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EIGENVALUE DISTRIBUTIONS OF RANDOM PERMUTATION MATRICES

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Let M be a randomly chosen $n \times n$ permutation matrix. For a fixed arc of the unit circle, let X be the number of eigenvalues of M which lie in the specified arc. We calculate the large n asymptotics for the mean and variance of X, and show that $(X - E[X])/(\mathrm{Var}(X))^{1/2}$ is asymptotically normally distributed. In addition, we show that for several fixed arcs $I_1, \ldots, I_m$, the corresponding random variables are jointly normal in the large n limit.

1. Introduction. There has been a great deal of recent interest in random matrices, particularly the distribution of eigenvalues. Random matrices have applications in fields ranging from physics to number theory, and much recent work has explored the fact that the distributions which arise from a number of different matrix ensembles are strikingly similar. The main matrix ensembles which have been studied are continuous: the Gaussian ensembles and related perturbations of the Gaussian measure, and the compact Lie groups U_n, O_n, and Sp_n. The group of permutation matrices S_n sits in U_n and O_n, and it is interesting to study the eigenvalue distribution for this finite group (with uniform probability measure) to see how much of the structure on the larger groups can be seen in a finite subgroup.

One immediate question to ask is how many eigenvalues lie in some fixed arc of the circle. A fairly natural guess is to say that the number will be proportional to the size of the arc. Roughly speaking this is true, but the answer depends on a number of things – the size of the interval, where the interval is located on the unit circle, and even whether the interval is open or closed.

Fix an arc of the unit circle $I = (e^{2\pi i\alpha}, e^{2\pi i\theta}]$. In this paper, we study the number of eigenvalues of a random permutation matrix which lie in this arc. To do this, define a random variable X_n^I on S_n to be the number of eigenvalues in I. We will show for large n that the mean and variance asymptotics of X_n^I are

$$E_{S_n}[X_n^I] = n(\theta - \alpha) - c_1 \log n + o(\log n) \tag{1.1}$$

and

$$\mathrm{Var}(X_n^I) = c_2 \log n + o(\log n), \tag{1.2}$$

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where c_1 and c_2 are explicit functions of α and θ. In addition, we will show that the normalized random variable

$$Y_n^I = \frac{X_n^I - E[X_n^I]}{(c_2 \log n)^{1/2}} \Rightarrow \mathcal{N}(0,1), \tag{1.3}$$

as $n \to \infty$, and give a rate of convergence.

This behavior leads to another question. Fix intervals $I_1 = (e^{2\pi i\alpha_1}, e^{2\pi i\theta_1}]$, $I_2 = (e^{2\pi i\alpha_2}, e^{2\pi i\theta_2}], \ldots, I_m = (e^{2\pi i\alpha_m}, e^{2\pi i\theta_m}]$, and let $X_n^{I_1}, X_n^{I_2}, \ldots, X_n^{I_m}$ be the corresponding random variables. Individually, each of the random variables

$$Y_n^{I_k} = \frac{X_n^{I_k} - E[X_n^{I_k}]}{\left(c_2^{(k)} \log n\right)^{1/2}}$$

converges to a normal random variable, and it is reasonable to ask what the joint behavior of the random variables is. We will show that in the large n limit, the normalized random variables follow a joint normal distribution.

This paper is organized as follows. The next section introduces some standard results for probability on the symmetric group, which will be useful throughout the rest of the paper. In Section 3, equations (1.1) and (1.2) will be proved, and the values of the constants c_1 and c_2 will be given in some special cases. Evaluation of the constants requires some number theory; these details can be found in the Appendix. The asymptotic joint normality will be proved in the fourth section. Finally, in the last section, these results will be compared with results for the unitary group.

2. Probability on the symmetric group. One advantage to working with S_n is that random permutations are well understood. Many questions about permutations concern only the cycle structure of the permutations; questions about the eigenvalues of a permutation matrix are among these. This allows us to take advantage of the extensive work on cycle lengths of random permutations. The rest of this section will be spent introducing some of the standard tools and results concerning cycle lengths, then showing what these tools say about X_n^I.

2.1. *The cycle index theorem.* Let σ be a permutation in S_n, and let $(a_1, a_2, \ldots, a_n)$ be the cycle structure of σ (so σ has a_1 fixed points, a_2 2-cycles, etc.). If a permutation is chosen at random, there are many questions one can ask about the permutation. A few examples of questions which depend only on the cycle structure are: the number of fixed points; the number of cycles; the length of the longest cycle. The number of fixed points, first studied by Montmort in 1708, is one of the earliest results in probability [16]. The second and third questions are handled by Goncharov in [11]. All three questions are discussed in the article by Shepp and Lloyd [17].

These and many other questions about cycles can be answered easily using the cycle index theorem.

DEFINITION 1. *The cycle index polynomial for* S_n *is*

$$\psi_n(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_i^{a_i(\sigma)}.$$

The cycle index polynomials can be used to define a generating function

$$\Psi(z, x_1, \ldots) = \sum_{n=0}^{\infty} z^n \psi_n(x_1, \ldots x_n), \tag{2.1}$$

using $\psi_0 = 1$. The cycle index theorem says that this generating function factors:

THEOREM 1 (Cycle index theorem).

$$\Psi(z, \mathbf{x}) = \prod_{i=1}^{\infty} e^{x_i z^i / i}.$$

The proof amounts to expanding the exponentials in the right side of the theorem, and comparing powers of z. (The details can be found in the introductory section of [17].)

Observe that

$$\psi_n(x_1, \ldots, x_n) = E_{S_n}\left[\prod_{j=1}^{n} x_j^{a_j}\right]. \tag{2.2}$$

The cycle index theorem provides a simple form for the generating function, which makes it easy to calculate information about the cycle structure of a random permutation. One application is the following theorem, from [11], which will be of use for calculating the mean and variance of X_n^I.

THEOREM 2. *If a permutation* σ *is chosen at random from* S_n, *then for large* n, *the numbers* $a_1, a_2, \ldots, a_k$ *are asymptotically independent Poisson random variables with parameters* $1, \frac{1}{2}, \ldots, \frac{1}{k}$. *In particular*,

$$E[a_j] = \begin{cases} \dfrac{1}{j}, & \text{if } j \le n, \\ 0, & \text{otherwise}, \end{cases}$$

$$E[a_j a_k] = \begin{cases} \dfrac{1}{jk}, & \text{if } j + k \le n, \\ 0, & \text{otherwise}, \end{cases}$$

if $j \ne k$, *and*

$$\operatorname{Var}(a_j) = \begin{cases} \dfrac{1}{j}, & \text{if } j \le n/2, \\ \dfrac{1}{j} - \dfrac{1}{j^2}, & \text{if } n/2 < j \le n, \\ 0, & \text{otherwise}. \end{cases}$$

To see how the moment formulas can be proved, note that

$$E_{S_n}[a_k] = \left(\frac{\partial}{\partial x_k}\psi_n\right)(1, 1, \ldots, 1). \tag{2.3}$$

Thus the expected value $E_{S_n}[a_k]$ can be computed using the cycle index theorem by differentiating Ψ once with respect to x_k, setting all of the x_j's equal to 1, and then reading off the coefficient of z^n. In fact, all of the moments can be calculated using a similar procedure. Diaconis and Shahshahani [7] used this method to calculate all of the moments and prove that as $n \to \infty$, these moments converge to the moments of independent Poisson $1/j$ random variables.

2.2. *The Feller coupling.* In order to make use of the fact that the cycle structure a_j's are close to independent Poisson, it is necessary to obtain some estimates for how close they really are. A number of useful estimates due to Arratia, Barbour, and Tavaré can be found in [1]. A summary of their results is given below.

The work of those articles is based on the Feller coupling, a procedure for generating random permutations. The Feller coupling was introduced in [9] for studying the number of cycles of a random permutation. The procedure works as follows. Let $\xi_1, \ldots, \xi_n$ be independent Bernoulli random variables with $P(\xi_j = 1) = 1/j$. The permutation σ will be built in cycle notation using $\xi_1, \ldots, \xi_n$ in reverse order. Start the first cycle with a 1. If $\xi_n = 1$, close the current cycle, and start the next cycle with the smallest unused integer. If $\xi_n = 0$, choose a random integer uniformly from the remaining integers and put it in the current cycle, to the right of the last number added. Keep building in this way, using the sequence $\xi_{n-1}, \xi_{n-2}, \ldots, \xi_1$ to determine when to start new cycles. The result of the procedure will be a uniformly distributed permutation in cycle notation.

For $\sigma \in S_n$ generated by the Feller coupling, the cycle structure can be expressed explicitly in terms of the ξ_j's, which provides a means of constructing a Poisson process which is close (in total variation distance) to the a_j's. The cycle lengths are the spacings between consecutive 1's in the sequence $\xi_1, \ldots, \xi_n$, so a_j will be the number of times the spacing is equal to j:

$$a_j(\sigma) = \xi_{n-j+1}(1 - \xi_{n-j+2}) \cdots (1 - \xi_n) \tag{2.4}$$

$$+ \sum_{i=1}^{n-j} \xi_i(1 - \xi_{i+1}) \cdots (1 - \xi_{i+j-1})\xi_{i+j}. \tag{2.5}$$

For the large n limit, set

$$W_j = \sum_{i=1}^{\infty} \xi_i(1 - \xi_{i+1}) \cdots (1 - \xi_{i+j-1})\xi_{i+j} \tag{2.6}$$

and

$$W_{jm} = \sum_{i=m+1}^{\infty} \xi_i(1-\xi_{i+1})\cdots(1-\xi_{i+j-1})\xi_{i+j}. \tag{2.7}$$

In [1], it is shown that the W_j's are independent Poisson random variables with $E[W_j] = 1/j$ (see pages 523–524). This coupling of $\{W_j\}$ with $\{a_j\}$ makes it easy to compare the two in a very explicit way.

One of the results in [1] compares the joint distribution of $(a_1, \dots, a_{b(n)})$ for $0 \le b(n) \le n$ with $(W_1, \dots, W_{b(n)})$ while allowing $b(n)$ to grow with n. The authors show that as $n \to \infty$, if $b(n)/n \to 0$ then the total variation distance between $(a_1, \dots, a_{b(n)})$ and $(W_1, \dots, W_{b(n)})$ goes to zero. For the purposes of this paper, several of their estimates will be useful. The following bound can be found in [5].

LEMMA 1. *Define*

$$J_n = \min\{i \ge 1 : \xi_{n-i+1} = 1\} \qquad \textit{and} \qquad K_n = \min\{i \ge 1 : \xi_{n+i} = 1\}.$$

Then

$$-W_{jn} - 1_{\{J_n + K_n = j+1\}} \le a_j - W_j \le 1_{\{J_n = j\}}.$$

This bound implies that

$$\sum_{j=1}^{n} |a_j - W_j| \le 2 + \sum_{j=1}^{n} W_{jn}. \tag{2.8}$$

In addition, the independence of the ξ_j's can be used to show that

$$\sum_{j=1}^{n} E[W_j] \le 1, \tag{2.9}$$

which implies the following:

LEMMA 2. *Suppose the a_j's and the W_j's are coupled as above, and set*

$$R_n = \sum_{j=1}^{n} \frac{a_j - W_j}{(\log n)^{1/2}}.$$

Then $R_n \to_p 0$.

(For a proof, see pages 525–526 of [1].) A lot of work has been done with these estimates; [3] and [5] give a number of applications, and [2] has much sharper bounds on the total variation distance between the cycle structure numbers and the Poisson process.

2.3. *Using cycles to study X_n^I.* For each element $\sigma \in S_n$ there is a natural way to assign a corresponding permutation matrix M_σ, namely

$$(M_\sigma)_{ij} = \begin{cases} 1, & \text{if } j = \sigma(i), \\ 0, & \text{otherwise.} \end{cases} \tag{2.10}$$

Because of this relationship, the eigenvalues of M_σ depend only on the cycle structure of σ. Each k-cycle in σ corresponds to a set of k eigenvalues: $\{1, e^{2\pi i/k}, e^{4\pi i/k}, \dots, e^{2(k-1)\pi i/k}\}$. Thus if the cycle structure of σ is $(a_1, a_2, \dots, a_n)$, then M_σ has a_k copies of these eigenvalues.

The random variable X_n^I can be written in terms of the cycle structure $(a_1, a_2, \dots, a_n)$. To simplify the notation, recall that $I = (e^{2\pi i\alpha}, e^{2\pi i\theta}]$, and assume that $0 \le \alpha < 1$ and $\alpha \le \theta < \alpha + 1$. Of the k eigenvalues corresponding to a k-cycle, $\lfloor k\theta \rfloor - \lfloor k\alpha \rfloor$ of these lie in I. Thus X_n^I can be written

$$X_n^I(\sigma) = \sum_{k=1}^{n} a_k(\sigma)(\lfloor k\theta \rfloor - \lfloor k\alpha \rfloor) \tag{2.11}$$

$$= (\theta - \alpha) \sum_{k=1}^{n} k a_k - \sum_{k=1}^{n} a_k(\{k\theta\} - \{k\alpha\}) \tag{2.12}$$

$$= n(\theta - \alpha) - \sum_{k=1}^{n} a_k(\{k\theta\} - \{k\alpha\}) \tag{2.13}$$

where the last line follows from the fact that for any permutation in S_n, $\sum k a_k = n$ (and $\{k\theta\}$ denotes the fractional part of $k\theta$). Equation (2.13) is the basis for the main calculations in this paper.

The mean and variance of X_n^I can be computed using Theorem 2:

$$E_{S_n}[X_n^I] = n(\theta - \alpha) - \sum_{k=1}^{n} E_{S_n}[a_k](\{k\theta\} - \{k\alpha\}) \tag{2.14}$$

$$= n(\theta - \alpha) - \sum_{k=1}^{n} \frac{1}{k}(\{k\theta\} - \{k\alpha\}) \tag{2.15}$$

and

$$\operatorname{Var}(X_n^I) = \operatorname{Var}\left(\sum_{j=1}^{n} a_j(\{j\theta\} - \{j\alpha\})\right) \tag{2.16}$$

$$\begin{aligned} &= \sum_{j=1}^{n} \frac{1}{j}(\{j\theta\} - \{j\alpha\})^2 \\ &\quad - \sum_{j=1}^{n} \sum_{k=n-j+1}^{n} \frac{1}{jk}(\{j\theta\} - \{j\alpha\})(\{k\theta\} - \{k\alpha\}). \end{aligned} \tag{2.17}$$

Thus calculating the asymptotics of the mean and variance for X_n^I is a matter of obtaining asymptotics for the sums in (2.15) and (2.17). This will be done in the next section.

For two intervals I_1 and I_2, the formula for covariance of $X_n^{I_1}$ and $X_n^{I_2}$ is similar to the variance formula:

$$\begin{aligned}\operatorname{Cov}(X_n^{I_1}, X_n^{I_2}) &= \sum_{j=1}^{n} \frac{1}{j}(\{j\theta_1\} - \{j\alpha_1\})(\{j\theta_2\} - \{j\alpha_2\}) \\ &\quad - \sum_{j=1}^{n} \sum_{k=n-j+1}^{n} \frac{1}{jk}(\{j\theta_1\} - \{j\alpha_1\})(\{k\theta_2\} - \{k\alpha_2\}).\end{aligned} \tag{2.18}$$

3. Mean and variance asymptotics. The goal of this section is the calculation of the asymptotics of $E[X_n^I]$ and $\operatorname{Var}(X_n^I)$ from equations (2.15) and (2.17). The main part of the calculation is finding asymptotics for sums of the form

$$\sum_{j=1}^{n} \frac{1}{j}(\{j\theta\} - \{j\alpha\})^m. \tag{3.1}$$

Fortunately, the problem of evaluating these sums can be translated into a standard number theory question by means of an elementary analysis theorem.

This section will be organized in two parts. The technical theorem will be proved in the first part, and used in the second.

3.1. *Logarithmic summability.* The first part of the theorem below is a special case of a more general theorem. The general case can be found on page 15 of [14].

THEOREM 3. *Let $(\omega_k)_{k\geq 1}$ be a sequence of real numbers such that $0 \leq \omega_k \leq K$ for some constant $K < \infty$, for all k.*

1. *If there is a number $L < \infty$ for which*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \omega_k = L,$$

then

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\omega_k}{k} = L.$$

2. *Suppose in addition that there are positive constants A and b such that for all $n \geq 1$,*

$$\left| \frac{1}{n} \sum_{k=1}^{n} \omega_k - L \right| < An^{-b}.$$

Then there is a constant C which depends on A and b such that for $n > 1$,

$$\left| \frac{1}{\log n} \sum_{k=1}^{n} \frac{\omega_k}{k} - L \right| \leq \frac{C}{\log n}.$$

PROOF (Part 1). Define two sequences (s_n) and (t_n) by

$$s_n = \frac{1}{n}\sum_{k=1}^{n} \omega_k, \tag{3.2}$$

and

$$t_n = \frac{1}{H_n}\sum_{k=1}^{n} \frac{\omega_k}{k}, \tag{3.3}$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Also define a transformation M by

$$M_{ij} = \begin{cases} \dfrac{1}{(j+1)H_i}, & \text{if } i < j, \\ \dfrac{1}{H_i}, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

It is easy to check that the row sums of M are all 1 and that for fixed j, $\lim_{i\to\infty} M_{ij} = 0$. It is also fairly straightforward to show that $Ms = t$. Because the row sums of M are all 1, we have

$$t_n - L = M_{n1}(s_1 - L) + M_{n2}(s_2 - L) + \cdots + M_{nn}(s_n - L). \tag{3.5}$$

Suppose $\varepsilon > 0$ is given. Find an integer N_0 such that for all $k > N_0$, $|s_k - L| < \frac{\varepsilon}{2}$. Next choose N_1 big enough that for all $n > N_1$ and all $k \le N_0$, $M_{nk} \le \varepsilon/4N_0K$. Then for all $n > \max(N_0, N_1)$,

$$|t_n - L| \le \sum_{k=1}^{N_0} M_{nk}|s_k - L| + \sum_{k=N_0+1}^{n} M_{nk}|s_k - L| \tag{3.6}$$

$$\le \frac{\varepsilon}{4N_0K} N_0 2K + \frac{\varepsilon}{2} = \varepsilon. \tag{3.7}$$

Thus

$$\lim_{n\to\infty} \frac{1}{H_n}\sum_{k=1}^{n} \frac{\omega_k}{k} = L. \tag{3.8}$$

Finally, by observing that $H_n \sim \log n + O(1)$, we have

$$\lim_{n\to\infty} \frac{1}{\log n}\sum_{k=1}^{n} \frac{\omega_k}{k} = L. \tag{3.9}$$

PROOF (Part 2). Finding a rate for the convergence in this theorem is mostly a matter of making use of the definition of M_{ij} and bounding the terms carefully. Recall equation (3.5),

$$t_n - L = M_{n1}(s_1 - L) + \cdots + M_{nn}(s_n - L). \tag{3.10}$$

Then using (3.4)

$$|t_n - L| \le \sum_{k=1}^{n} M_{nk} |s_k - L| \tag{3.11}$$

$$\le \frac{1}{H_n}\left(\frac{A}{n^b} + \sum_{k=1}^{n-1} \frac{A}{(k+1)k^b}\right) \tag{3.12}$$

$$\le \frac{A}{H_n}\left(1 + \sum_{k=1}^{n-1} \frac{1}{k^{1+b}}\right). \tag{3.13}$$

Because b is positive, the sum $\sum_{k=1}^{\infty} 1/k^{1+b}$ converges. Since $H_n \sim \log n + O(1)$, we can find a constant C so that

$$|t_n - L| \le \frac{C}{\log n}, \tag{3.14}$$

which is the desired result. □

3.2. *Applying the theorem.* Using the functions $f(x) = x$ and $f(x) = x^2$, Theorem 3 can be applied to see that

$$\sum_{j=1}^{n} \frac{1}{j}(\{j\theta\} - \{j\alpha\}) = c_1 \log n + o(\log n), \tag{3.15}$$

$$\sum_{j=1}^{n} \frac{1}{j}(\{j\theta\} - \{j\alpha\})^2 = c_2 \log n + o(\log n), \tag{3.16}$$

where

$$c_1 = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (\{j\theta\} - \{j\alpha\}) \tag{3.17}$$

and

$$c_2 = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (\{j\theta\} - \{j\alpha\})^2. \tag{3.18}$$

The mean follows immediately:

$$E_{S_n}[X_n^I] = n(\theta - \alpha) - c_1 \log n + o(\log n). \tag{3.19}$$

For the variance of X_n^I, there is a second sum in (2.17) to study,

$$\sum_{j=1}^{n} \sum_{k=n-j+1}^{n} \frac{1}{jk}(\{j\theta\} - \{j\alpha\})(\{k\theta\} - \{k\alpha\}), \tag{3.20}$$

but this sum is bounded for all n, because

$$\sum_{j=1}^{n} \sum_{k=n-j+1}^{n} \frac{1}{jk} = \sum_{j=1}^{n} \frac{1}{j^2} \le \frac{\pi^2}{6}. \tag{3.21}$$

Thus the variance is

$$\mathrm{Var}(X_n^I) = c_2 \log n + o(\log n). \tag{3.22}$$

This still leaves the problem of evaluating the constants c_1 and c_2. The mean constant c_1 can always be evaluated; the variance constant is not so easy to calculate in general. Calculation of the constants uses some standard number theory results. The details are somewhat tedious, but the ideas are interesting, so this is done in the Appendix. The values of the constants for a few special cases are given below.

EXAMPLE 1. If $\alpha = 0$, then X_n^I is the number of eigenvalues in $(1, e^{2\pi i\theta}]$, and the formulas for the constants simplify to

$$c_1 = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \{j\theta\},$$

$$c_2 = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \{j\theta\}^2.$$

This case is discussed in detail in Chapter 5 of [19]. The values of the constants are $c_1 = 1/2$ and $c_2 = 1/3$ for irrational θ, and $c_1 = (q-1)/2q$ and $c_2 = (q-1)(2q-1)/6q^2$ for rational $\theta = p/q$. [See (A.7) – (A.10) in the Appendix.]

At first glance, this may look very strange. However, since all of the eigenvalues of a permutation matrix are rational multiples of 2π, it is not so surprising that there would be some sort of boundary effect which would not be present when the interval ends at an irrational multiple of 2π. (Notice that as the denominator of a rational θ becomes larger, c_1 approaches 1/2 and c_2 approaches 1/3.)

EXAMPLE 2. If α and θ are both irrational, then $c_1 = 0$. If, in addition, α and θ are linearly independent over $\mathbb{Q}$, then $c_2 = 1/6$. These results follow follow from (A.9) and (A.10) and from Corollary 1.

Choosing endpoints to be irrational and linearly independent of each other should eliminate most of the effects due to the rational spacing of the eigenvalues, allowing any underlying structure to be seen. When comparing results for permutation matrices with results for unitary matrices, this is the case which will be used.

EXAMPLE 3. Suppose that α is irrational and $\theta = p/q$ is rational. Then

$$c_1 = \frac{q-1}{2q} - \frac{1}{2} = -\frac{1}{2q} \tag{3.23}$$

and

$$c_2 = \frac{1}{6} + \frac{1}{6q^2}. \tag{3.24}$$

[See equations (A.7) – (A.10) and Theorem 11.]

EXAMPLE 4. If $\alpha = p/q$ and $\theta = r/s$ are both rational, then from (A.7)

$$c_1 = \frac{1}{2q} - \frac{1}{2s}. \tag{3.25}$$

There is no simple form for c_2 in most cases, but using (A.8) and (A.17), c_2 can be calculated by evaluating the sum in

$$c_2 = \frac{(q-1)(2q-1)}{6q^2} + \frac{(s-1)(2s-1)}{6s^2} - \frac{2}{qs}\sum_{j=1}^{qs}\left\{\frac{jp}{q}\right\}\left\{\frac{jr}{s}\right\}. \tag{3.26}$$

EXAMPLE 5. For the general case with α and θ irrational and linearly dependent over $\mathbb{Q}$, $c_1 = 0$ follows from (A.9), but formulas for c_2 can be calculated for only some of the cases (the same cases covered in Theorem 12). If θ and α are related by $\theta = r\alpha + p/q$, then

$$c_2 = \frac{1}{6} - \frac{1}{6rq^2}. \tag{3.27}$$

This follows from (A.10) and Theorem 12. While there is no formula for c_2 in the most general case, the pattern established so far makes it seem reasonable to expect that in general c_2 is around 1/6.

REMARK. A half-open interval $I = (e^{2\pi i\alpha}, e^{2\pi i\theta}]$ was chosen because of the simplicity of the formula (2.13) for X_n^I. If α and θ are irrational, then it makes no difference whether the interval is open or closed (since all of the eigenvalues are roots of unity).

If $\alpha = p/q$, then for the closed interval $J_c = [e^{2\pi i\alpha}, e^{2\pi i\theta}]$,

$$X_n^{J_c} = n(\theta - \alpha) - \sum_{k=1}^{n} a_k(\{k\theta\} - \{k\alpha\}) + \sum_{k|q, k\le n} a_k; \tag{3.28}$$

if $\theta = r/s$, then for the open interval $J_o = (e^{2\pi i\alpha}, e^{2\pi i\theta})$,

$$X_n^{J_c} = n(\theta - \alpha) - \sum_{k=1}^{n} a_k(\{k\theta\} - \{k\alpha\}) - \sum_{k|s, k\le n} a_k. \tag{3.29}$$

All of the results in this paper are true in these cases as well; the main difference is a change in the coefficient of the $\log n$ term. In the case where both α and θ are rational (as in Example 4), $c_1(\text{open}) = (1/2q) + (1/2s)$ and $c_1(\text{closed}) = -(1/2q) - (1/2s)$. Generally, the constants are more difficult to compute in the open and closed cases than in the half-open interval used throughout.

4. Limiting normality.

4.1. *Joint limiting distribution.* Fix a finite number of intervals on the unit circle, $I_1 = (e^{2\pi i\alpha_1}, e^{2\pi i\theta_1}]$, $I_2 = (e^{2\pi i\alpha_2}, e^{2\pi i\theta_2}], \dots, I_m = (e^{2\pi i\alpha_m}, e^{2\pi i\theta_m}]$, and let $X_n^{I_1}, X_n^{I_2}, \dots, X_n^{I_m}$ be the corresponding random variables. Also, let

$c_1^{(k)}$ and $c_2^{(k)}$ be the mean and variance constants for $X_n^{I_k}$. The goal of this section will be to investigate the behavior of the normalized random variables

$$Y_n^{I_k} = \frac{X_n^{I_k} - E\left[X_n^{I_k}\right]}{\left(c_2^{(k)} \log n\right)^{1/2}}.$$

Let $\mathbf{Z} = (Z_1, \ldots, Z_m)$ be jointly distributed normal random variables with covariance matrix

$$D_{kk} = 1 \tag{4.1}$$

and

$$D_{kl} = \frac{1}{\left(c_2^{(k)} c_2^{(l)}\right)^{1/2}} \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (\{j\theta_k\} - \{j\alpha_k\})(\{j\theta_l\} - \{j\alpha_l\}). \tag{4.2}$$

The following will be the main result of this section.

THEOREM 4. *As* $n \to \infty$, $(Y_n^{I_1}, \ldots, Y_n^{I_m}) \Rightarrow (Z_1, \ldots, Z_m)$.

Using the Cramér-Wold device, Theorem 4 will follow from:

THEOREM 5. *For each* $(t_1, \ldots, t_m) \in \mathbb{R}^m$,

$$t_1 Y_n^{I_1} + \cdots + t_m Y_n^{I_m} \Rightarrow t_1 Z_1 + \cdots + t_m Z_m,$$

as $n \to \infty$.

The proof of Theorem 5 will make use of the Feller coupling estimates described in Section 2.2. Suppose the random permutation has been chosen by the Feller coupling procedure, and let $\{W_j\}$ be the independent Poisson random variables constructed in Section 2.2. For each $(t_1, \ldots, t_m) \in \mathbb{R}^m$, set

$$V_{n,j}^{(t)} = \frac{W_j - \frac{1}{j}}{(\log n)^{1/2}} \left[\sum_{k=1}^{m} \frac{t_k}{\left(c_2^{(k)}\right)^{1/2}} (\{j\theta_k\} - \{j\alpha_k\}) \right] \tag{4.3}$$

and

$$T_n^{(t)} = \sum_{j=1}^{n} V_{n,j}^{(t)}. \tag{4.4}$$

$T_n^{(t)}$ is the analogue of $t_1 Y_n^{I_1} + \cdots + t_m Y_n^{I_m}$.

First, observe that

$$\left| t_1 Y_n^{I_1} + \cdots + t_m Y_n^{I_m} - T_n^{(t)} \right| \le \left(\sum_{k=1}^{m} \frac{|t_k|}{\left(c_2^{(k)}\right)^{1/2}} \right) \left| \sum_{j=1}^{n} \frac{a_j - W_j}{(\log n)^{1/2}} \right|. \tag{4.5}$$

By Lemma 2, this converges in probability to 0. By Slutsky's theorem (see page 72 of [8]), to prove Theorem 5 it is only necessary to understand the limiting behavior of $T_n^{(t)}$.

Since the W_j's are independent, the Lindeberg-Feller theorem should apply. The following fact, which is the main condition needed for the Lindeberg-Feller theorem, can be verified easily by standard arguments.

LEMMA 3. *Fix an integer $0 < K < \infty$. Let $\{\omega_j\}_{j\geq 1}$ be any sequence of real numbers such that $|\omega_j| \leq K$ for all j. Then for any $\varepsilon > 0$,*

$$\sum_{j=1}^{n} E\left[\frac{\left|W_j - \frac{1}{j}\right|^2 \omega_j^2}{\log n}; \left|W_j - \frac{1}{j}\right| > \frac{\varepsilon (\log n)^{1/2}}{|\omega_j|}\right] \to 0,$$

as $n \to \infty$.

Now that the ideas have been outlined, the details can be done carefully.

PROOF OF THEOREM 5. Set

$$\omega_j = \sum_{k=1}^{m} \frac{t_k}{\left(c_2^{(k)}\right)^{1/2}} (\{j\theta_k\} - \{j\alpha_k\}). \tag{4.6}$$

It is easy to see that

$$|\omega_j| \leq \sum_{k=1}^{m} \frac{|t_k|}{\left(c_2^{(k)}\right)^{1/2}}, \tag{4.7}$$

so applying Lemma 3, the array $V_{n,j}^{(t)}$ satisfies the conditions of the Lindeberg-Feller theorem and

$$T_n^{(t)} \Rightarrow \mathscr{N}(0, \sigma^2), \tag{4.8}$$

where

$$\sigma^2 = \lim_{n\to\infty} \sum_{j=1}^{n} E\left[(V_{n,j}^{(t)})^2\right] \tag{4.9}$$

$$= \lim_{n\to\infty} \frac{1}{\log n} \sum_{j=1}^{n} \frac{1}{j} \left(\sum_{k=1}^{m} \frac{t_k}{\left(c_2^{(k)}\right)^{1/2}} (\{j\theta_k\} - \{j\alpha_k\}) \right)^2. \tag{4.10}$$

The limit can be found by applying Theorem 3:

$$\sigma^2 = \sum_{k=1}^{m} \sum_{l=1}^{m} t_k t_l D_{kl}. \tag{4.11}$$

Since

$$\left|t_1 Y_n^{I_1} + \cdots + t_m Y_n^{I_m} - T_m^{(t)}\right| \to_p 0, \tag{4.12}$$

we have

$$t_1 Y_n^{I_1} + \cdots + t_m Y_n^{I_m} \Rightarrow \mathcal{N}\left(0, \sum_{j=1}^{m} \sum_{k=1}^{m} t_j t_k D_{jk}\right), \tag{4.13}$$

as $n \to \infty$, which is the distribution of $t_1 Z_1 + \cdots + t_m Z_m$. □

Except in a few special cases, the entries of the covariance matrix D are not easy to compute. One case will be worked out in the next section, for comparison with similar results for continuous groups.

REMARK. Hambly, Keevash, O'Connell and Stark [12] prove a similar theorem for the logarithm of the characteristic polynomial of a random permutation matrix; their work gives another proof of Theorem 4 for a single interval.

REMARK. Theorem 3, Lemma 2 and Lemma 3 imply a more general result concerning linear combinations of the a_j's. Let $\{\beta\}_{j\geq 1}$ be a sequence of real numbers which satisfy $0 \leq \beta_j \leq K$ for some $K < \infty$. Suppose these numbers also satisfy

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \beta_j = A > 0, \tag{4.14}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \beta_j^2 = B > 0, \tag{4.15}$$

where A and B are both finite. Finally, set

$$\zeta_n = \sum_{j=1}^{n} \beta_j a_j. \tag{4.16}$$

Then $E[\zeta_n] = A \log n + o(\log n)$, $\mathrm{Var}(\zeta_n) = B \log n + o(\log n)$ and

$$\frac{\zeta_n - E[\zeta_n]}{(B \log n)^{1/2}} \Rightarrow \mathcal{N}(0, 1) \tag{4.17}$$

as $n \to \infty$. (A related result concerning linear combinations of cycles can be found in example D of [3].)

4.2. *Rate of convergence for a single interval.* Let $\Phi(x)$ be the standard normal distribution function. For a single interval $I = (e^{2\pi i\alpha}, e^{2\pi i\theta}]$, Y_n^I converges to normal at a rate of $(\log n)^{-1/2}$.

THEOREM 6. *There exists a constant* $h = h(\alpha, \theta)$ *such that for all* x *and* n,

$$\sup_x \left| P\left[Y_n^I \le x\right] - \Phi(x) \right| \le \frac{h}{(\log n)^{1/2}}.$$

The proof of this theorem will make use of the following fact from [5] (see page 174).

LEMMA 4. *Let* B *be a random variable and* η *be a positive constant with* $\sup_x |P(B \le x) - \Phi(x)| \le \eta$. *Suppose* A_1 *and* A_2 *are independent random variables with* $|A_1 - B| \le A_2$ *and* $E[A_2] < \infty$. *Then*

$$\sup_x \left| P(A_1 \le x) - \Phi(x) \right| \le 3\left(\eta + \frac{4E[A_2]}{(2\pi)^{1/2}}\right).$$

PROOF OF THEOREM 6. Reusing the notation from Section 4.1, let

$$V_{n,j}^{(t)} = \frac{W_j - \frac{1}{j}}{(c_2 \log n)^{1/2}}(\{j\theta\} - \{j\alpha\}) \tag{4.18}$$

and

$$T_n = \sum_{j=1}^{n} V_{n,j}. \tag{4.19}$$

Note that for each j, $E[V_j] = 0$. Also,

$$\sum_{j=1}^{n} \mathrm{Var}(V_j) = \frac{1}{c_2 \log n} \sum_{j=1}^{n} \frac{1}{j}(\{j\theta\} - \{j\alpha\})^2 \sim 1 + o(1) \tag{4.20}$$

and

$$\sum_{j=1}^{n} E(|V_j|^3) = \frac{1}{(c_2 \log n)^{3/2}} \sum_{j=1}^{n} \left(\frac{2}{j^3}e^{-1/j} + \frac{1}{j}\right)(\{j\theta\} - \{j\alpha\})^3 \tag{4.21}$$

$$\sim \frac{c_3}{((c_2)^3 \log n)^{1/2}} + o\left(\frac{1}{(\log n)^{1/2}}\right), \tag{4.22}$$

where

$$c_3 = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (\{j\theta\} - \{j\alpha\})^3. \tag{4.23}$$

Applying the Berry-Esséen Theorem (page 544 of [10]) shows that there is a constant k for which

$$\sup_x \left| P(T_n \le x) - \Phi(x) \right| \le \frac{k}{(c_2 \log n)^{1/2}}. \tag{4.24}$$

To get back to Y_n, note that

$$|Y_n - T_n| \le \sum_{j=1}^{n} \frac{|a_j - W_j|}{(c_2 \log n)^{\frac{1}{2}}} \cdot \Big|\{j\theta\} - \{j\alpha\}\Big| \tag{4.25}$$

$$\le \frac{1}{(c_2 \log n)^{1/2}} \left[2 + \sum_{j=1}^{n} W_{jn} \right]. \tag{4.26}$$

Note that Y_n and $\sum W_{jn}$ are independent. Applying Lemma 4,

$$\sup_x \Big| P(Y_n^I \le x) - \Phi(x) \Big| \le \frac{3}{(\log n)^{1/2}} \left(k + \frac{12}{(2\pi c_2)^{1/2}} \right) \tag{4.27}$$

$$= \frac{h}{(\log n)^{1/2}}. \tag{4.28}$$

□

REMARK. If α and θ are rational numbers or irrational numbers of finite type, it should be possible to use discrepancy bounds on the rates of convergence for the constants c_2 and c_3 to find explicit values for the constants k and h in Theorem 6.

5. Discussion. The symmetric group has been connected to random matrix theory for continuous ensembles in a number of ways. One connection can be found in [7], which studies the trace of a randomly chosen matrix from one of the compact Lie groups and the trace of a random permutation matrix. Another connection arises from the following question: if a permutation is chosen at random from S_n, what is the length of the longest increasing substance? This problem has a long history, but recent work by Baik, Deift, and Johansson (see [4]) has shown that the length of the longest increasing subsequence follows the same distribution as the largest eigenvalue of a random matrix chosen from the Gaussian Unitary Ensemble (when both random variables are properly scaled).

To put the results of this paper into a bigger picture, it is interesting to compare them with similar results for some of the continuous matrix groups. The easiest family of groups to work with is U_n. We begin with a short description of what is known about the eigenvalue distribution on these groups, and then compare the unitary group with the permutation group.

5.1. *Eigenvalue distribution on U_n.* Since U_n is a compact Lie group, it has a unique left-invariant probability measure. For all of the results below, this is the probability measure which is used. For any interval $I = (e^{2\pi i\alpha}, e^{2\pi i\theta})$ on the unit circle, let T_n^I be the random variable on U_n which counts the number of eigenvalues in I. The mean and variance of T_n^I are known to be

$$E[T_n^I] = n(\theta - \alpha) \tag{5.1}$$

and

$$\mathrm{Var}(T_n^I) = \frac{1}{\pi^2} \log n + O(1). \tag{5.2}$$

(The mean is easy to see by symmetry; complete asymptotics for the variance were calculated by Rains in [15].) Furthermore, the normalized random variable

$$V_n^I = \frac{T_n^I - n(\theta - \alpha)}{\frac{1}{\pi}(\log n)^{1/2}} \tag{5.3}$$

converges in distribution to a standard normal random variable as $n \to \infty$ (see Chapter 2 of [19]).

The joint behavior of $V_n^{I_1}, \dots, V_n^{I_k}$ is somewhat surprising. The theorem below is stated for two intervals for simplicity (for a proof, see Chapter 3 of [19]).

THEOREM 7. *Let I_1 and I_2 be two open arcs on the unit circle.*

1. *If I_1 and I_2 do not have a common endpoint (the intervals may overlap), then $V_n^{I_1}$ and $V_n^{I_2}$ converge in distribution to independent normal random variables with mean* 0 *and variance* 1 *as $n \to \infty$.*
2. *If $I_1 = (e^{i\alpha}, e^{i\beta})$ and $I_2 = (e^{i\beta}, e^{i\gamma})$, then as $n \to \infty$, $V_n^{I_1}$ and $V_n^{I_2}$ converge in distribution to jointly distributed normal random variables with mean* 0, *variance* 1 *and covariance* $-1/2$.
3. *If $I_1 = (e^{i\alpha}, e^{i\beta})$ and $I_2 = (e^{i\alpha}, e^{i\gamma})$, then as $n \to \infty$, $V_n^{I_1}$ and $V_n^{I_2}$ converge in distribution to jointly distributed normal random variables with mean* 0, *variance* 1 *and covariance* $1/2$.

REMARK. Similar behavior can be seen in a number of random matrix settings. See, for example [6]. Other related results can be found in [18].

5.2. *Comparison of results.* The random variable T_n^I is the analogue of X_n^I. The mean and variance asymptotics given in (5.1) and (5.2) for T_n^I are similar to the mean and variance asymptotics for X_n^I given in (3.19) and (3.22). In both cases, the first order term of the mean is proportional to the size of the interval; also, in both cases the variance grows as $\log n$, which is fairly slow. (This is a reflection of the fact that the eigenvalues of a random permutation or unitary matrix are much more evenly spaced than n independent randomly chosen points would be.) One striking difference between the two cases is that the coefficient of $\log n$ in the variance does not depend on the interval for the unitary group, while it does for the permutation group. While this may be surprising at first, since the eigenvalues of permutation matrices are always rational multiples of 2π, there can be a significant contribution to X_n^I at the endpoints of the interval.

The limiting normality in the single-variable case holds for both groups (for permutation matrices, this follows from Theorem 4, with $m = 1$). The interesting question is whether the behavior exhibited in Theorem 7 can also be seen for permutation matrices. In general, the behavior is not the same. For example, if $I_1 = (1, i]$ and $I_2 = (-1, -i]$, then $Y_n^{I_1}$ and $Y_n^{I_2}$ are positively correlated, even though there is no common endpoint. This is not too surprising, because of the contribution of eigenvalues at the endpoints. However, under special circumstances these effects can be minimized and the behavior which appears does match the behavior random unitary matrices.

THEOREM 8. *Let α, β, γ, and δ be irrational numbers which are linearly independent over the rational numbers.*

1. *If $I_1 = (e^{2\pi i\alpha}, e^{2\pi i\beta}]$ and $I_2 = (e^{2\pi i\gamma}, e^{2\pi i\delta}]$ do not have a common endpoint (the intervals may overlap), then $Y_n^{I_1}$ and $Y_n^{I_2}$ converge in distribution to independent normal random variables with mean 0 and variance 1, as $n \to \infty$.*
2. *If $I_1 = (e^{i\alpha}, e^{i\beta}]$ and $I_2 = (e^{i\beta}, e^{i\gamma}]$, then as $n \to \infty$, $Y_n^{I_1}$ and $Y_n^{I_2}$ converge in distribution to jointly distributed normal random variables with mean 0, variance 1 and covariance $-1/2$.*
3. *If $I_1 = (e^{i\alpha}, e^{i\beta}]$ and $I_2 = (e^{i\alpha}, e^{i\gamma}]$, then as $n \to \infty$, $Y_n^{I_1}$ and $Y_n^{I_2}$ converge in distribution to jointly distributed normal random variables with mean 0, variance 1 and covariance $1/2$.*

PROOF. All three results will follow from Theorem 4, by evaluation of the covariance constants D_{12}. Because of the assumptions on α, β, γ, and δ, in all cases, the mean and variance constants were mentioned in Example 2: $c_1^{(1)} = c_1^{(2)} = 0$ and $c_2^{(1)} = c_2^{(2)} = 1/6$. The covariance constants in the three parts of the theorem can be calculated from formulas in the Appendix [see equation (A.10) and Corollary 1]. □

Theorem 8 can be extended (under appropriate conditions) for any number of intervals. The covariance structure will be the same, provided that all endpoints are linearly independent irrational numbers. Though the general case is not nearly so nice, the intervals in Theorem 8 are fairly typical in the sense that if four endpoints are chosen at random from the unit circle, they will be irrational and linearly independent with probability 1.

The results in this paper still leave many questions unanswered. One subject of much interest in the mathematical physics setting is the distribution of the spacing between the eigenvalues. While this has been studied for many of the continuous matrix groups and ensembles, this question has not been studied for the permutation matrices. Another question, which is closely related to the work in this paper, is whether there is another limiting process which is buried under the structure which appears in Theorems 7 and 8. Both questions could be quite useful for understanding how permutations fit into the random matrix picture.

APPENDIX: CALCULATION OF CONSTANTS

The mean, variance, and covariance constants derived in this paper are

$$c_1 = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (\{j\theta\} - \{j\alpha\}), \tag{A.1}$$

$$c_2 = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (\{j\theta\} - \{j\alpha\})^2 \tag{A.2}$$

and

$$D_{12} = \frac{1}{\left(c_2^{(1)} c_2^{(2)}\right)^{1/2}} \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (\{j\theta_1\} - \{j\alpha_1\})(\{j\theta_2\} - \{j\alpha_2\}). \tag{A.3}$$

These constants can be easy or difficult to compute depending on the values of α and θ. When separated into pieces, these sums contain three basic elements:

$$s_1(\beta) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \{j\beta\}, \tag{A.4}$$

$$s_2(\beta) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \{j\beta\}^2 \tag{A.5}$$

and

$$s_3(\beta, \gamma) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \{j\beta\}\{j\gamma\}. \tag{A.6}$$

These three limits are the building blocks for the examples in Section 3 and for Theorem 8.

This Appendix will be spent calculating s_1, s_2, and s_3 in as many cases as possible.

A.1. *The single-variable constants, s_1 and s_2.* First suppose β is rational, say $\beta = p/q$. In this case, the numbers $\{j\beta\}$ cycle through some rearrangement of the numbers $1/q, 2/q, \dots, (q-1)/q, 0$, and the numbers $\{j\beta\}^2$ cycle through some rearrangement of the numbers $1/q^2, 4/q^2, \dots, (q-1)^2/q^2, 0$. Thus

$$s_1(\beta) = \frac{1}{q} \sum_{j=0}^{q-1} \frac{j}{q} = \frac{q-1}{2q} \tag{A.7}$$

and

$$s_2(\beta) = \frac{1}{q} \sum_{j=0}^{q-1} \frac{j^2}{q^2} = \frac{(q-1)(2q-1)}{6q^2}. \tag{A.8}$$

Next, if β is irrational then

$$s_1(\beta) = 1/2, \tag{A.9}$$

$$s_2(\beta) = 1/3. \tag{A.10}$$

This can be shown using a classical number theory result, essentially due to Weyl:

THEOREM 9. *Let f be any Riemann integrable function on* $[0,1]$, *and let* θ *be any irrational number. Then for any real number* b,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(\{k\theta+b\}) = \int_0^1 f(x)dx.$$

PROOF. If $b=0$ (or any other integer), this follows directly from Theorem 1.1, page 2 and Example 2.1, page 8 of [13]; if $b\neq 0$, a simple modification of Example 2.1 is required. □

REMARK. Before looking at s_3, it is worth mentioning the rates of convergence of the limits in s_1 and s_2. First, it is relatively easy to see that when $\beta = p/q$,

$$\left|\frac{1}{n}\sum_{j=1}^{n}\{j\beta\} - \frac{q-1}{2q}\right| \le \frac{q}{n} \tag{A.11}$$

and

$$\left|\frac{1}{n}\sum_{j=1}^{n}\{j\beta\}^2 - \frac{(q-1)(2q-1)}{6q^2}\right| \le \frac{q}{n}. \tag{A.12}$$

Also, using the notion of discrepancy and several good number theory estimates, it can be shown that for almost every irrational β, there are finite constants $K_1(\beta)$ and $K_2(\beta)$, and $0 < b(\beta) \le 1$ for which

$$\left|\frac{1}{n}\sum_{j=1}^{n}\{j\beta\} - \frac{1}{2}\right| \le \frac{K_1}{n^b} \tag{A.13}$$

and

$$\left|\frac{1}{n}\sum_{j=1}^{n}\{j\beta\}^2 - \frac{1}{3}\right| \le \frac{K_2}{n^b}, \tag{A.14}$$

for all n. Here is why these bounds are relevant. The numbers s_1 and s_2 arise from sums of the form $\sum_{j=1}^{n}\{j\beta\}/j$ and $\sum_{j=1}^{n}\{j\beta\}^2/j$. By combining the estimates above with part 2 of Theorem 3, these sums have asymptotics

$$\sum_{j=1}^{n}\frac{\{j\beta\}}{j} = s_1(\beta)\log n + O(1), \tag{A.15}$$

$$\sum_{j=1}^{n}\frac{\{j\beta\}^2}{j} = s_2(\beta)\log n + O(1) \tag{A.16}$$

for all rational β and almost every irrational β. The details of these estimates and asymptotics can be found in section 5.4 of [19].

A.2. *The two-variable constant s_3.* For any β and γ, $s_3(\beta, \gamma)$ can be calculated numerically to any desired degree of accuracy. However, a closed formula can be found for only some cases. These cases are summarized below.

We first mention the situation when both $\beta = p/q$ and $\gamma = r/s$ are rational. There is no single closed formula which covers all of the cases, but s_3 can always be computed exactly as the average

$$s_3(\beta, \gamma) = \frac{1}{qs} \sum_{j=1}^{qs} \left\{ \frac{jp}{q} \right\} \left\{ \frac{jr}{s} \right\}. \tag{A.17}$$

The easiest case which has a simple answer is when β and γ are irrational numbers which are linearly independent over $\mathbb{Q}$. In this case, there is a two-dimensional analogue of Theorem 9.

THEOREM 10. *Let f and g be any Riemann integrable functions on $[0, 1]$, and let β and γ be any irrational numbers which are linearly independent over the rational numbers. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\{k\beta\}) g(\{k\gamma\}) = \left(\int_0^1 f(x)dx \right) \left(\int_0^1 g(y)dy \right).$$

PROOF. See Theorem 6.1 and Example 6.1 on page 48 of [13]. □

Using Theorem 10 with $f(x) = x$ and $g(y) = y$ proves the following.

COROLLARY 1. *If β and γ are linearly independent irrational numbers,*

$$s_3(\beta, \gamma) = \frac{1}{4}.$$

Another straightforward case is if one of the numbers is rational and the other is irrational:

THEOREM 11. *Suppose $\beta = p/q$ and suppose γ is irrational. Then*

$$s_3(\beta, \gamma) = \frac{1}{4} - \frac{1}{4q}.$$

PROOF. For $k = 1, 2, \ldots, q$, set $\alpha_k = \{kp/q\}$. Rewrite the sum in s_3 as

$$\begin{aligned} s_3 = \lim_{n\to\infty} \Biggl(& \frac{1}{q\lfloor \frac{n}{q} \rfloor} \sum_{j=0}^{\lfloor \frac{n}{q} \rfloor} \sum_{k=1}^{q} \alpha_k \{(qj + k)\gamma\} \\ & + \left(\frac{1}{n} - \frac{1}{q\lfloor \frac{n}{q} \rfloor} \right) \sum_{j=0}^{\lfloor \frac{n}{q} \rfloor} \sum_{k=1}^{q} \alpha_k \{(qj + k)\gamma\} \\ & + \frac{1}{n} \sum_{j=\lfloor \frac{n}{q} \rfloor + 1}^{n} \{j\gamma\}\{j\beta\} \Biggr). \end{aligned} \tag{A.18}$$

The second and third terms are bounded by q/n, so both terms disappear in the limit. Thus

$$s_3 = \frac{1}{q}\sum_{k=1}^{q}\alpha_k\left(\lim_{n\to\infty}\frac{1}{q\lfloor\frac{n}{q}\rfloor}\sum_{j=0}^{\lfloor\frac{n}{q}\rfloor}\sum_{k=1}^{q}\{(qj+k)\gamma\}\right). \tag{A.19}$$

Now apply Theorem 9 to each of the limits, using $\theta = q\gamma$, $b = k\gamma$, and $f(x) = x$. This shows that

$$s_3(\beta,\gamma) = \frac{1}{2q}\sum_{k=1}^{q}\left\{\frac{kp}{q}\right\} = \frac{q-1}{4q}. \tag{A.20}$$

□

Finally, if β and γ are irrational numbers which are linearly dependent over $\mathbb{Q}$, then there are integers p, q, r, and s (with p and q relatively prime, r and s relatively prime, and q and s positive) so that

$$\gamma = \frac{r}{s}\beta + \frac{p}{q}. \tag{A.21}$$

Although the closed form of $s_3(\beta,\gamma)$ will not be given in general, one can hope that some intuition can be taken from the linearly independent case in Corollary 1. A reasonable guess is that s_3 is approximately 1/4 with a correction term which will depend on p, q, r, and s. The following theorem, which covers the cases with $s = 1$, supports this guess.

THEOREM 12. *Suppose that β is irrational and $\gamma = r\beta + p/q$. Then*

$$s_3(\beta,\gamma) = \frac{1}{4} + \frac{1}{12rq^2}.$$

PROOF. The constant r will be assumed to be positive. (Since s_3 deals only with fractional parts $\{j\gamma\}$, if r is negative, γ can be replaced with $\tilde{\gamma} = |r|(1-\beta) + p/q$. It is easy to check that $\{j\gamma\} = \{j\tilde{\gamma}\}$.)

First, consider the easier case $s_3(\beta, r\beta)$. Note that

$$\{jr\beta\} = r\{j\beta\} - \sum_{k=1}^{r-1} 1_{\{j\beta\}\geq k/r}. \tag{A.22}$$

Substituting this into the formula for s_3 shows that

$$s_3(\beta, r\beta) = \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\left(r\{j\beta\}^2 - \sum_{k=1}^{r-1}\{j\beta\}1_{\{j\beta\}\geq k/r}\right) \tag{A.23}$$

$$= \frac{r}{3} - \sum_{k=1}^{r-1}\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\{j\beta\}1_{\{j\beta\}\geq k/r}. \tag{A.24}$$

For each of the limits above, apply part 1 of Theorem 9 using

$$f(x) = \begin{cases} x, & \text{if } x \geq k/r, \\ 0, & \text{otherwise.} \end{cases} \tag{A.25}$$

Then

$$s_3(\beta, r\beta) = \frac{r}{3} - \sum_{k=1}^{r-1} \left(\frac{1}{2} - \frac{k^2}{2r^2} \right) \tag{A.26}$$

$$= \frac{1}{4} + \frac{1}{12r}. \tag{A.27}$$

In the general case,

$$s_3(\beta, \gamma) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left\{ jr\beta + \frac{jp}{q} \right\} \{j\beta\}. \tag{A.28}$$

Start by writing

$$\left\{ jr\beta + \frac{jp}{q} \right\} = \{jr\beta\} + \left\{ \frac{jp}{q} \right\} - 1_{\{jr\beta\} \geq 1 - \{jp/q\}}. \tag{A.29}$$

Then

$$s_3 = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \{jr\beta\}\{j\beta\} + \left\{ \frac{jp}{q} \right\} \{j\beta\} \tag{A.30}$$

$$- \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \{j\beta\} 1_{\{jr\beta\} \geq 1 - \{jp/q\}}$$

$$= \frac{1}{2} + \frac{1}{12r} - \frac{1}{4q} - \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \{j\beta\} 1_{\{jr\beta\} \geq 1 - \{jp/q\}}, \tag{A.31}$$

where the last line follows from (A.27) and Theorem 11. The remaining problem is to evaluate the limit

$$L = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \{j\beta\} 1_{\{jr\beta\} \geq 1 - \{jp/q\}}. \tag{A.32}$$

For $k = 1, 2, \ldots, q$, set $\alpha_k = \{kp/q\}$. So

$$\begin{aligned} L = \lim_{n\to\infty} \Bigg(& \frac{1}{q\lfloor \frac{n}{q} \rfloor} \sum_{j=0}^{\lfloor \frac{n}{q} \rfloor} \sum_{k=1}^{q} \{(qj+k)\beta\} 1_{\{jr\beta\} \geq 1 - \alpha_k} \\ & + \left(\frac{1}{n} - \frac{1}{q\lfloor \frac{n}{q} \rfloor} \right) \sum_{j=0}^{\lfloor \frac{n}{q} \rfloor} \sum_{k=1}^{q} \{(qj+k)\beta\} 1_{\{jr\beta\} \geq 1 - \alpha_k} \\ & + \frac{1}{n} \sum_{j=\lfloor \frac{n}{q} \rfloor + 1}^{n} \{j\beta\} 1_{\{jr\beta\} \geq 1 - \{jp/q\}} \Bigg). \end{aligned} \tag{A.33}$$

As in the proof of Theorem 11, the second and third limits are zero and

$$L = \sum_{k=1}^{q} \lim_{n\to\infty} \frac{1}{q\lfloor \frac{n}{q} \rfloor} \sum_{j=0}^{\lfloor \frac{n}{q} \rfloor} \sum_{k=1}^{q} \{(qj+k)\beta\} 1_{\{jr\beta\} \geq 1-\alpha_k}. \tag{A.34}$$

For each k, let

$$L_k = \lim_{n\to\infty} \frac{1}{q\lfloor \frac{n}{q} \rfloor} \sum_{j=0}^{\lfloor \frac{n}{q} \rfloor} \sum_{k=1}^{q} \{(qj+k)\beta\} 1_{\{jr\beta\} \geq 1-\alpha_k}. \tag{A.35}$$

Theorem 9 can be applied to each of these, using $\theta = q\beta$, $b = k\beta$. Using equation (A.22),

$$L_k = \int_0^1 f_k(x)dx, \tag{A.36}$$

where

$$f_k(x) = \begin{cases} x, & \text{if } rx - \sum_{\ell=1}^{r-1} 1_{x \geq \frac{\ell}{r}} \geq 1-\alpha_k, \\ 0, & \text{otherwise.} \end{cases} \tag{A.37}$$

Thus

$$L_k = \sum_{\ell=0}^{r-1} \int_{\frac{1-\alpha_k+\ell}{r}}^{\frac{\ell+1}{r}} x dx \tag{A.38}$$

$$= \frac{1}{2} - \frac{r-1}{2r}(1-\alpha_k) - \frac{(1-\alpha_k)^2}{2r}. \tag{A.39}$$

This means that

$$L = \frac{1}{q} \sum_{k=1}^{q} \frac{1}{2} - \frac{r-1}{2r}(1-\alpha_k) - \frac{(1-\alpha_k)^2}{2r}. \tag{A.40}$$

Using the fact that the α_k's cycle through $0, \ldots, (q-1)/q$,

$$L = \frac{1}{q} - \frac{r-1}{2rq} \sum_{k=1}^{q} \frac{k}{q} - \frac{1}{2rq} \sum_{k=1}^{q} \frac{k^2}{q^2} \tag{A.41}$$

$$= \frac{1}{4} - \frac{1}{4q} + \frac{1}{12r} - \frac{1}{12rq^2}. \tag{A.42}$$

Finally putting this into (A.31),

$$s_3(\beta, \gamma) = \frac{1}{4} + \frac{1}{12rq^2}. \tag{A.43}$$

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