \omega_{i,j} d\phi_i^j$, then we define (for $0 < s < t < \infty$),

$$\int_s^t \omega \circ dX = \sum_{i\in I} \sum_{j=1}^N \int_s^t \Psi_i \omega_{i,j}(X_\ell) \circ df_i^j(X_\ell). \tag{16}$$

In order to make clear the summation in the above formula, one should decompose each integral in the right-hand side and separately add the components. Writing

$$b(f_i^j) = df_i^j(b), \qquad g(\nabla p_\ell, \nabla f_i^j) = df_i^j(\nabla p_\ell),$$

and $\omega = \sum_i \Psi_i \omega = \sum_i \sum_j \Psi_i \omega_{i,j} df_i^j$, one can see that the sums of β and α-components are

$$\sum_i \sum_j \int_s^t \Psi_i \omega_{i,j}(X_\ell) d\beta(f_i^j)_\ell^s = \int_s^t \omega(b)(X_\ell)\, d\ell,$$

$$\sum_i \sum_j \int_s^t \Psi_i \omega_{i,j}(X_\ell) d\alpha(f_i^j)_\ell^s = \int_s^t p_\ell^{-1}(X_\ell)\omega(\nabla p_\ell)(X_\ell)\, d\ell.$$

Denoting by $\beta\omega_t^s = \beta_t^s$, respectively, $\alpha\omega_t^s = \alpha_t^s$ these sums, one sees that, as in the case of expression (14), one has

$$E^o(\operatorname{Var} \beta_t^s) \le (t-s)\|\omega\|_{L^2}\|b\|_\infty \sup_{s\le u\le t} \|p_u\|_\infty^{1/2} < \infty,$$

$$E^o(\operatorname{Var} \alpha_t^s) \le (t-s)^{1/2}\|\omega\|_{L^2}\left(\int_s^t |\nabla p_\ell|^2\, dV\, dl\right)^{1/2} < \infty.$$

The sums defining α_t^s and β_t^s converge in the mean of variation. The sums corresponding to the forward and backward martingale components,

$$M\omega_t^s = M_t^s = \sum_i \sum_j \int_s^t \Psi_i \omega_{ij}(X_\ell)\, d(Mf_i^j)_\ell^s,$$

$$\overline{M}\omega_s^t = \overline{M}_s^t = \sum_i \sum_j \int_0^{t-s} \Psi_i \omega_{i,\,j}(X_{t-\ell})\, d(\overline{M}f_i^j)_{t-\ell}^t$$

converge as L^2-martingales. The brackets of these martingales are

$$\langle M_\bullet^s \rangle_t = 2\int_s^t |\omega|^2(X_\ell)\, d\ell,$$

$$\langle \overline{M}_{t-\bullet}^t \rangle_{t-s} = 2\int_s^t |\omega|^2(X_\ell)\, d\ell.$$

With this notation, relation (16) may be written as

$$\int_s^t \omega \circ dX = \tfrac{1}{2} M_t^s - \tfrac{1}{2}\overline{M}_s^t - \alpha_t^s + \beta_t^s. \tag{17}$$

By using Lemmas 4.3 and 4.4, one deduces that the above definition is independent of the family of charts (U_i, ϕ_i), of the partition (Ψ_i) as well as of the family of functions (σ_i). Also, because of Proposition 4.2, $M_t^s + \overline{M}_s^t$ has uniformly zero energy, and consequently M and $\overline{M}$ are uniquely determined in relation (17).

This definition may be extended. In order to do so, we are going to point out the local character of the stochastic integral. So, let U be an open set such that ω vanishes on it and set

$$\Lambda_{s,\,t} = \{X_s \in U\} \cap \{T_{U^c} \circ \theta_s > t - s\},$$

for $0 < s < t$. Define the backward stopping time

$$R = \inf\{u > 0\colon X_{t-u} \in U^c\}.$$

Then obviously $\Lambda_{s,\,t} = \{X_t \in U\} \cap \{R > t-s\}$. On the set $\Lambda_{s,\,t}$ the following integrals vanish: α_t^s, β_t^s, $\langle M_\bullet^s \rangle_t$, $\langle \overline{M}_{t-\bullet}^t \rangle_{t-s}$. This implies that each term in the right-hand side of (17) vanishes on the set $\Lambda_{s,\,t}$.

Now, let us assume that ω is satisfying the weaker condition that $|\omega| \in L^2_{\ell\mathrm{oc}}$. Choose an increasing sequence (U_n), of relatively compact open sets such that $E = \bigcup_n U_n$. Then $1_{U_n}|\omega|$ is in L^2, so that the integral of the form $1_{U_n}\omega$ is well defined. Then we define the integral of ω by the limit

$$\int_s^t \omega \circ d\,X = \lim_n \int_s^t 1_{U_n}\omega \circ d\,X,$$

on the set $\{t < \zeta\}$. The sequence of integrals in the right side is stationary almost surely on this set, because it may be written as a union

$$\{t < \zeta\} = \bigcup_n \{t < T_{U_n^c}\}$$

and on the set $\{t < T_{U_n^c}\}$ the integrals corresponding to m's greater than n all coincide. In fact, all the component sequences are almost surely stationary. Let us set

$$\beta(\omega, \zeta)_t^s = 1_{\{t<\zeta\}} \int_s^t \omega(b)(X_\ell)\, d\ell = 1_{\{t<\zeta\}} \lim_n \beta(1_{U_n}\omega)_t^s,$$

$$\alpha(\omega, \zeta)_t^s = 1_{\{t<\zeta\}} \int_s^t p_\ell^{-1} \omega(\nabla p_\ell)(X_\ell)\, d\ell = 1_{\{t<\zeta\}_m} \lim_n \alpha(1_{U_n}\omega)_t^s,$$

$$M(\omega, \zeta)_t^s = 1_{\{t<\zeta\}} \lim_n M(1_{U_n}\omega)_t^s,$$

$$\overline{M}(\omega, \zeta)_s^t = 1_{\{t<\zeta\}} \lim_n \overline{M}(1_{U_n}\omega)_s^t.$$

Of course, we have

$$1_{\{t<\zeta\}} \int_s^t \omega \circ d\, X = \tfrac{1}{2} M(\omega, \zeta)_t^s - \tfrac{1}{2} \overline{M}(\omega, \zeta)_s^t - \alpha(\omega, \zeta)_t^s + \beta(\omega, \zeta)_t^s. \tag{18}$$

REMARK 4.1. For s fixed, the process $(M(\omega, \zeta)_{T_{n\wedge t}}^s,\ t \in [s, \infty))$ is an L^2 martingale, with $T_n = T_{U_n^c}$, $n \in \mathbb{N}$. The process $(M(\omega, \zeta)_t^s,\ t > s)$, however, is not necessarily a local martingale, because $\lim_n T_n = \zeta$, which in general is different from ∞. Instead, the process $\{\overline{M}(w, \zeta)_{t-u}^t,\ \Im_{t'-u},\ u \in [0, t)\}$ is a local martingale with parameter set $[0, t)$ (t fixed) This is easily checked, taking into account that $\{t < \zeta\} = \{X_t \in E\} \in \Im'$ and using the reducing sequence of backward stopping times $R_n = \inf\{u \in (0, t){:}\ X_{t-u} \in U_n^c\}$, (under the natural convention $\inf \varnothing = t$).

5. Limits of stochastic integrals of 1-forms. In this section we are going to study the possibility of taking the limit as $s \searrow 0$ or $t \nearrow \zeta$ in the integral

$$\int_s^t w \circ dX.$$

We approach this limit on the components of stochastical integrals. In order to do so, we need a jointly continuous [in (s, t)] version of each component. The following simple lemma allows us to assume that the components of the stochastic integral are always jointly continuous.

LEMMA 5.1. *Let $(\Omega, \mathscr{E}, P)$ be a probability space, $S \le T$ two real-valued random variables and $(M_t^s,\ S < s \le t \le T)$ a two-parameter real-valued process defined on the random interval $(S, T]$. Assume that for each fixed s the process $(M_t^s, s \le t \le T)$ is continuous on the set $\{S < s \le T\}$ and for each triple u, s, t such that $u \le s \le t$, the following equality holds almost surely on the set $\{S < u, t \le T\}$:*

$$M_t^s = M_t^u - M_s^u.$$

Then the process has a version which is jointly continuous in (s, t) and the above relation is satisfied everywhere.

A two-parameter process as in the conclusion of Lemma 5.1 will be called smooth in what follows.

THEOREM 5.2. *Let ω be a differential form such that $|\omega| \in L^2_{\mathrm{loc}}$ and suppose that U is an open neighborhood of o such that the following conditions are satisfied with $T = T_{U^c}$, the hitting time of U^c*:

$$\text{(i)} \qquad \int_0^T |\omega|^2(X_s)\,ds < \infty \quad P^o \text{ almost surely},$$

$$\text{(ii)} \qquad \int_0^T p_s^{-1}|\omega(\nabla p_s)|(X_s)\,ds < \infty \quad P^o \text{ almost surely}.$$

If we choose a smooth version of the stochastic integral and fix $t > 0$, then the following limit exists and is finite P^o almost surely on $\{t < \zeta\}$:

$$\text{(iii)} \qquad \lim_{s\searrow 0} \int_s^t \omega \circ dX.$$

PROOF. By the above lemma we may suppose that all components of the integral in (iii) are smooth on the set $\{t < \zeta\}$. Condition (ii) ensures the existence of the limit

$$\lim_{s\searrow 0} \alpha(\omega, \zeta)_t^s$$

and similarly, condition (i) together with the inequality $|\omega(b)| \le |\omega|_g |b|_g$ ensure the existence of the limit

$$\lim_{s\searrow 0} \beta(\omega, \zeta)_t^s.$$

The process $(\overline{M}(\omega, \zeta)_{t-u}^t, \mathfrak{F}'_{t-u},\ u \in [0, t))$ is a local martingale. Its bracket on $[0, t)$ is

$$1_{\{t<\zeta\}} \int_0^t |\omega|^2(X_{t-\ell})\,d\ell.$$

Condition (i) ensures that it is P^o-almost surely finite. This implies the existence of the limit $\lim_{s\searrow 0} \overline{M}(\omega, \zeta)_s^t$.

Now, let us look at the process $(M(\omega, \zeta)_t^s, s \in (0, t])$. Take V, a relatively compact open set, and put $R = T_{V^c}$. Then we have

$$M(\omega, \zeta)_{R\wedge t}^s = M(1_V \omega)_{R\wedge t}^s,$$

which is an L^2-martingale for any fixed $s > 0$ and $t \in [s, \infty)$. On the set $\{t < R\}$ one obviously has

$$\lim_{s\searrow 0} M(\omega, \zeta)_t^s = \lim_{s\searrow 0} M(1_V \omega)_{R\wedge t}^s,$$

whenever one of the limits exists. On the other hand, condition (i) leads to the fulfillment of condition (iii′) of Lemma 6.2 for the process $\{M(1_V \omega)_t^s, 0 < s \le t\}$. This lemma implies the existence of the limit in the above right-hand side. Since V is arbitrary, this completes the proof. □

As we remarked in Section 4, the stochastic integral of a form ω has a local character. Now we underline this aspect by remarking that if we restrict the process to an open set U (by killing at the boundary), then the stochastic integral of $\omega_{|U}$ made with this process coincides with the stochastic integral of ω made with the global process before the first exit time $T = T_{U^c}$. So, if U is a small open neighborhood of o, which is included in a chart, it can be included in $\mathbb{R}^N$. Then the infinitesimal generator $L_{|U}$ may be extened to $\mathbb{R}^N$ such that it satisfies the conditions of Section 8. By Theorem 8.1 we get the following result.

COROLLARY 5.3. *Suppose that the form ω is such that $|\omega| \in L^2_{\text{loc}}$ and*

$$\int_0^{1\wedge T} f^{-1}(s)\,|\omega|^2(X_s)\,d\,s < \infty \quad P^o\text{-almost surely},$$

where $f \in C^1((0,1])$ is a function with the following properties: $f > 0$,

$$\limsup_{s\searrow 0} f(s)|\ln s| < \infty,$$

$$\int_0^1 |f'(s)|\,|\ln s|\,d\,s < \infty.$$

Then the conclusion of the above theorem holds.

Remark that, because of Aronson's estimates, the condition in the above corollary is satisfied if $1_U|\omega| \in L^p$ for some neighborhood U, $p > N$ and $f(s) = |\ln s|^{-2}$.

Now, let us direct our attention to the case where ω is of the particular form $\omega = df$ for some $f \in H$. It is not difficult to see that for such a form the stochastic integral is expressed as

$$\int_s^t \omega \circ d\,X = f(X_t) - f(X_s).$$

(To check this, it suffices to take first $f \in C_c^\infty$ and then to make an approximation.) The existence of the limit as $s \searrow 0$ is equivalent to the existence of the limit $\lim_{s\to 0} f(X_s)$, under P^o. Let us suppose that this last limit exists P^o-almost surely. By Blumenthal's 0–1 law, it should be a constant; denote it by ℓ. Then, for each $\varepsilon > 0$, consider the set

$$U_\varepsilon = \{x \in E\colon |f(x) - \ell| < \epsilon\}.$$

The first exit time $T_{U_\epsilon^c}$ is strictly positive under P^o, because of our supposition. In other words U_ϵ is a fine neighborhood of o and this tells us that f admits ℓ as fine limit in o. Thus the above results offer us criteria of existence of fine limit of a function at a point, in terms of its (generalized) first derivatives. If the function $|df| = |\nabla f|$ is in $L^p(U)$, for some open neighborhood U of o, with $p > N$, it is known by Sobolev estimates that f admits a limit in the ordinary topology. This agrees with the above remark, which ensures only the fine limit of the function. However, the above results offer more insight in cases where f is not continuous. So, let us consider the classical case $E = \mathbb{R}^3$ and $L = \Delta$ (Laplace operator $= \sum_{i=1}^{N} \partial^2/\partial x^{i2}$) when the process is Brownian motion. We consider a kind of Lebesgue spine or thorn set,

$$C_\delta = \{(x, y, z) \in \mathbb{R}^3 \colon x > 0,\ (y^2 + z^2)^{1/2} \le \exp -x^{-\delta}\}.$$

Define a function $f\colon \mathbb{R}^3 \to \mathbb{R}$ by setting $r = (x^2 + y^2 + z^2)^{1/2}$, $\theta = \arctan(y^2 + z^2)^{1/2}/x$, $x > 0$ and

$$f(x, y, z) = r^\alpha (\ln 1/\theta)^{\alpha/\theta}\delta$$

for $x > 0$ and $\theta < 1$, and set $f(x, y, z) = 0$ if $x \le 0$ or $\theta \ge 1$.

We assert that

$$\lim_{C_\delta \ni (x, y, z) \to 0} \inf f(x, y, z) = 1.$$

To see this it suffices to treat only the points in the plane $z \equiv 0$, because of rotational symmetry around the x-axis. Let us look in the plane $z \equiv 0$ at the graph of the function $y = \exp -1/x^\delta$. For small values of θ, the line $y = \tan\theta x$ intersects the graph. Let $(x(\theta), y(\theta), 0)$ be the nearest intersection point. This point is in the boundary of C_δ. The coordinate $x(\theta)$ satisfies the relation

$$x(\theta)^\delta \ln \frac{1}{x(\theta)} + x(\theta)^\delta \ln \frac{1}{\tan\theta} = 1.$$

This relation shows that if $\theta \to 0$, we have $x(\theta) \to 0$, and hence $x(\theta)^\delta \ln 1/x(\theta) \to 0$, which leads to $x(\theta)^\delta \ln 1/tg\theta \to 1$ or $x(\theta)^\delta \ln 1/\theta \to 1$.

On the other hand, the relation $y(\theta)/x(\theta) \to 0$ (as $\theta \to 0$), implies $r(\theta)/x(\theta) \to 1$. Therefore we have

$$f(x(\theta), y(\theta), 0) = r(\theta)^\alpha (\ln 1/\theta)^{\alpha/\delta} \to 1, \qquad \theta \to 0.$$

This proves the assertion.

Moreover, by a straightforward computation, one can prove that for $\alpha < \delta/2$, the following relation holds:

$$E^\circ\left[\int_0^\infty (\ln t)^2 1_B(X_t) |\nabla f|^2(X_t)\, dt\right] < \infty,$$

with B the unit ball. This relation implies the finiteness condition of Corollary 5.3 with $\omega = df$, and hence f is finely continuous at 0. Thus the preceding reasoning offers a proof of the fact that C_δ is thin at 0.

Now we continue with a discussion of the limit of the stochastic integral when t tends to the lifetime of the process ζ. If $|w| \in L^2$, then all components M_t^s, $\overline{M}_s^t$, α_t^s, β_t^s are continuous in t when s is fixed, and so the existence of the limit of the stochastic integral as $t \nearrow \zeta$ is automatic on the set $\{s < \zeta < \infty\}$. It remains then to investigate the existence of the limit as $t \nearrow \infty$ on the set $\{\zeta = \infty\}$. If ω is such that $|\omega|$ belongs only to L^2_{loc}, the stochastic integral is defined up to ζ and it is a consistent problem to study the limit at ζ, on the finite as well on the infinite part. The next theorem offers a criterion.

THEOREM 5.4. *Let ω be such that $|\omega| \in L^2_{\mathrm{loc}}$ and satisfy the following conditions for certain $s > 0$:*

(i) $$\int_s^\infty |\omega|^2(X_\ell)\,d\ell < \infty \quad P^o\text{-almost surely},$$

(ii) $$\int_s^\infty |\omega(b)|(X_\ell)\,d\ell < \infty \quad P^o\text{-almost surely},$$

(iii) $$\int_s^\infty p_\ell^{-1}|\omega(\nabla p_\ell)|(X_\ell)\,d\ell < \infty \quad P^o\text{-almost surely}.$$

If a smooth version of the stochastic integral is choosen, then the following limit exists and is finite P^o-almost surely on the set $\{s < \zeta\}$:

$$\lim_{t \nearrow \zeta} \int_s^t \omega \circ dX.$$

[Remark that in the above three conditions the integral is in fact taken only on the interval (s, ζ).]

PROOF. We look at each component appearing in relation (17). Conditions (ii) and (iii) imply the convergence of terms α_t^s and β_t^s. The forward martingale component $M(\omega, \zeta)_t^s$ is like a local martingale and condition (i) ensures that its bracket $\langle M_\bullet^s \rangle_\zeta$ is almost surely finite. Then it is easy to deduce the convergence of this component. Now let us look at the backward martingale component, $\overline{M}(\omega, \zeta)_s^t$. Define for $0 < u \leq v \leq 1$,

$$N_v^u = \overline{M}_{(\zeta - v)\vee s}^{(\zeta - u)\vee s} \quad \text{on } \{s < \zeta < \infty\},$$
$$N_v^u = 0 \quad \text{on } \{\zeta \leq s\} \cup \{\zeta = \infty\}.$$

Because ζ is a stopping time for the backward filtration, it follows that $(\zeta - u)\vee s$ is alike. Then it is not difficult to check that the process (N_v^u) with the filtration $\mathscr{G}_u = \Im'_{(\zeta - u)\vee s}$ satisfies conditions (i′), (ii), (iii′) of Lemma 6.2. Therefore, the following limit exists almost surely on the set $\{s < \zeta < \infty\}$:

$$\lim_{u \searrow 0} N_1^u = \lim_{u \searrow 0} \overline{M}_{(\zeta - 1)\vee s}^{(\zeta - u)} = \lim_{t \nearrow \zeta} \overline{M}_{(\zeta - 1)\vee s}^t.$$

Since one has $\overline{M}_s^t = \overline{M}_{(\zeta-1)\vee s}^t + \overline{M}_s^{(\zeta - 1)\vee s}$ (we assume that a smooth version was choosen), the desired conclusion is obtained on $\{s < \zeta < \infty\}$.

In order to get the same result on the set $\{\zeta = \infty\}$ one should again use Lemma 6.2. However, another method of reversing the time should be employed. Let φ: $(0, 1] \to [s, \infty)$ be a bijection (order reversing), for example, $\varphi(t) = s - \ln t$. Define for $0 < u \leq v \leq 1$,

$$N_v^u = \overline{M}_{\varphi(v)}^{\varphi(u)} \quad \text{on } \{\zeta = \infty\},$$
$$N_v^u = 0 \quad \text{on } \{\zeta < \infty\},$$

and $\mathscr{G}_u = \Im'_{\varphi(u)}$. We get the existence of the desired limit as before, thus completing the proof. □

We complete this section with some remarks concerning possibilities of checking the conditions (i), (ii), and (iii) of the theorem in concrete situations. Let us suppose that the process is transient and denote by U its potential kernel

$$Uf(x) = E^x\left(\int_0^\infty f(X_t)\,dt\right).$$

If there exists a compact set K such that

$$U(1_{K^c}|\omega|^2)(o) < \infty,$$
$$U(1_{K^c}|\omega(b)|)(o) < \infty,$$

then clearly conditions (i) and (ii) follow.

Now let us assume that the semigroup P_t is symmetric and its density p_t satisfies an estimate of the following type: $\|p_t\|_\infty \le ct^{-d/2}$, for large t, with some constant $d > 0$. Also suppose that $|\omega| \in L^2$. Then we write

$$E^o\left(\int_s^\infty p_\ell^{-1}|\omega(\nabla p_\ell)|(X_\ell)\,d\ell\right) = \int_s^\infty \int |\omega(\nabla p_\ell)|\,dV\,d\ell$$
$$\le \left(\int |\omega|^2 dV\right)^{1/2} \int_s^\infty \left(\int |\nabla p_\ell|^2\,dV\right)^{1/2} d\ell$$

The tail of the last integral may be majorized by a series

$$\sum_{n\ge n_0} \int_{2^n}^{2^{n+1}} \left(\int |\nabla p_\ell|^2\right)^{1/2} d\ell \le \sum 2^{n/2}\left(\int_{2^n}^{2^{n+1}} \int |\nabla p_\ell|^2\,d\ell\right)^{1/2}.$$

Because of the relation

$$\frac{d}{d\ell}p_{2\ell}(o) = \frac{d}{d\ell}\int p_\ell^2 = -2\int |\nabla p_\ell|^2,$$

it follows that the preceding sum is dominated by

$$C\sum_{n\ge n_0} 2^{n/2}2^{-nd/4}.$$

If $d > 2$, this sum is finite and condition (iii) follows.

6. Technical probabilistic lemmas. In this section we prove the probabilistic arguments used in the proofs of the limit theorems of the previous section.

LEMMA 6.1. *Let* $(\Omega, \Im, P)$ *be a probability space and* $\Im_s$, $s \in (0,1]$ *an increasing family of sub* σ*-algebras. Let* $(M_t^s,\ 0 < s \le t \le 1)$ *be a jointly continuous two-parameter process satisfying the following conditions.*

(i) *for each fixed* $s \in (0,1)$, *the process* $(M_t^s, \Im_t,\ t \in [s,1])$ *is an* L^2*-martingale.*

(ii) *For* $0 < u \le s \le t \le 1$, *it satisfies the next relation*

$$M_t^s = M_t^u - M_s^u \quad \textit{everywhere}.$$

(iii) $\lim_{s \searrow 0} E((M_1^s)^2) < \infty$.

Under these conditions, the following lim t *exists almost surely and in* L^2:

$$\lim_{s \searrow 0} M_1^s.$$

PROOF. Remark that relation (ii) above implies that $M_s^s = 0$. Also from relation (ii) and the martingale property, it follows that

$$E((M_t^u)^2) = E((M_t^s)^2) + E((M_s^u)^2).$$

Therefore the limit in (iii) is increasing. Let us take a sequence $s_n \to 0$ and set $H_n = M_1^{s_n}$. By the preceding relation we deduce

$$\|H_{n+p} - H_n\|_2^2 = \|M_{s_n}^{s_{n+p}}\|_2^2 = \|M_1^{s_{n+p}}\|_2^2 - \|M_1^{s_n}\|_2^2,$$

which shows that the sequence (H_n) is a Cauchy sequence in L^2, and consequently it has a limit H. Then we put $H_s = H - M_1^s$ and assert that the process $(H_s, \Im_s,\ s \in (0,1])$ is an L^2-martingale. First, the relation

$$H_s = \lim_{n \to \infty} M_1^{s_n} - M_1^s = \lim M_s^{s_n},$$

shows that H_s is $\Im_s$-measurable. Then, for $0 < s \le t \le 1$, we have

$$E(H_t - H_s | \Im_s) = E(M_t^s | \Im_s) = 0.$$

We conclude by the usual convergence theorem for martingales, that $\lim_{s \searrow 0} H_s$ exists almost surely and in L^2, which implies the conclusion of the lemma. □

LEMMA 6.2. *Assume that condition* (ii) *of Lemma* 6.1 *remains valid and conditions* (i) *and* (iii) *are replaced by the following ones*.

(i′) *The process* $(M_t^s, \Im_t,\ t \in [s,1])$ *is a local martingale for each fixed* $s \in (0,1)$.

(ii′) *There exists a sequence* $s_n \searrow 0$ *such that*

$$\lim_{s_n \searrow 0} \langle M_\bullet^{s_n} \rangle_1 < \infty \quad \textit{almost surely}.$$

Then the following limit exists and is almost surely finite:

$$M = \lim_{s \searrow 0} M_1^s.$$

Moreover, *there exists a strictly positive stopping time* $T > 0$ (*almost surely*) *such that the following limit relation holds in* L^2:

$$\lim_{s \searrow 0} (M_1^{s \wedge T} - M) = 0.$$

(*Note that* M *is not necessarily in* L^2.)

PROOF. Let us denote $K = \lim_{n\to\infty} \langle M^{s_n}_\bullet \rangle_1$, where (s_n) is choosen such that $s_n \searrow 0$. We assert that there exists a continuous, increasing and adapted process $(K_s,\ s \in [0,1])$, such that $K_0 = 0$ and

$$K_s = K - \langle M^s_\bullet \rangle_1,$$

almost surely.

It suffices to define the process K^n, which has the same increments on the interval $(s_n, 1]$, by

$$K^n_s = K - (\langle M^{s_n}_\bullet \rangle_1 - \langle M^{s_n}_\bullet \rangle_s), \qquad s > s_n.$$

Clearly this process is continuous and increasing. To see that it is adapted one uses the relation

$$K^n_s = \lim_m \langle M^{s_m}_\bullet \rangle_1 - (\langle M^{s_n}_\bullet \rangle |^1_s),$$

which by condition (2) becomes, for $m > n$, equal to

$$\lim_m [\langle M^{s_m}_\bullet \rangle_1 - (\langle M^{s_m}_\bullet \rangle\ |^1_s)] = \lim_{m\to\infty} \langle M^{s_m} \rangle_s.$$

If $m > n$ and $s \in (s^n, 1]$, then $K^m_s = K^n_s$, almost surely. Then one may define a continuous, increasing and adapted process $(K_s,\ s \in [0,1])$ such that $K_0 = 0$ and $K_s = K^n_s$ almost surely for $s \in (s_n, 1]$. Now, let us define the stopping time,

$$T = \inf\{s \in (0,1]\colon K_s \geq 1\},$$

(with the convention $\inf \varnothing = 1$). Then set

$$N^s_t = M^s_{(t\wedge T)\vee s},$$

obtaining a process which clearly satisfies condition (2) of the preceding lemma. Let us check conditions (1) and (3) now. From the relation

$$K_t - K_s = \langle M^s_\bullet \rangle_1 - \langle M^t_\bullet \rangle_1 = \langle M^s_\bullet \rangle_t, \qquad 0 < s \leq t \leq 1,$$

it follows that

$$\langle N^s_\bullet \rangle_1 = \langle M^s_\bullet \rangle_{T\vee s} = K_{T\vee s} - K_s \leq K_T \leq 1,$$

almost surely, which implies that $E((N^s_1)^2) \leq 1$, checking conditions (1) and (3). By the preceding lemma, the following limit exists almost surely and in L^2:

$$N = \lim_{s\searrow 0} N^s_1.$$

Because $M^s_{T_{\vee s}} = N^s_1$, we may write $M^s_1 = N^s_1 + M^{T\vee s}_1$. On the other hand, because $T > 0$ almost surely, it follows that $\lim_{s\searrow 0} M^{T\vee s}_{1,} = M^T_1$ almost surely and consequently

$$M = \lim_{s\searrow 0} M^s_1 = N + M^T_1 \quad \text{almost surely.}$$

The first assertion of our lemma is proved. Further, noting that $N_1^s = M_T^{s\wedge T}$, we have

$$M_1^{s\wedge T} = N_1^s + M_1^T.$$

This relation, together with the preceding one, leads to the second assertion of the lemma. □

7. The case of a negatively curved manifold. Let us suppose that E together with g is a complete and connected Riemannian manifold, V is the volume measure associated to g and $L = \Delta$, so that the associated process is symmetric. Let $o \in E$ be fixed and denote by $r(x) = d(o, x)$, the geodesic distance between x and o. Assume further that the sectional curvature k is negatively bounded: $-b \leq k \leq -a$, with two constants $0 < a < b < \infty$. Under these assumptions it is known that there exists a constant $\theta > 1$, such that

$$\theta^{-1} < \liminf_{t\to\infty} r(X_t)/t \leq \limsup_{t\to\infty}(X_t)/t < \theta, \tag{19}$$

almost surely and a constant $\gamma > 0$ such that the volume of a ball B_δ centered at o and of radius δ satisfies a growth bounding condition such as

$$V(B_\delta) \leq Ce^{\gamma_\delta}. \tag{20}$$

A result of Li and Karp [9] ensures that the process is conservative; that is, $P_t 1 = 1$ under this growth condition. The stochastic calculus under an infinite measure such as P^V was presented in [14] and [13]. Clearly, the definition of the stochastic integral of 1-forms coincides with that presented in Section 4. The following theorem offers a strong control over the rate of convergence of the stochastic integrals of forms.

THEOREM 7.1. *Let ω be a bounded differential form such that the function φ: $(0, \infty) \to \mathbb{R}$, defined by $\varphi(t) = \sup\{|\omega|(x)/r(x) \geq t\}$ satisfies the condition*

$$\int_0^\infty \varphi(t)\,dt < \infty.$$

Then the following limit exists and is almost surely finite:

$$\lim_{t\to\infty}\int_0^t \omega \circ dX.$$

Moreover, the oscillation of the stochastic integral over the interval $[2^n,\ 2^{n+1}]$ *is asymptotically almost surely dominated by*

$$\operatorname{osc}_{2^n \leq t \leq 2^{n+1}}\left(\int_{2^n}^t \omega \circ d\,X\right) \leq C\int_{2^n}^{2^{n+1}} \varphi(\theta^{-1}\,t)\,dt.$$

PROOF. Under the measure P^V, the stochastic integral is written as the difference of a forward and a backward martingale, that is, the finite variation term α disappears (the term β disappears, too, because the drift b vanishes),

$$I_t = \int_0^t \omega \circ d\,X = \tfrac{1}{2}M_t^0 - \tfrac{1}{2}\overline{M}_0^t. \tag{$*$}$$

Let (λ_n) be a sequence of nonnegative numbers to be specified later to suit our proof and set

$$T_n = \inf\{t > 0/r(X_t) \geq \theta 2^n\},$$
$$\Omega_n = \left\{r(X_0) \leq \theta 2^n, 2^n < T_n, \int_{2^{n-1}}^{2^n} |\omega|^2(X_t)\,dt \leq K_n, \operatorname{osc}_{-2^{n-1}\leq t\leq 2^n}(I_t) > \lambda_n\right\},$$
$$K_n = \int_{2^{n-1}}^{2^n} \varphi^2(\theta^{-1}t)\,dt.$$

Observe that in the definition of Ω_n only the condition concerning the oscillation of I_t is nontrivial asymptotically, because relation (19) implies that for P^V-almost surely for large n and t one has $2^n < T_n$ and

$$|\omega|(X_t) \leq \varphi(\theta^{-1}t).$$

Now we are going to estimate the P^V-measure of Ω^n by using the method of Takeda [22]. The Dirichlet space associated with Δ, V and the Neumann boundary conditions generate a semigroup and a process on the closed ball $\bar{B}_{\theta 2^n}$. Until the first hitting time of the boundary this process coincides with the process we already have. The advantage of the new process is that its invariant measure is the restriction of V to $B_{\theta 2^n}$, which is a bounded measure. Let us denote by μ_n this measure normalized such that it becomes a probability measure and write $\hat{P}^{\mu_n}$ for the associated probability measure on the path space of the process with Neumann conditions. Under $\hat{P}^{\mu_n}$, the stochastic integral I_t preserves the above decomposition $(*)$ with the same forward and backward martingales so that we may write

$$\begin{aligned} P^V(\Omega_n) = V(B_{\theta 2^n})\hat{P}^{\mu_n}(\Omega_n) \leq V\,(B_{\theta 2^n}) \\ \times \hat{P}^{\mu_n}\Bigg(\operatorname{osc}_{2^{n-1}\leq t\leq 2^n}(M_t^0) > \lambda_n \text{ or } \operatorname{osc}_{0\leq t\leq 2^{n-1}}(\overline{M}^{2^n}_{2^n-t}) > \lambda_n; \\ \int_{2^{n-1}}^{2^n} |\omega|^2(X_\ell)\,d\ell \leq K_n\Bigg). \end{aligned}$$

By Lemma 7.2 we have

$$\hat{P}^{\mu_n}(\Omega_n) \leq C(\sqrt{K_n}/\lambda_n)\exp -(\lambda_n^2/2K_n)$$

and hence, on account of (20), we get

$$P^V(\Omega_n) \leq C\,(\sqrt{Kn}/\lambda_n)\exp[\gamma\theta 2^n - (\lambda_n^2/2K_n)].$$

In order to make convergent the series

$$\sum_n P^V(\Omega_n),$$

we choose $\lambda_n = \tau\sqrt{2^n K_n}$, with $\tau > 2\gamma\theta$. It follows by a Borel–Cantelli argument that for large n we have

$$\operatorname{osc}_{2^{n-1}\leq t\leq 2^n}(I_t) \leq \lambda_n.$$

Because obviously $K_n \le 2^{n-1}\varphi(\theta^{-1}2^{n-1})^2$, it follows that

$$\lambda_n \le C\int_{2^{n-1}}^{2^n} \varphi(\theta^{-1}t)\,dt,$$

and hence

$$\sum_n \lambda_n \le C\int_0^\infty \varphi(t)\,dt < \infty.$$

The assertions of the theorem clearly follow from these estimates. □

LEMMA 7.2. *Let* $(\Omega, \Im, \Im_t, P)$ *be a filtered probability space and* (M_t) *be a continuous martingale. Then the following inequality holds*:

$$P(\mathrm{osc}_{0\le t\le \tau} M_t > \lambda;\ \langle M\rangle_\tau < K) \le C(\sqrt{K}/\lambda)\exp -(\lambda^2/2K).$$

For the proof, see [11], page 80.

Another criterion for the existence of the limit of a stochastic integral for large t is the following proposition.

PROPOSITION 7.3. *Let* $\omega \in L^2$ *be such that* $r|\omega|^2 \in L^1$. *Then the following limit exists and is* P^V*-almost surely finite*

$$\lim_{t\nearrow\infty} \int_0^t \omega \circ d\,X.$$

PROOF. Let us set $A_t = \{x \in E\colon \theta^{-1}t \le r(x) \le \theta t\}$ and compute

$$\begin{aligned} E^V\left(\int_0^\infty 1_{A_t}|\omega|^2(X_t)\,dt\right) &= \int_0^\infty \int_{A_t} |\omega|^2\,dV\,dt \\ &= \int (\theta - \theta^{-1}) r|\omega|^2\,dV < \infty. \end{aligned}$$

Again we look at the decomposition $(*)$. The bracket of M (and of $\overline{M}$) is finite

$$\langle M^0_\bullet\rangle_\infty = \int_0^\infty |\omega|^2(X_t)\,dt < \infty \quad \text{almost surely},$$

because its tail coincides with the tail of the integral

$$\int_0^\infty 1_{A_t}|\omega|^2(X_t)\,dt,$$

which is almost surely finite according to the above computation. To be more specific, let us denote by

$$T = \sup\{t\colon r(X_t) \le \theta^{-1}t \text{ or } \theta t \le r(X_t)\}.$$

By (19), we know that $T < \infty$ almost surely. On the set $\{T < t\}$ one has $1_{A_t}(X_t) = 1$, almost surely, and this fact ensures that the two tails coincide.

The finiteness of the bracket suffices to conclude the convergence of the forward martingale

$$\lim_{t\nearrow\infty} M_t^0.$$

It remains to treat the backward martingale. Set $B_t = 1_{A_t}(X_t)$ and observe that the process (B_t) is previsible and $B_t = 1$ almost surely on the set $\{T < t\}$. Note also that $t-T$ on $\{T < t\}$ is a stopping time with respect to the filtration $(\mathscr{F}'_{t-u},\ u \in [0,t])$. Let us put

$$\overline{N}_s^t = \int_0^{t-s} B_{t-u}\ d_u\ \overline{M}_{t-u}^t.$$

Certainly we have $\overline{N}_T^t = \overline{M}_T^t$ almost surely on $\{T \le t\}$. The bracket of $\overline{N}$ is

$$\langle \overline{N}_{t-\bullet}^t \rangle_u = \int_{t-u}^t B_\ell |\omega|^2(X_\ell)\, d\ell$$

and it follows that

$$\sup_t E^V(\langle \overline{N}_{t-\bullet}^t \rangle_t) = (\theta - \theta^{-1}) \int r|\omega|^2\, dV < \infty.$$

Now, choose a smooth version of $\overline{N}$ and inverting the time with a bijection φ: $(0,1] \to [0,\infty)$, apply Lemma 7.4. This implies, almost surely, the existence and finitenes of the limit

$$\lim_{t\to\infty} \bar{N}_0^t.$$

Because for smooth versions one has (on $\{T < t\}$)

$$\overline{M}_0^t = \overline{M}_T^t + \overline{M}_0^T, \qquad \overline{N}_0^t = \overline{N}_T^t + \overline{N}_0^T,$$

it follows that

$$\lim_{t\nearrow\infty} \overline{M}_0^t = \overline{M}_0^T + \lim_{t\nearrow\infty} \overline{N}_0^t - \overline{N}_0^T. \qquad \square$$

LEMMA 7.4. *Let* $(\Omega, \Im, Q)$ *be a measure space with a filtration* $(\Im_t;\ t \in (0,1])$ *such that* Q *is* σ *finite with respect to each* σ*-algebra* $\Im_t$. *Let* $(M_t^s,\ 0 < s \le t \le 1)$ *be a continuous smooth process such that the following conditions are satisfied*:

(i) M_t^s *is* $\Im_t$ *measurable*, $E((M_t^s)^2) < \infty$, *and the martingale relation*

$$E(M_{t_2}^s; A) = E(M_{t_1}^s; A)$$

holds for any $0 < s \le t_1,\ \le t_2 \le 1,\ A \in \Im_{t_1}$;

(ii) $M_{t_2}^{t_1} = M_{t_2}^s - M_{t_1}^s$;

(iii) $\lim_{s\searrow 0} E((M_1^s)^2) < \infty$.

Then the following limit exists in L^2 *and almost surely*:

$$\lim_{s\to 0} M_1^s.$$

PROOF. First note that for $0 < s \le t \le 1$ one has

$$E((M_1^s)^2) = E((M_1^t)^2) + E((M_t^s)^2).$$

This and condition (3) imply that

$$\lim_{t \searrow s} E((M_1^s - M_1^t)^2) = \lim_{t \searrow s} E((M_t^s)^2) = 0.$$

Thus we have obtained the existence of the L^2-limit,

$$M_1^0 := \lim_{s \searrow 0} M_1^s.$$

For the almost sure convergence we introduce the variables $M_t^0 = M_1^0 - M_1^t$. One easily checks that $M_t^0 = \lim_{s\searrow 0} M_t^s$ and then deduces that $(M_t^0, \mathscr{F}_t,\ t \in (0,1])$ is a generalized martingale in the sense of (1).

The problem is that the restriction of Q to $\mathscr{F}_0$ is not known to be σ-finite. Therefore we consider the process $((M_t^0)^2, \mathscr{F}_t)$, which is a generalized submartingale. Doob's down-crossing inequality applies to it and gives the almost sure convergence, completing the proof of the lemma. □

Without assuming that the curvature of the manifold has an upper bound, we may obtain another criterion based on Theorem 5.4 and an estimate of Li-Yau [10]. The result of Li-Yau assumes only that the Ricci curvature is bounded from below by a constant $-K$, with $K \ge 0$. Then it states that any positive solution u of the equation $\Delta u - \partial_t u = 0$ on $[0,\infty) \times E$ satisfies the estimate

$$\left(\frac{\nabla u}{u}\right)^2 - \alpha \frac{\Delta u}{u} \le \frac{N\alpha}{2}\left(\frac{1}{t} + \frac{K}{2(\alpha-1)}\right),$$

where α is an arbitrary constant such that $\alpha > 1$. As a consequence of this estimate, we have the following lemma.

LEMMA 7.5. *If $f\colon [1,\infty) \to R_+$ is such that $\int_1^\infty f(t)\,dt < \infty$, then the following integrability condition holds*:

$$E^o\left[\int_1^\infty f(t) |\nabla \ln p_t| 2(X_t)\,dt\right] < \infty.$$

PROOF. The proof follows from the preceding estimate and the following relation

$$E^o\left[\frac{\Delta p_t}{p_t}(X_t)\right] = \int \Delta p_t \, dV = 0$$

In order to prove this relation we first apply Theorem 6.3 of [18] and deduce that $\Delta p_t \in L^1$. Then we write p_t in the form

$$p_t = U_\lambda \varphi,$$

with $\varphi = (\lambda - \Delta)p_t$, so that $\Delta p_t = -\varphi + \lambda U_\lambda \varphi$.Then one takes a sequence $(\varphi_n) \subset \mathscr{C}_c^\infty$, such that $0 \le \varphi_n \le 1$ and $\varphi_n \nearrow 1$ and writes

$$\int \lambda U_\lambda \varphi = \lim_{n\to\infty} \int \varphi_n \lambda U_\lambda \varphi = \lim_{n\to\infty} \int \varphi \lambda U_\lambda \varphi_n = \int \varphi.$$

The last equality is a consequence of the fact that $\lambda U_\lambda 1 = 1$. $\square$

This lemma and Theorem 5.4 immediately lead to the next corollary.

COROLLARY 7.6. *If $\omega \in L^2_{\mathrm{loc}}$ and $f\colon [1,\infty) \to R_+$ is strictly positive, decreasing, and the next conditions are satisfied,*

$$\int_1^\infty f(t)\,dt < \infty,$$

$$\int_1^\infty f^{-1}(t)|\omega|^2(X_t)\,dt < \infty \quad P^o\text{-almost surely},$$

then the following limit exists and is P^o-almost surely finite:

$$\lim_{t\nearrow\infty} \int_1^t \omega \circ d\,X.$$

Compared with Theorem 7.1, this corollary has a more general hypothesis. However, the conclusion of the theorem is more precise.

8. The case of a divergence form operator in $\mathbb{R}^N$. In this section we suppose that $E = \mathbb{R}^N$, $N \ge 2$. First, we recall some well-known results of Aronson and Nash and refer to [20] for an elegant, concise exposition on this subject. Let $a = (a^{ij})$ be a measurable $N \times N$ matrix, ρ a measurable function and $b = (b^k)$ a measurable vector field on $\mathbb{R}^N$. Assume that the following conditions hold:

$$\frac{1}{\lambda}|\xi|^2 \le \sum_{i,j} a^{ij}(x)\xi^i\xi^j \le \lambda|\xi|^2\colon x, \xi \in \mathbb{R}^N, \tag{21}$$

$$\frac{1}{\lambda} \le \rho(x) \le \lambda\colon x \in \mathbb{R}^N, \tag{22}$$

$$|b^k(x)| \le \Lambda\colon x \in \mathbb{R}^N, k = 1\dots N, \tag{23}$$

with some constants $\lambda \ge 1$, $\Lambda \ge 0$. The operator

$$L = \frac{1}{\rho}\sum_{i,j}\frac{\partial}{\partial x^i}\rho a^{ij}\frac{\partial}{\partial x^j} + \sum_k b^k \frac{\partial}{\partial x^k}$$

generates a Feller continuous conservative semigroup (P_t) which admits a density $p_t(x, y)$, with respect to the volume measure $V(dx) = \rho(x)\,dx$. This semigroup is strongly continuous on both $L^2(dV)$ and $H = W_0^{1,2}(\mathbb{R}^N)$. The density satisfies the estimates

$$\frac{1}{C} e^{-\eta t} q_{t/\theta}(x-y) \leq p_t(x, y) \leq C e^{\eta t} q_{t\theta}(x-y)\,, \tag{24}$$

for each $(t, x, y) \in (0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N$, where $q_t(x) = (4\pi t)^{-N/2} \exp -|x|^2/4t$ and $C, \theta \geq 1$, $\eta \geq 0$ are constants which depend only on N, λ and Λ (see II.3.8. of [20]). As a function of (t, x, y) the density $p_t(x, y)$ is locally Hőlder continuous. More precisely, the following estimate holds:

$$|p_t(x, y) - p_{t'}(x', y')| \leq C\left(\frac{\sqrt{|t-s|} \vee |x-x'| \vee |y-y'|}{\delta}\right)^{\alpha} \tag{25}$$

for all $(t, x, y), (s, z, y) \in [\delta, 1/\delta] \times \mathbb{R}^N \times \mathbb{R}^N$, with constants C and α dependent only on N, λ, Λ and $\delta > 0$. This estimate is presented in Appendix B. If b is identically zero, then the semigroup is symmetric and the above estimates (24) hold with $\eta = 0$.

Now we are going to prove an estimate for the gradient of the density, which turns out to be suitable for the local treatment of stochastic integrals of forms.

THEOREM 8.1. *Let $f \in \mathscr{C}^1(0, 1]$ be a function such that $f \geq 0$,*

$$\gamma := \limsup_{s \to 0} f(s) |\log s| < \infty$$

and

$$\int_0^1 |f'(s) \log s| \, ds < \infty.$$

Then there exists a constant M which depends only on N, λ and Λ such that the function $p_l(y) = p_l(x_0, y)$ satisfies

$$\int_0^t f(l) \int \frac{|\nabla p_l|^2}{p_l} \, dV \, dl$$
$$\leq M\left[\gamma + f(t)(|\log t| + 1) + \int_0^t |f'(l)|(|\log l| + 1) \, dl + \int_0^t f(l) \, dl\right],$$

for each fixed $x_0 \in \mathbb{R}^N$ and $t \in (0, 1]$.

PROOF. For $\sigma \in (0, 1)$ and $x_0 \in \mathbb{R}^N$, set

$$\Gamma_\sigma = \left\{ \phi = \frac{1}{\tau} \int_0^\tau P_l^* \psi \, dl : / : \psi \in \mathscr{C}_c(\mathbb{R}^N), \; \psi \geq 0, \right.$$
$$\left. \int \psi \, dV = 1, \operatorname{supp} \psi \subset B_{\sqrt{\sigma}}(x_0), \; \tau \in (0, \sigma) \right\}.$$

Take $\epsilon \geq 0$ and $\phi \in \Gamma_\sigma$ and define

$$F_l = f(l) P_l^* \phi \log(P_l^* \phi + \epsilon).$$

By Lemma 8.2, one concludes that $F_l \in L^1(\mathbb{R}^N)$. Applying the Leibnitz–Newton formula on an interval $[s,t] \subset [\sigma,1]$ and then integrating with respect to V one gets

$$
\begin{aligned}
f(t)\int P_t^*\phi \log(P_t^*\phi+\epsilon)\,dV - f(s)\int P_s^*\phi\log(P_s^*\phi+\epsilon)\,dV \\
= \int_s^t f'(l)\int P_l^*\phi\log(P_l^*\phi+\epsilon)\,dV\,dl \\
+ \int_s^t f(l)\int (L^*P_l^*\phi)\log(P_l^*\phi+\epsilon)\,dV\,dl \\
+ \int_s^t f(l)\int \frac{P_l^*\phi}{P_l^*\phi+\epsilon} L^*P_l^*\phi\,dV\,dl.
\end{aligned}
$$

We are now going to analyze each term of this relation. The function $L^*P_l^*\phi$ is absolutely integrable because it equals

$$
(*) \qquad L^*P_l^*\phi = \frac{1}{\tau}P_l^*(P_\tau^*\psi - \psi),
$$

if ϕ is expressed as $(1/\tau)\int_0^\tau P_l^*\psi dl$ and ψ satisfies the conditions in the above definition of Γ_σ. Since $\int L^*P_l^*\phi\,dV = 0$ and $\log(P_l^*\phi+\epsilon) - \log\epsilon \in H$, the second term of the right-hand side of the equality can be written as

$$
-\int_s^t f(l)\int \frac{|\nabla P_l^*\phi|^2}{P_l^*\phi+\epsilon}\,dV\,dl + \int_s^t f(l)\int \frac{P_l^*\phi}{P_l^*\phi+\epsilon} b(P_l^*\phi)\,dV\,dl.
$$

The first term in this expression converges, as ϵ, s and σ tend to zero, to the left side of the estimate we have to prove. So, in order to obtain the desired estimate we are going to let $\epsilon \to 0$ and bound the other terms in the above equality. The left terms as well as the first on the right of the relation are bounded by Lemma 8.2. Let us look at the last term in the above relation. Letting $\epsilon \to 0$, on account of the expression of $L^*P_l^*\phi$, what we have to estimate is

$$
\begin{aligned}
\int \frac{1}{\tau}\int_s^t f(l)(P_{l+\tau}^*\psi - P_l^*\psi)\,dl\,dV \\
= \int \frac{1}{\tau}\bigg[\int_t^{t+\tau} f(l)P_l^*\psi\,dl - \int_s^{s+\tau} f(l)P_l^*\psi\,dl \\
+ \int_s^t (f(l) - f(l+\tau))P_{l+\tau}^*\psi\,dl\bigg]\,dV.
\end{aligned}
$$

Since $\int P_l^*\psi\,dV \le 1$, the absolute value of this expression is dominated by

$$
\frac{1}{\tau}\int_t^{t+\tau} f(l)\,dl + \int_s^{t+\tau} |f'(l)|\,dl.
$$

Further we have to bound the following term:

$$
\left|\int_s^t f(l)\int b(P_l^*\phi)\,dV\,dl\right| \le \frac{\Lambda\sqrt{N\lambda}}{2}\int_s^t f(l)\left(\delta\int \frac{|\nabla P_l^*\phi|^2}{P_l^*\phi}\,dV + \frac{1}{\delta}\int P_l^*\phi\,dV\right)dl.
$$

Choosing $\delta = 1/\Lambda\sqrt{N\lambda}$ and concluding all the above calculations, as $\epsilon \to 0$ one gets

$$\begin{aligned}\frac{1}{2}\int_s^t f(l)\int \frac{|\nabla P_l^*\phi|^2}{P_l^*\phi}\,dV\,dl \\ \leq M\left[f(t)(|\log t|+1)+f(s)(|\log s|+1)+\int_s^t |f'(l)|(|\log l|+1)\,dl\right] \\ +\frac{1}{\tau}\int_t^{t+\tau} f(l)\,dl+\int_s^{t+\tau}|f'(l)|\,dl+\frac{N\lambda\Lambda^2}{2}\int_s^t f(l)\,dl.\end{aligned}$$

Now, with s and t fixed, we choose two sequences $\sigma_n \to 0$ and (ϕ_n), with $\phi_n \in \Gamma_{\sigma_n}$ and pass to the limit, taking into account Lemma 8.3. Finally the estimate of the statement is obtained letting $s \to 0$. □

LEMMA 8.2. *There exists a constant $M = M(N, \lambda, \Lambda)$ such that*

$$\int P_t^*\phi|\log P_t^*\phi|\,dV \leq M(|\log t|+1),$$

for any $\phi \in \Gamma_\sigma$ and $0 < \sigma \leq t \leq 1$.

PROOF. If $0 < u \leq \sigma \leq t$, one deduces from (24) that

$$p_{t+u}(x, y) \leq 2^{N/2} C q_{2\theta t}(x-y),$$

and hence, with ψ corresponding to ϕ and using the notation (Q_u) for the semigroup with density $q_u(x-y)$,

$$P_t^*\phi(x) \leq 2^{N/2} C Q_{2\theta t}\psi(x).$$

Now, let us examine the function $u \to u\log u$. It is decreasing on $(0, e^{-1}]$, increasing on $[e^{-1}, \infty)$ and its minimum at e^{-1} is $-e^{-1}$. This shows that

$$(**) \qquad u|\log u| \leq v(|\log v|+1),$$

whenever $0 < u \leq v$. Therefore, what we have to estimate is

$$\int Q_{2\theta t}\psi|\log Q_{2\theta t}\psi|\,dx.$$

If $|x-x_0| \geq 2\sqrt{\sigma}$ and $|y-x_0| \leq \sqrt{\sigma}$, one has $2|x-y| \geq |x-x_0|$, which implies

$$q_{2\theta t}(x-y) \leq 2^N q_{8\theta t}(x-x_0).$$

Since $\operatorname{supp}\psi \subset B_{\sqrt{\sigma}}(x_0)$, one deduces for $|x-x_0| \geq 2\sqrt{\sigma}$,

$$Q_{2\theta t}\psi(x) \leq 2^N q_{8\theta t}(x-x_0).$$

On the other hand, for $|x-x_0| \leq 2\sqrt{\sigma}$ obviously,

$$Q_{2\theta t}\psi(x) \leq (8\pi\theta t)^{-N/2}$$

holds. By using again the inequality $(**)$, it remains to bound the integral

$$\int q_{8\theta t}(x)|\log q_{8\theta t}(x-x_0)|\,dx$$

on the complement of the ball $B_{2\sqrt{\sigma}}(x_0)$, which follows by a direct computation. The integral over this ball is easily bounded because $\sigma \le t$. □

LEMMA 8.3. *Let $x_0 \in R_N$ and Γ_σ be as in the proof of Theorem 7.1. Take a sequence $\sigma_n \to 0$ and $\phi_n \in \Gamma_{\sigma_n}$ for each $n \in N$. Then*

$$\lim_{n\to\infty} \int_s^t \int |\nabla(P_l^* \phi_n - p_l)|^2 \, dV \, dl = 0,$$

for any $0 < s < t < \infty$.

PROOF. Because of continuity of the density $p_l(\cdot, x)$, it follows that $P_l^*\phi_n(x) \to p_l(x)$, for each $x \in R^N$. On the other hand, the upper estimate in (24) implies convergence in L^2. This holds uniformly for $l \in [s, t]$ because

$$\|P_l^*\phi_n - p_l\|_2 = \|P_{l-s}^*(P_s^*\phi_n - p_s)\|_2 \le C\|P_s^*\phi_n - p_s\|_2 \to 0.$$

Then, by the same argument as in the proof of Lemma 2.2, one has for $h \in L^2$,

$$\int_s^t \int |\nabla P_l^* h|^2 \, dV \, dl \le \int (P_s^* h)^2 \, dV + \int (P_t^* h)^2 \, dV + \Lambda^2 \int_s^t \int (P_l^* h)^2 \, dV \, dl.$$

This inequality, applied to $h = P_{s/2}^*\phi_n - p_{s/2}$ on the interval $[s/2, t - s/2]$, concludes the proof of the lemma. □

As we have mentioned, if $b \equiv 0$, the semigroup is symmetric and the estimates (24) hold with $\eta = 0$. This fact enables one to prove a result similar to Theorem 8.1 concerning the behavior of ∇p_t for large t. The next theorem is one such result. The key point is that Lemma 8.2 extends to $t \in [\sigma, \infty)$. (In fact, all the reasoning used to prove Theorem 8.1 extends even to the case of a general manifold E, except for the estimate contained in Lemma 8.2)

THEOREM 8.4. *Assume that $b \equiv 0$ and let $f \in \mathscr{C}_+^1([1, \infty))$ be such that*

$$\gamma = \limsup_{t\to\infty} f(t) \log t < \infty, \qquad \int_1^\infty |f'(t)| \log t \, dt < \infty.$$

Then the following relation holds:

$$\int_t^\infty f(l) \int \frac{|\nabla p_l|^2}{p_l} \, dV \, dl \le M\left[\gamma + f(t)(\log t + 1) + \int_t^\infty |f'(l)|(\log l + 1) \, dl\right],$$

with $p_l = p_l(x_0, y)$, x_0 fixed arbitrary in R^N, $t \ge 1$ and the constant M depending only on N and λ.

From Theorems 5.2 and 8.1 one deduces a result concerning the limit at zero of stochastic integrals of forms. It is contained in Corollary 5.3. Similarly, by Theorems 5.4 and 8.4 one obtains the following result.

COROLLARY 8.5. *Assume that* $b \equiv 0$ *and* $\omega \in L^2_{\ell oc}(\mathbb{R}^N)$ *satisfies the following condition*:

$$\int_1^\infty f^{-1}(s)|\omega|^2(X_s)\,ds < \infty,$$

P^0*-almost surely, with a function* $f \in C^1([1,\infty))$ *such that* $f > 0$, $\limsup_{t\to\infty} f(t)\ln t < \infty$, $\int_1^\infty |f'(\ell)|\ln\ell\,d\ell < \infty$. *Then the following limit exists and is* P^0*-almost surely finite*:

$$\lim_{t\to\infty}\int_1^t \omega\circ d\,X.$$

APPENDIX A

Proof of Theorem 2.1. The method of our proof consists in obtaining first locally the properties stated in the theorem and then deducing the global assertions. Let (U,Φ) be a chart on E such that conditions (i) and (ii) are satisfied on it. To simplify the notation, we identify the set U with its image so that we look at U also as a subset of $\mathbb{R}^N$ and extend the functions g_{ij}, g^{ij}, ρ, b^i in $\mathbb{R}^N$ and assume that conditions (i) and (ii) are uniformly satisfied on $\mathbb{R}^N$ and the extension of b is uniformly bounded. Let g'_{ij}, g'^{ij}, ρ', b'^i be these extensions and define in $\mathbb{R}^N$ the operator

$$L' = \sum_{ij}\frac{1}{\rho'}\frac{\partial}{\partial x^i}\rho' g'^{ij}\frac{\partial}{\partial x^j} + \sum_i b'^i\frac{\partial}{\partial x^i}.$$

As mentioned in Section 8, the semigroup generated by L' in $L^2(\mathbb{R}^N, dV')$ admits a density $p'_t(x, y)$ which satisfies estimates (24) and (25). In particular, the semigroup (P'_t) generated by L', has strong Feller property and generates a diffusion process in $\mathbb{R}^N$. On the other hand, the weak solutions of L' produce a harmonic space in the sense of Brelot (see [7]; Theorems 8.20 and 8.30 are particularly useful in checking the axioms). It turns out that the notions of balayage in the sense of the harmonic space and of the process coincide. For example, if $N \geq 3$, one checks that the potential kernel

$$G'f(x) = \int_E\int_0^\infty p_t(x, y)f(y)\,dt\,V(dy)$$

transforms each function $f \in \mathscr{C}_c(\mathbb{R}^N)$, $f \geq 0$ into a potential with respect to the harmonic space. In general, if $D \in \mathbb{R}^N$ is a bounded open set and $f \in \mathscr{C}_c(D)$, $f \geq 0$, then the function u which solves the problem

$$L'u = -f \quad \text{in } D,$$
$$u = 0 \quad \text{on } \partial D,$$

is a potential in D. This is observed first for open sets with smooth boundary and then follows for general D by approximation from inside. Therefore, we have

$$u(x) = E^x\left[\int_0^T f(X_s)\,ds\right], \qquad x \in D,$$

where $T = T_{D^c}$. The part of the diffusion X in D admits a density which we denote by p_t^D, density that is related to p'_t by the equation

$$p_t^D(x, y) = p'_t(x, y) - E^x[p'_{t-T}(X_T, y);\ T < t], \qquad x,\ y \in D.$$

The main point is that if $D \subset U$, then the function p^D is uniquely determined by L viewed as an operator in D and is independent of the extension L'. (The function p^D is in fact the fundamental solution of $L - \partial_t$ with zero boundary conditions on ∂D.) From the above formula one sees that p^D is continuous on $(0, \infty) \times D \times D$ and also satisfies the inequality $p_t^D(x, y) \leq p'_t(x, y)$. This, together with the estimate (24), implies that p_t^D may be extended for $t \in (-\infty, 0]$ with the value 0, obtaining a lower semicontinuous function on $\mathbb{R} \times D \times D$ which is continuous outside the set $\{(0, x, u) \mid x \in D\}$.

Concluding, we deduce that the part of X on U is a diffusion on U generated by L and if U' is another chart domain satisfying (i) and (ii), the two processes on U and U' produce by restriction on $U \cap U'$ processes having the same transition function. Then, by a result of Courrège–Priouret [4], one gets a diffusion on E such that its part on each chart domain U as above has the transition function expressed with the density p^U as $p_t^U(x, y)V(dy)$, $x,\ y \in U$. The transition function of the global process X given by this result is uniquely determined. In fact, this transition function corresponds to the semigroup (P_t) generated by L. To see this, one should use the fact that locally the process X is associated to the harmonic space produced by L on E and deduce that for each relatively compact open set $D \subset E$, the function

$$u(x) = E^x\left[\int_0^T e^{-dt} f(X_t)\,dt\right], \qquad x \in D,$$

with $T = T_{D^c}$ is the solution of the problem

$$(L - \alpha)u = -f \qquad \text{in } D,$$
$$u = 0 \qquad \text{on } \partial D.$$

Taking an increasing sequence (D_n) of relatively compact open sets such that $\bigcup_n D_n = E$, one deduces that the function

$$u(x) = E^x\left[\int_0^\infty e^{-\alpha t} f(X_t)\,dt\right], \qquad x \in E$$

is in H and satisfies the equation $(L-\alpha)u = f$ for, say, $f \in \mathscr{C}_c(E)$. This implies

$$u = \int_0^\infty e^{-\alpha t} P_t f\,dt,$$

which in turn, by uniqueness of the Laplace transform, leads to

$$P_t f(x) = E^x(f(X_t)) \quad \text{for } V\text{-a.e. } x \in E.$$

In order to go further and obtain the density $p_t(x, y)$ on E, we are going to use some potential theoretic facts on the space–time set $\mathbb{R} \times E$ endowed with the structure of the harmonic space (in the sense of Constantinescu-Cornea [3]) associated to the space–time process. This is a process on $\mathbb{R} \times E$ which has two independent components: the first is the uniform motion to the left and the second is the process X in E. This means that this process is the diffusion generated by the operator $L - \partial t$. For the chart domain U we may use the density $p_t^U(x, y)$ to define the Green function corresponding to L' in $\mathbb{R} \times U$ by putting

$$q^U(s, x, t, y) = p_{s-t}^U(x, y), \qquad (s, x), (t, y) \in \mathbb{R} \times U,$$

where p^U is the above-mentioned extension so that one gets the value 0 when $s \leq t$. The function q^U is lower semicontinuous on $\mathbb{R} \times U \times \mathbb{R} \times U$ and continuous outside the diagonal. For each fixed $(t, y) \in \mathbb{R} \times U$ the function $q^U(\cdot, \cdot, t, y)$ is a potential in $\mathbb{R} \times U$ with support at (t, y). This potential can be rised to a potential $q(\cdot, \cdot, t, y)$ in $\mathbb{R} \times E$ uniquely determined by the following conditions:

1. $q(\cdot, \cdot, t, y)$ has support at the point (t, y).
2. the following relation holds for any $(s, x) \in \mathbb{R} \times U$:

$$q(s, x, t, y) = q^U(s, x, t, y) + E^x(q(s - T, x_T, t, y)),$$

where $T = T_{U^c}$ is the hitting time of U^c.

The second term in the right-hand side of this relation, as a function of (s, x) represents the balayage of the potential $q(\cdot, \cdot, t, y)$ on the set $\mathbb{R} \times U^c$. This function is obtained from $q^U(\cdot, \cdot, t, y)$ by a formula which shows that the association $q^U \to q$ is continuous (see Theorem 3.1′ on page 32 of [19]).

This way one obtains the Green function corresponding to $L - \partial t$ in $\mathbb{R} \times E$, which is lower semicontinuous on $(\mathbb{R} \times E) \times (\mathbb{R} \times E)$ and continuous outside the diagonal. In other words, q represents the density of the potential kernel of the space–time process, which can be expressed by the following relation:

$$E^x\left[\int_0^\infty \varphi(s - u, X_u)\, du\right] = \int\int_{\mathbb{R} \times E} q(s, x, t, y)\varphi(t, y)\, dt\, V(dy),$$

for any $\varphi \in \mathscr{C}_c(\mathbb{R} \times E)$ and $(s, x) \in \mathbb{R} \times E$. From this relation, one easily deduces that $q(s, x, t, y) = q(s + u, x, t + u, y)$, $u \in \mathbb{R}$ and then that the function $p_t(x, y) = q(t, x, 0, y)$ represents the density of the semigroup (P_t). Since $q(\cdot, \cdot, 0, y)$ is a potential with support at $(0, y)$ one deduces that this function vanishes on $(-\infty, 0) \times E$. Taking also into account the maximum principle in the form

$$\begin{aligned} &\sup\{q(s, x, 0, y) \mid (s, x) \in \mathbb{R} \times E \backslash (-\varepsilon, \varepsilon) \times U\} \\ &\quad = \sup\{q(s, x, o, y) \mid (s, x) \in \partial(-\varepsilon, \varepsilon) \times U\}, \end{aligned}$$

we get relations (i) and (ii) of the statement. $\square$

APPENDIX B

Continuity of the semigroup density in $\mathbb{R}^N$. In what follows, we assume that $E = \mathbb{R}^N$ and will show that the density $p_t(x, y)$ of the semigroup is locally Hölder continuous by proving the estimate (25). Since the arguments of the proof involve the operator L as well as its adjoint L^*, we start in a general setting and consider an operator of the form

$$\mathscr{L}_u = \frac{1}{\rho}\frac{\partial}{\partial x^i}\rho a^{ij}\frac{\partial u}{\partial x^j} + b^k\frac{\partial u}{\partial x^k} - \frac{1}{\rho}\frac{\partial}{\partial x^k}(\rho\tilde{b}^k u) + cu.$$

All the coefficients $\rho, a^{ij}, b^k, \tilde{b}^k, c$ are assumed to be bounded with bounded derivatives of all orders. Moreover conditions (21), (22) and (23) are supposed to be satisfied with some constants λ and Λ, the field $\tilde{b}$ is assumed to be bounded by (23) and $| c |$ to be bounded by the same constant Λ. As in the preceding sections, we denote $V(dx) = \rho(x)\,dx$ and

$$\Delta = \frac{1}{\rho}\frac{\partial}{\partial x^i}\rho a^{ij}\frac{\partial}{\partial x^j}.$$

We also write $B = b^k(\partial/\partial x^k)$, $\tilde{B} = \tilde{b}^k(\partial/\partial x^k)$ so that the formal adjoint of $\tilde{B}$ is expressed as $\tilde{B}^* u = -(1/\rho)(\partial/\partial x^k)(\rho\tilde{b}^k u)$ and thus one may write $\mathscr{L} = \Delta + B + \tilde{B}^* + c$. One immediately sees that the adjoint of $\mathscr{L}$ is of the same type: $\mathscr{L}^* = \Delta + B^* + \tilde{B} + c$. It is known that the semigroup generated by $\mathscr{L}$ admits a density $p_t(x, y)$ which is in the class $\bigcup_n \mathscr{C}_b^\infty((1/n, \infty) \times \mathbb{R}^N \times \mathbb{R}^N)$ and satisfies the estimate (24) (see [20]). Moreover, this density satisfies the estimate (25). This follows from the next proposition applied to $p_\bullet(\cdot, y)$, with respect to $\partial_t - \mathscr{L}$ and to $p_\bullet(x, \cdot)$ with respect to $\partial_t - \mathscr{L}^*$. The statement of this proposition is given in terms of cylindrical sets of the form

$$Q(\sigma, \xi, r) = [\sigma - r^2, o] \times \bar{B}_r(\xi),$$

where $\sigma \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, $r \in \mathbb{R}_+$. It is a variation of Nash's theorem, which is well known (see [8]). An elegant proof for the case $\tilde{B} \equiv 0$ may be found in [20] (Theorem II.2.12). However, for the reader's ease, we include a short proof here which reduces the result to that treated in [20].

PROPOSITION B.1. *For each $\delta > 0$, there exist two constants $\alpha \in (0, 1]$ and $C > 0$, which depend only on N, λ, Λ and δ, such that any solution of $\partial_t u - \mathscr{L}u = 0$, $u \in \mathscr{C}^{1,2}(Q(\sigma, \xi, r))$ satisfies the estimate*

$$| u(s, x) - u(t, y) | \le C\left(\frac{| s - t |^{1/2} \vee | x - y |}{r}\right)^\alpha \|u\|_{Q(\sigma, \xi, r)},$$

for any $(s, x), (t, y) \in Q(\sigma, \xi, \delta r)$, $\sigma \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, $r \in (0, 1/\delta]$.

PROOF. We are going to show that the problem can be reduced to the case $\tilde{B} \equiv 0$. Take $\xi \in \mathbb{R}^N$ and set $\zeta = \xi + (3/\delta)e_1$ with $e_1 = (1, 0, \dots, 0)$. Then choose $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^N)$ so that $\operatorname{supp}\varphi \subset B_{1/\delta}(\xi)$, $\varphi \geq 0$ and $\int \varphi\, dV = 1$ and set

$$w(x) = \int_0^\infty e^{-\tau t} \int_{\mathbb{R}^N} p_t(x, y)\varphi(y)V(dy)\, dt,$$

where $\tau = \eta + 1$, and η is the constant appearing in the estimate (24). Strightforward computations show that w has the following properties:

1. $w \in \mathscr{C}_b^\infty(\mathbb{R}^N)$ and $(\mathscr{L} - \tau)w = -\varphi$; in particular one has $(\mathscr{L} - \tau)w = 0$ on $B_{1/\delta}(\xi)$.
2. There exist two constants $0 < m \leq M < \infty$ which depend only on N, λ, Λ and δ so that

   $$m \leq w(x) \leq M$$

   for any $x \in B_{1/\delta}(\xi)$.

Theorem 8.22 of [7] gives another constant C which depends only on N, λ, Λ and δ so that

$$(*) \qquad | w(x) - w(y) | \leq C \mid x - y \mid^\alpha, \; x, y \in B_{1/\delta}(\xi).$$

Now take arbitrary $\sigma \in \mathbb{R}$, $r \in (0, 1/\delta)$ and $v \in \mathscr{C}^{1,2}(Q(\sigma, \xi, r))$. Then one has

$$(\partial_t - \mathscr{L})(e^{\tau t} wv) = e^{\tau t} w(\partial_t - L)v,$$

where $L = \Delta' + B - \tilde{B}$ and

$$\Delta' = \frac{1}{w^2\rho} \frac{\partial}{\partial x^i} w^2 \rho a^{ij} \frac{\partial}{\partial x^j}.$$

Now, the main point is that the operator Δ', defined on $B_{1/\delta}(\xi)$, is similar to Δ and the function $\rho' = w^2\rho$ can be extended to $\mathbb{R}^N$ so that it satisfies an estimate like (22) and with this extension Δ' becomes like Δ in $\mathbb{R}^N$. Therefore, one may apply Theorem II.2.12. of [20] with respect to L. If u is a solution of $(\partial_t - L)u = 0$ in $Q(\sigma, \xi, r)$, then the function $v = e^{\tau(\sigma - t)}u/w$ is a solution of $(\partial_t - L)v = 0$, and according to the theorem mentioned in [20], the function v satisfies the estimate from the statement. Then, taking into account relation $(*)$, one easily deduces that $u = e^{\tau(t-\sigma)}wv$ should satisfy a similar inequality.

Finally, the estimate (24) for the density of the operator L of Section 8 (with measurable coefficients) is obtained as in Theorem II.3.8 of [20] by approximation. □

APPENDIX C

Quasi-continuity. In this Appendix, we prove the following lemma.

LEMMA C.1. *If a function f is quasi-continuous with respect to the capacity given by $\mathscr{E}$, then the process $f(X_t)$ is continuous.*

PROOF. The property asserted in this lemma is clearly a local one, so that it is enough to treat the case $E = \mathbb{R}^N$. In this situation one can show that if (A_n) is a decreasing sequence of open sets such that $\operatorname{cap}(A_n) \to 0$, then the sequence $T_n = T_{A_n}$ of first hitting times converges to infinity a.s. To see this one should use the estimates (24). With $\gamma > \eta$, one constructs a γ-coexcessive function which is strictly positive,

$$u(x) = \int_{\mathbb{R}^N} \int_0^\infty e^{-\gamma t} p_t(y, x) V(dy).$$

Then the semigroup $e^{-\gamma t} P_t$ is in duality with the semigroup $u^{-1} e^{-\gamma t} P_t^*(u\cdot)$ under the duality measure $u \cdot V$. Since the second semigroup became sub-Markovian, one may apply the duality theory of [2]. On the other hand, the Green function of this pair is comparable [via (24)] with the Green function of the symmetric semigroup $e^{-\gamma t} Q_t$. This leads to $E^\bullet(e^{-\gamma T_n}) \to 0$. □

APPENDIX D

The domain of L^* in $L^1(dV)$.

LEMMA D.3. *Let $u \in H \cap L^1$ and $h \in L^1$ be such that the following relation is satisfied for each $\varphi \in \mathscr{C}_c^\infty$:*

$$-\mathscr{E}(\varphi, u) + (b\varphi, u) = (\varphi, h).$$

Then u belongs to the domain in $L^1(dV)$ of the operator L^* [or, in other words, to the domain of the infinitesimal generator of the semigroup (P_t^*) considered on the space $L^1(dV)$, which is denoted by $\mathscr{D}_{L^1}(L^*)$] and $L^* u = h$.

PROOF. The relation in the hypothesis extends to any $\varphi \in H \cap L^\infty$. In particular, for a constant $\lambda > 0$ and $\varphi \in \mathscr{C}_c^\infty$, we may write

$$-\mathscr{E}(U_\lambda \varphi, u) + (bU_\lambda \varphi, u) = (U_\lambda \varphi, h).$$

From this equality we deduce

$$(\varphi, u) = \overline{\mathscr{E}}_\lambda(U_\lambda \varphi, u) = (U_\lambda \varphi, \lambda u - h) = (\varphi, U_\lambda^*(\lambda u - h)),$$

which implies $u = U_\lambda^*(\lambda u - h)$ in L^1. □

From this lemma one immediately gets the following propositon.

PROPOSITION D.4. *If $u \in \mathscr{D}(L^*) \cap L^\infty$ and $v \in \mathscr{D}(\Delta) \cap L^\infty$, then $uv \in \mathscr{D}_{L^1}(L^*)$ and the following relation holds*:

$$L^*(uv) = (L^* u)v + u(\Delta - b)v + 2(\nabla u, \nabla v).$$

In particular, one has the following result.

PROPOSITION D.5. *If $u, v \in \mathscr{D}(\Delta) \cap L^\infty$, then $uv \in \mathscr{D}_{L^1}(\Delta)$ and*

$$\Delta uv = u\Delta v + v\Delta u + 2\langle \nabla u, \nabla v \rangle.$$

Observe also that the above statements remain true if the condition on the function v is replaced by $v \in \mathscr{D}_{L^1}(\Delta) \cap H \cap L^\infty$.

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