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THE MAXIMUM OF A BRANCHING RANDOM WALK WITH SEMIEXPONENTIAL INCREMENTS

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We consider an infinite Galton–Watson tree Γ and label the vertices v with a collection of i.i.d. random variables $(Y_v)_{v\in\Gamma}$. In the case where the upper tail of the distribution of Y_v is semiexponential, we then determine the speed of the corresponding tree-indexed random walk. In contrast to the classical case where the random variables Y_v have finite exponential moments, the normalization in the definition of the speed depends on the distribution of Y_v. Interpreting the random variables Y_v as displacements of the offspring from the parent, $(Y_v)_{v\in\Gamma}$ describes a branching random walk. The result on the speed gives a limit theorem for the maximum of the branching random walk, that is, for the position of the rightmost particle. In our case, this maximum grows faster than linear in time.

1. Introduction and statement of the result. Let Γ be an infinite tree with vertices v and with a distinguished vertex ρ called the root. Let $(X_v)_{v\in\Gamma}$ be a collection of i.i.d. random variables. Given Γ and the collection $(X_v)_{v\in\Gamma}$, we define the tree-indexed random walk $(S_v)_{v\in\Gamma}$ by $S_v := \sum_{w\le v} X_w$ where $w \le v$ means that v is a descendant of w (i.e., w is on the shortest path from ρ to v.) We denote the rays of the tree by ξ (rays are infinite paths from ρ to which do not backstep) and the set of rays by $\partial\Gamma$. For a vertex v, $|v|$ denotes the distance of v to the root, that is, the number of edges on the shortest path from ρ to v. There are different ways to define a speed for the random walk $(S_v)_{v\in\Gamma}$. In [2], [9] and [11], the following notions of speed were considered:

$$\text{Cloud speed: } s_{\text{cloud}} := \limsup_{n\to\infty} \frac{1}{n} \max_{v:|v|=n} S_v,$$

$$\text{Burst speed: } s_{\text{burst}} := \sup_{\xi\in\partial\Gamma} \limsup_{v\in\xi} \frac{S_v}{|v|},$$

$$\text{Sustainable speed: } s_{\text{sust}} := \sup_{\xi\in\partial\Gamma} \liminf_{v\in\xi} \frac{S_v}{|v|}.$$

One has always

$$s_{\text{sust}} \le s_{\text{burst}} \le s_{\text{cloud}}. \tag{1}$$

The inequalities may be strict in general; we refer to [11] for examples. However, it was shown that for Galton–Watson trees, these speeds coincide. A Galton–Watson tree is defined as follows: let Z be a random variable with

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values in $\mathbb{N}_0$. Consider the branching process (Z_n) with $Z_0 = 1$ and $Z_{n+1} = \sum_{i=1}^{Z_n} Z_i^{(n)}$ where $Z_1^{(n)}, Z_2^{(n)}, \dots$ are i.i.d. random variables with the same distribution as Z and draw edges between the offspring and their parent, the first ancestor being the root.

Let $m := E[Z]$. Assume

$$X_v \text{ is not a.s. constant, } E\big[X_v\big] = 0 \quad \text{and} \quad E\big[\exp(\lambda X_v)\big] < \infty \text{ for all } \lambda \geq 0. \tag{2}$$

Let $\widetilde{S}_n := \sum_{i=1}^{n} X_i$ be the sum of i.i.d. random variables X_i with the same distribution as X_v, and let $I(a)$ be the corresponding rate function defined by

$$I(a) := -\lim \frac{1}{n} \log P[\widetilde{S}_n > na]. \tag{3}$$

Let $s^* := \sup\{s | I(s) \leq \log m\}$.

THEOREM 1 (Hammersley [7], Kingman [8], Biggins [3]). *Let Γ be a Galton–Watson tree with mean $m > 1$. Suppose the vertices of Γ are labeled by i.i.d. random variables (X_v) satisfying* (2). *On the event that Γ survives, we have a.s.,*

$$s_{\text{sust}} = s_{\text{burst}} = s_{\text{cloud}} = s^*. \tag{4}$$

We refer to [11] for the proof.

REMARKS. (i) Interpreting the random variables (X_v) as displacements of the offspring from the parent, the tree-indexed random walk is a branching random walk, and Theorem 1 says that, under the hypothesis 2, the position of the rightmost particle moves linearly in time, with speed s^*.

(ii) For connections to first passage percolation, we refer to [2].

We will here consider one of the cases where (2) is not satisfied. Let $(Y_v)_{v\in\Gamma}$ be a collection of i.i.d. random variables with finite expectations. We will assume that the distribution of Y_v has the following semiexponential upper tail. Let $r \in]0, 1[$. Assume

$$P[Y_v \geq t] = a(t)\exp(-L(t)t^r) \tag{5}$$

for t large enough, where a and L are slowly varying functions such that $L(t)/t^{1-r}$ is nonincreasing for t large enough. Let ψ be a positive function such that

$$\frac{L(\psi(n))\psi(n)^r}{n} \xrightarrow[n\to\infty]{} 1. \tag{6}$$

Let

$$S^{\psi}_{\text{cloud}} := \limsup_{n\to\infty} \frac{1}{\psi(n)} \max_{v:|v|=n} S_v,$$

$$S^{\psi}_{\text{burst}} := \sup_{\xi\in\partial\Gamma} \limsup_{v\in\xi} \frac{S_v}{\psi(|v|)},$$

$$S^{\psi}_{\text{sust}} := \sup_{\xi\in\partial\Gamma} \liminf_{v\in\xi} \frac{S_v}{\psi(|v|)}.$$

EXAMPLES.

1. $L(t) \equiv b$. Then we can take $\psi(n) = (1/b^{1/r})n^{1/r}$.
2. $L(t) = \log t$. Then we can take $\psi(n) = (rn/\log n)^{1/r}$.
3. $L(t) = \log\log t$. Then we can take $\psi(n) = (rn/\log\log n)^{1/r}$.

We will prove the following.

THEOREM 2. *Let Γ be a Galton–Watson tree with mean $m > 1$. Suppose the vertices of Γ are labeled by i.i.d. random variables (Y_v) satisfying* (5), *and ψ satisfies* (6). *On the event that Γ survives, we have a.s.*,

$$s^{\psi}_{\text{sust}} = s^{\psi}_{\text{burst}} = s^{\psi}_{\text{cloud}} = (\log m)^{1/r}. \tag{7}$$

REMARKS. (i) Exactly as for Theorem 1, there is an interpretation in terms of branching random walk: under the hypothesis (5), the position of the rightmost particle moves faster than linear in time, namely like $\psi(n)(\log m)^{1/r}$at time n. In the case where Z has a distribution with a regularly varying tail, a limit theorem for the distribution of the position of the rightmost particle has been proved in [4].

(ii) Let $M_n := \max_{v:|v|=n} S_v$ be the position of the rightmost particle in the branching random walk. We can interpret (4) as well as (7) by saying that the growth of M_n is the same as if the rays of the tree were independent. Differences between a collection of independent rays and the rays of the tree are expected to appear if one investigates the fluctuations of M_n. In the case where Z has a distribution with a regularly varying tail, this was analyzed in [5].

(iii) Note that the limit value $(\log m)^{1/r}$ only depends on r whereas the function ψ, which gives the normalization, also depends on L. A similar phenomenon occurs if one considers functional Erdős–Renyi laws for random variables satisfying (5): the normalization that is, in this case the "size of the window," depends on L but the limit set only depends on r; we refer to [6]. The reason one has to normalize with $\psi(n)$ instead of n is the bigger influence of extreme values; roughly speaking, the tail of the sum of i.i.d. random variables Y_v is given by the tail of the maximum.

(iv) Except that we assumed that Y_v has a finite expectation, we did not make requirements on the lower tail of the distribution of Y_v, that is, on the decay of $P[Y_v \le t]$ for $t \to -\infty$.

EXAMPLE. Let Γ be a binary tree and $(X_v)_{v\in\Gamma}$ be an i.i.d. collection of random variables with distribution $N(0,1)$. Due to Theorem 1, we have a.s.,

$$\sup_{\xi\in\partial\Gamma}\lim_{v\in\xi}\frac{S_v}{|v|}=\lim_{n\to\infty}\frac{1}{n}\max_{v:|v|=n}S_v=(2\log 2)^{1/2}. \tag{8}$$

Let $\beta>2$ and $S_v^{\beta}:=\sum_{u\le v}X_u^{\beta}$. Due to Theorem 2 with $Y_v=X_v^{\beta}$, $r=2/\beta$ and $\psi(n)=2^{\beta/2}n^{\beta/2}$, we have a.s.,

$$\sup_{\xi\in\partial\Gamma}\lim_{v\in\xi}\frac{S_v^{\beta}}{|v|^{\beta/2}}=\lim_{n\to\infty}\frac{1}{n^{\beta/2}}\max_{v:|v|=n}S_v^{(\beta)}=(2\log 2)^{\beta/2}. \tag{9}$$

2. Proof of Theorem 2. The key to the proof of Theorem 2 is provided by the following logarithmic tail asymptotics for i.i.d. random variables with a semiexponential upper tail as in (5).

THEOREM 3. *Let $Y, Y_1,\dots,Y_n$ be i.i.d. random variables with finite expectation $E[Y]$. Let $r\in]0,1[$. Assume*

$$P[Y\ge t]=a(t)\exp(-L(t)t^r) \tag{10}$$

for t large enough, where a and L are slowly varying functions such that $L(t)/t^{1-r}$ is nonincreasing for t large enough. Let ψ be a positive function satisfying (6). *Let $S_n:=\sum_{i=1}^n Y_i$. Then we have, for $x>0$,*

$$\lim_{n\to\infty}\frac{1}{n}\log P[S_n\ge\psi(n)x]=-x^r. \tag{11}$$

Under stronger assumptions on L and a, one can give much more precise expansions instead of logarithmic asymptotics. This was carried out by A. V. Nagaev; we refer to the survey paper [10] by S.V. Nagaev, page 764.

PROOF.
Lower bound. Let $\varepsilon>0$. Since $Y_1,\dots,Y_n$ are independent, we have

$$P[S_n\ge\psi(n)x]\ge P[Y_1\ge\psi(n)(x+\varepsilon)]P\left[\sum_{i=2}^n Y_i\ge\psi(n)(-\varepsilon)\right]. \tag{12}$$

Since $\psi(n)/n\to\infty$, the second term on the r.h.s. of (12) goes to 1 due to the law of large numbers. For the first term on the r.h.s. of (12), we have

$$P[Y_1\ge\psi(n)(x+\varepsilon)]=a(\psi(n)(x+\varepsilon))\exp(-L(\psi(n)(x+\varepsilon))\psi(n)^r(x+\varepsilon)^r).$$

Taking logarithms, dividing by n and letting $n\to\infty$ yields, due to (6),

$$\liminf_n\frac{1}{n}\log P[S_n\ge\psi(n)x]\ge-(x+\varepsilon)^r. \tag{13}$$

For $\varepsilon\to 0$, the lower bound in (11) follows.

Upper bound. We can assume w.l.o.g. (replacing Y with Y^+) that Y is nonnegative. Note that

$$P[S_n \geq \psi(n)x] \leq P\Big[\max_{1\leq i\leq n} Y_i \geq \psi(n)x\Big] + P\Big[\sum_{i=1}^n Y_i^{(n)} \geq \psi(n)x\Big], \tag{14}$$

where $Y_i^{(n)} := Y_i I_{\{Y_i \leq \psi(n)x\}}$. For the first term on the r.h.s. of (14), we have

$$P\Big[\max_{1\leq i\leq n} Y_i \geq \psi(n)x\Big] \leq n a(\psi(n)x)\exp(-L(\psi(n)x)\psi(n)^r x^r), \tag{15}$$

hence

$$\limsup_n \frac{1}{n}\log P\Big[\max_{1\leq i\leq n} Y_i \geq \psi(n)x\Big] \leq -x^r. \tag{16}$$

For the second term on the r.h.s. of (14), apply Chebyshev's inequality with $c > 0$ to get

$$\begin{aligned} P\Big[\sum_{i=1}^n Y_i^{(n)} \geq \psi(n)x\Big] &\leq E\Big[\exp\Big(c\frac{L(\psi(n))}{\psi(n)^{1-r}}Y^{(n)}\Big)\Big]^n \\ &\quad \times \exp(-cL(\psi(n))\psi(n)^r x), \end{aligned} \tag{17}$$

where $Y^{(n)} := YI_{\{Y\leq \psi(n)x\}}$. We will show that for $c < x^{r-1}$,

$$\limsup_n \log E\Big[\exp\Big(c\frac{L(\psi(n))}{\psi(n)^{1-r}}Y^{(n)}\Big)\Big] \leq 0. \tag{18}$$

Equations (6), (14), (16) and (17) then imply

$$\limsup_n \frac{1}{n}\log P[S_n \geq \psi(n)x] \leq -x^r. \tag{19}$$

PROOF OF (18). Let $k \in \mathbb{N}$ be such that $k > r/(1-r)$. Using the estimates $\log x \leq x - 1$ and $e^x - 1 \leq x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots + (1/(k+1)!)x^{k+1}e^x$, we have

$$\begin{aligned} &\log E\Big[\exp\Big(c\frac{L(\psi(n))}{\psi(n)^{1-r}}Y^{(n)}\Big)\Big] \\ &\quad \leq c\frac{L(\psi(n))}{\psi(n)^{1-r}}E[Y^{(n)}] \\ &\qquad + \frac{1}{2}c^2\frac{L(\psi(n))^2}{\psi(n)^{2(1-r)}}E[(Y^{(n)})^2] + \frac{1}{6}c^3\frac{L(\psi(n))^3}{\psi(n)^{3(1-r)}}E[(Y^{(n)})^3] + \cdots \\ &\qquad + \frac{1}{(k+1)!}c^{k+1}\frac{L(\psi(n))^{k+1}}{\psi(n)^{(k+1)(1-r)}}E\Big[(Y^{(n)})^{k+1}\exp\Big(c\frac{L(\psi(n))}{\psi(n)^{1-r}}Y^{(n)}\Big)\Big] \end{aligned} \tag{20}$$

Since we assumed Y to be nonnegative, and because of (10), we have that for each m, $\limsup_n E[(Y^{(n)})^m] < \infty$, and we see from (20) that it suffices to show that, for $c < x^{r-1}$,

$$\limsup_n \frac{L(\psi(n))^{k+1}}{\psi(n)^{(k+1)(1-r)}} E\left[(Y^{(n)})^{k+1} \exp\left(c\frac{L(\psi(n))}{\psi(n)^{1-r}} Y^{(n)}\right)\right] \leq 0. \tag{21}$$

Since

$$\frac{L(\psi(n))^{k+1}}{\psi(n)^{(k+1)(1-r)-1}} \xrightarrow[n\to\infty]{} 0 \tag{22}$$

due to our assumption on k, it is enough to show that, for $c < x^{r-1}$,

$$\limsup_n \frac{1}{\psi(n)} E\left[(Y^{(n)})^{k+1} \exp\left(c\frac{L(\psi(n))}{\psi(n)^{1-r}} Y^{(n)}\right)\right] < \infty. \tag{23}$$

Fix $\varepsilon > 0$. Due to the Cauchy–Schwarz inequality,

$$\begin{aligned} &E\left[(Y^{(n)})^{k+1} \exp\left(c\frac{L(\psi(n))}{\psi(n)^{1-r}} Y^{(n)}\right)\right] \\ &\quad \leq E\left[(Y^{(n)})^{(k+1)(1+\varepsilon)/\varepsilon}\right]^{\varepsilon/1+\varepsilon} E\left[\exp\left((1+\varepsilon)c\frac{L(\psi(n))}{\psi(n)^{1-r}} Y^{(n)}\right)\right]^{1/1+\varepsilon} \end{aligned} \tag{24}$$

The first term on the r.h.s. of (24) remains bounded for $n \to \infty$. To prove (23), it therefore suffices to show that for $c < x^{r-1}$, we have, for each $T > 0$,

$$\limsup_n \frac{1}{\psi(n)} E\left[\exp\left((1+\varepsilon)c\frac{L(\psi(n))}{\psi(n)^{1-r}} Y^{(n)}\right); Y^{(n)} \geq T\right] < \infty. \tag{25}$$

Now we use the fact that for a nonnegative random variable X and $t > 0$, $0 < T < K < \infty$, we have

$$\begin{aligned} &E\left[\exp(tXI_{\{x\leq K\}})I_{\{X\geq T\}}\right] \\ &\quad = \int_T^K te^{ts} P[X > s]\, ds + e^{tT} P[X \geq T] - (e^{tK} - 1)P[X > K]. \end{aligned} \tag{26}$$

Plugging in $K = \psi(n)x$ and $t = (1+\varepsilon)c(L(\psi(n))/\psi(n)^{1-r})$, we get

$$\begin{aligned} &\frac{1}{\psi(n)} E\left[\exp\left((1+\varepsilon)c\frac{L(\psi(n))}{\psi(n)^{1-r}} Y^{(n)}\right); Y^{(n)} \geq T\right] \\ &\leq \frac{1}{\psi(n)} \int_T^{\psi(n)x} (1+\varepsilon)c\frac{L(\psi(n))}{\psi(n)^{1-r}} \exp\left((1+\varepsilon)c\frac{L(\psi(n))}{\psi(n)^{1-r}} s\right) P\left[Y^{(n)} > s\right] ds \\ &\quad + \frac{1}{\psi(n)} \exp\left((1+\varepsilon)c\frac{L(\psi(n))}{\psi(n)^{1-r}} T\right) \end{aligned} \tag{27}$$

The second term on the r.h.s. of (27) goes to 0 for $n \to \infty$. The first term on the r.h.s. of (27) is, for n large enough,

$$\begin{aligned} &\le \frac{1}{\psi(n)} \int_T^{\psi(n)x} (1+2\varepsilon) c \exp\left((1+\varepsilon) c \frac{L(\psi(n))}{\psi(n)^{1-r}} s\right) a(s) \exp(-L(s)s^r)\, ds \\ &= C_1 \int_{T/\psi(n)x}^1 \exp\left((1+\varepsilon) c \frac{L(\psi(n))}{\psi(n)^{1-r}} \psi(n)xs\right) a(\psi(n)xs) \\ &\qquad \times \exp(-L(\psi(n)xs)\psi(n)^r x^r s^r)\, ds \qquad (28) \\ &= C_1 \int_{T/\psi(n)x}^1 a(\psi(n)xs) \\ &\qquad \times \exp\big(\psi(n)^r \big[(1+\varepsilon) c L(\psi(n)) xs - L(\psi(n)xs) x^r s^r\big]\big)\, ds, \end{aligned}$$

where $C_1 = (1+2\varepsilon)cx$. Due to our assumptions on L, we now see that if $c(1+\varepsilon) < x^{r-1}$, the integral remains bounded for $n \to \infty$, and this proves (23). Finally, since $\varepsilon > 0$ was arbitrary, we conclude that (18) holds for $c < x^{r-1}$.

REMARK. We see from the proof that it suffices to have

$$a_1(t)\exp(-L_1(t)t^r) \le P[Y \ge t] \le a_2(t)\exp(-L_2(t)t^r) \qquad (29)$$

with a_1, a_2, L_1, L_2 as in Theorem 3.

We now follow the strategy in [11]. Let $\widetilde{S}_n := \sum_{i=1}^n Y_i$ where $Y_i, 1 \le i \le n$, are i.i.d. with the same distribution as Y_v.

Step 1. Exactly as in [11], we show that a.s. on nonextinction of Γ,

$$s_{\text{cloud}}^{\psi} \le (\log m)^{1/r}. \qquad (30)$$

Let q be the probability of extinction of Γ. We denote by Γ_n the vertices of Γ at level n. Due to Theorem 3, for each $\varepsilon > 0$, there is $\delta > 0$ such that, for n large enough,

$$P\big[\widetilde{S}_n > \psi(n)(\log m + \varepsilon)^{1/r}\big] \le \exp(-n(\log m + \delta)) = m^{-n}\exp(-n\delta). \qquad (31)$$

We have

$$\begin{aligned} &P\big[S_v > \psi(n)(\log m + \varepsilon)^{1/r} \text{ for some } v \in \Gamma_n \mid \text{nonextinction}\big] \\ &\quad = \frac{1}{1-q} \sum_{k=1}^{\infty} P\left[S_v > \psi(n)(\log m + \varepsilon)^{1/r} \text{ for some } v \in \Gamma_n \big| |\Gamma_n| = k\right] \\ &\qquad\qquad \times P[|\Gamma_n| = k] \qquad (32) \\ &\quad \le \frac{1}{1-q} \sum_{k=1}^{\infty} k P[|\Gamma_n| = k] m^{-n} e^{-n\delta} = \frac{1}{1-q} m^n m^{-n} e^{-n\delta} = \frac{1}{1-q} e^{-n\delta}, \end{aligned}$$

where the inequality is due to (31). Equation (30) is straightforward from here by applying the Borel–Cantelli lemma.

Step 2. We show that a.s. on nonextinction of Γ,

$$s^{\psi}_{\text{sust}} \geq (\log m)^{1/r}. \tag{33}$$

This time, we have to modify the argument in [11].

(i) Assume that the Galton–Watson tree has finite variance: that is, $\sigma^2 := \mathrm{Var}(Z) < \infty$. Let $\delta > 0$ and $a = (\log m)^{1/r} - 2\delta$. Let $1 < \gamma < (\log m)(a+\delta)^{-r}$ and $M > 0$. We denote the integer part of γ^k by $\lfloor \gamma^k \rfloor$. We will consider the following tree $\Gamma(\gamma, M)$ which is embedded in Γ. The vertices w_k at level k of $\Gamma(\gamma, M)$ satisfy

$$w_0 = \rho,\ |w_k| = |w_{k-1}| + \lfloor \gamma^k \rfloor,$$
$$S_{w_k} > S_{w_{k-1}} + \psi(\lfloor \gamma^k \rfloor)a,$$
$$S_u > S_{w_{k-1}} - \lfloor \gamma^k \rfloor M \text{ for all } u \text{ in the path from } w_{k-1} \text{ to } w_k.$$

We will first show that, for M large enough, on nonextinction of $\Gamma, \Gamma(\gamma, M)$ survives with positive probability. Here $\Gamma(\gamma, M)$ corresponds to a branching process with time-dependent branching, which is constructed by:

(a) Considering the levels $\lfloor \gamma \rfloor + \lfloor \gamma^2 \rfloor + \cdots + \lfloor \gamma^k \rfloor$ of the original tree Γ;

(b) Performing a (dependent) percolation where one keeps each path from level $\lfloor \gamma \rfloor + \lfloor \gamma^2 \rfloor + \cdots + \lfloor \gamma^{k-1} \rfloor$ to level $\lfloor \gamma \rfloor + \lfloor \gamma^2 \rfloor + \cdots + \lfloor \gamma^k \rfloor$ as an edge of $\Gamma(\gamma, M)$ with probability

$$p_k := P\left[\widetilde{S}_{\lfloor \gamma^k \rfloor} > \psi(\lfloor \gamma^k \rfloor)a,\ \min_{1 \leq j \leq \lfloor \gamma^k \rfloor} \widetilde{S}_j > -\lfloor \gamma^k \rfloor M\right].$$

Note that

$$P\left[\min_{1 \leq j \leq \lfloor \gamma^k \rfloor} \widetilde{S}_j > -\lfloor \gamma^k \rfloor M\right] \geq P\left[\frac{1}{\lfloor \gamma^k \rfloor} \sum_{j=1}^{\lfloor \gamma^k \rfloor} Y_j^- < M\right] \tag{34}$$

and, for M large enough, the last probability goes to 1 for $k \to \infty$ due to the law of large numbers. Together with Theorem 3, this implies that $p_k \geq \exp(-\lfloor \gamma^k \rfloor (a+\delta)^r)$ for k large enough. Denote by V_k the size of the generation at level $|w_k|$ in Γ, before percolation, and by $\overline{V}_k$ the size of the generation at level $|w_k|$ in Γ after percolation. Here $(V_k)_{k \geq 0}$ and $(\overline{V}_k)_{k \geq 0}$ are branching processes with time-dependent branching. More precisely, $V_0 = \overline{V}_0 = 1$ and

$$V_k = \sum_{j=1}^{V_{k-1}} X_{k,j}, \qquad \overline{V}_k = \sum_{j=1}^{\overline{V}_{k-1}} \overline{X}_{k,j}, \tag{35}$$

where $X_{k,j}$, $j = 1, 2, \ldots, V_k$ are i.i.d. ($X_{k,j}$ is the number of descendants in the $\lfloor \gamma^k \rfloor$th generation of the jth vertex at level $k-1$ of $\Gamma(\gamma, M)$), and $\overline{X}_{k,j}$, $j = 1, 2, \ldots, \overline{V}_k$ are i.i.d. with

$$\overline{X}_{k,j} = \sum_{i=1}^{X_{k,j}} Y_{i,k}, \tag{36}$$

where $Y_{i,k}$, $i = 1, 2, \dots, X_{k,j}$ are (dependent) random variables with $P[Y_{i,k} = 1] = p_k = 1 - P[Y_{i,k} = 0]$. Let $\bar{m}_k := E[\overline{X}_{k,j}]$ and $\bar{\sigma}_k^2 := \mathrm{Var}(\overline{X}_{k,j})$. We will show that

$$W_k := \frac{\overline{V}_k}{\bar{m}_k \bar{m}_{k-1} \cdots \bar{m}_1}, \qquad k = 1, 2, \dots \tag{37}$$

is a uniformly integrable positive martingale. We then conclude that for $W := \lim_{k\to\infty} W_k$, we have $E[W] = E[W_k] = 1$ and therefore $P[W > 0] > 0$ which implies that $P[(\overline{V}_k)$ survives$] > 0$. To prove uniform integrability, we will show that

$$\sup_k E[W_k^2] < \infty. \tag{38}$$

Note first that, due to (36),

$$\bar{m}_k = p_k E[X_{k,1}] = p_k m^{\lfloor \gamma^k \lfloor} \text{ and } E[\overline{X}_{k,1}^2] \le p_k E[X_{k,1}^2], \quad k = 1, 2, \dots, \tag{39}$$

hence

$$\bar{\sigma}_k^2 \le p_k \mathrm{Var}(X_{k,1}) + (p_k - p_k^2) E[X_{k,1}]^2, \qquad k = 1, 2, \dots. \tag{40}$$

Recursive calculation of the variance of a branching process with time-dependent branching yields

$$\begin{aligned} &\frac{\mathrm{Var}(\overline{V}_k)}{\bar{m}_k^2 \bar{m}_{k-1}^2 \cdots \bar{m}_1^2} \\ &\qquad = \sum_{j=1}^{k} \frac{\bar{\sigma}_j^2}{\bar{m}_j^2 \bar{m}_{j-1} \cdots \bar{m}_1} \\ &\qquad = \left(\frac{\bar{\sigma}_1^2}{\bar{m}_1^2} + \frac{\bar{\sigma}_2^2}{\bar{m}_2^2 \bar{m}_1} + \cdots + \frac{\bar{\sigma}_k^2}{\bar{m}_k^2 \bar{m}_{k-1} \cdots \bar{m}_1} \right), \quad k = 1, 2, \dots. \end{aligned} \tag{41}$$

For the number of descendants $X_{k,1}$, we have

$$\mathrm{Var}(X_{k,1}) = \frac{\sigma^2 m^{\lfloor \gamma^k \rfloor - 1} (m^{\lfloor \gamma^k \rfloor} - 1)}{m - 1}, \qquad k = 1, 2, \dots \tag{42}$$

(see also [1], page 4). Due to (37) and (41), we have

$$\begin{aligned} E[W_k^2] &\le \frac{\mathrm{Var}(\overline{V}_k)}{\bar{m}_k^2 \bar{m}_{k-1}^2 \cdots \bar{m}_1^2} + 1 \\ &= \sum_{j=1}^{k} \frac{\bar{\sigma}_j^2}{\bar{m}_j^2 \bar{m}_{j-1} \cdots \bar{m}_1} + 1, \qquad k = 1, 2, \dots. \end{aligned} \tag{43}$$

Since $E[X_{k,1}] = m^{\lfloor \gamma^k \rfloor}$, we see from (42) that

$$\frac{\mathrm{Var}(X_{k,1})}{E[X_{k,1}]^2} \le C_1, \qquad k = 1, 2, \dots, \tag{44}$$

where C_1 is a constant depending on m and σ but not on k. Using (39), (40) and (44), we see that

$$\frac{\bar{\sigma}_j^2}{\bar{m}_j^2} \leq \frac{1}{p_j} C_1 + \left(\frac{1}{p_j} - 1\right) \leq (C_1 + 1)\frac{1}{p_j}, \qquad j = 1, 2, \ldots \tag{45}$$

and, using (45) and

$$p_j \geq \exp(-\lfloor \gamma^j \rfloor (a+\delta)^r), \qquad E[\overline{X}_{j,1}] = p_j m^{\lfloor \gamma^j \rfloor},\ j = 1, 2, \ldots, \tag{46}$$

we see that the r.h.s. of (43) can be bounded uniformly,

$$\begin{aligned} \sup_k \sum_{j=1}^k \frac{\bar{\sigma}_j^2}{\bar{m}_j^2 \bar{m}_{j-1} \cdots \bar{m}_1} &\leq C_2 \sup_k \sum_{j=1}^k \frac{1}{p_j p_{j-1} m^{\lfloor \gamma^{j-1} \rfloor} p_{j-2} m^{\lfloor \gamma^{j-2} \rfloor} \cdots p_1} \\ &\leq C_2 \sum_{j=1}^\infty m^j \left(\frac{e^{(a+\delta)^r}}{m^{1/\gamma}}\right)^{\lfloor \gamma^j \rfloor + \lfloor \gamma^{j-1} \rfloor + \cdots + \lfloor \gamma \rfloor} < \infty, \end{aligned} \tag{47}$$

where $C_2 = (C_1 + 1)$. The last term in (47) is finite since $(a+\delta)^r < (1/\gamma)\log m$, and together with (43), this proves (38). Hence, we have shown that $P[(\overline{V}_k)$ survives$] = P[\Gamma(\gamma, M)$ survives$] > 0$. However, on nonextinction of $\Gamma(\gamma, M)$, we have

$$s_{\text{sust}}^{\psi} \geq \liminf_{k\to\infty} \frac{a\big(\psi(\lfloor\gamma\rfloor) + \cdots + \psi(\lfloor\gamma^{k-1}\rfloor)\big) - \lfloor\gamma^k\rfloor M}{\psi(\lfloor\gamma\rfloor) + \cdots + (\lfloor\gamma^k\rfloor)} = \frac{a}{\gamma^{1/r}}. \tag{48}$$

For $\gamma \to 1$, this yields $s_{\text{sust}}^{\psi} \geq a$ with positive probability. Now, exactly as in [11],

$$A := \{\Gamma | \Gamma \text{ finite or } s_{\text{sust}}^{\psi} \leq a \text{ on } \Gamma\} \tag{49}$$

is an inherited property. Applying the 0–1 law proved in [11], Proposition 3.2, we conclude that $P[s_{\text{sust}}^{\psi} \geq a] = 1$ a.s. on nonextinction. For $\delta \to 0$, (33) follows.

(ii) Finally, if the variance of Z is infinite, we can approximate in the following way: let $Z^{(K)} = Z I_{\{Z \leq K\}}$ where we choose $K \in \mathbb{N}$ large enough such that $m^{(K)} := E[Z^{(K)}] > 1$. Then $Z^{(K)}$ has finite variance. The corresponding tree $\Gamma^{(K)}$ is the tree remaining if we delete in Γ the $K+1$th, $K+2$th,…, children of every vertex. Now, by (i), we have $s_{\text{sust}}^{\psi} \geq (\log m^{(K)})^{1/r}$ a.s. on nonextinction of $\Gamma^{(K)}$. For $K \to \infty$, we have $m^{(K)} \to m$ and therefore $s_{\text{sust}}^{\psi} \geq (\log m)^{1/r}$ a.s. on nonextinction of Γ.

Step 3. Together with the first step, we have now shown that a.s. on nonextinction of Γ, we have

$$(\log m)^{1/r} \leq s_{\text{sust}}^{\psi} \leq s_{\text{burst}}^{\psi} \leq s_{\text{cloud}}^{\psi} \leq (\log m)^{1/r}. \tag{50}$$

□

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REFERENCES

[1] ATHREYA, K. B. and NEY, P. E. (1972). *Branching Processes*. Springer, New York.

[2] BENJAMINI, I. and PERES, Y. (1994). Tree-indexed random walks and first-passage percolation. *Probab. Theory Related Fields* **98** 91–112.

[3] BIGGINS, J. D. (1997). Chernoff's theorem in the branching random walk. *J. Appl. Probab* **14** 630–636.

[4] DURRETT, R. (1983). Maxima of branching random walks. *Z. Wahrsch. Verw. Gebiete* **62** 165–170.

[5] DURRETT, R. (1979). Maxima of branching random walks vs. independent random walks. *Stochastic Process. Appl.* **9** 117–135.

[6] GANTERT, N. (1998). Functional Erdős–Renyi laws for semiexponential random variables. *Ann. Probab.* **26** 1356–1369.

[7] HAMMERSLEY, J. M. (1974). Postulates for subadditive processes. *Ann. Probab.* **2** 652–680.

[8] KINGMAN, J. F. C. (1975). The first birth problem for an age-dependent branching process. *Ann. Probab.* **3** 790–801.

[9] LYONS, R. and PEMANTLE, R. (1992). Random walks in a random environment and first-passage percolation on trees. *Ann. Probab.* **20** 125–136.

[10] NAGAEV, S. V. (1979). Large deviations of sums of independent random variables. *Ann. Probab.* **7** 745–789.

[11] PERES, Y. (2000). Probability on trees: an introductory climb. *Lectures on Probability and Statistics Lecture Notes in Math.* **1717**. Springer, New York.

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