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THE SURVIVAL OF NONATTRACTIVE INTERACTING PARTICLE SYSTEMS ON Z

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We consider interacting particle systems on Z which allow five types of pairwise interaction: Annihilation, Birth, Coalescence, Death and Exclusion with corresponding rates a, b, c, d, e. We show that whatever the values of a, c, d, e, if the birthrate is high enough there is a positive probability the particle system will survive starting from any finite occupied set. In particular: *an IPS with rates a, b, c, d, e has a positive probability of survival if*

$$b > 4d + 6a, \qquad c + a \geq d + e,$$

or

$$b > 7d + 3a - 3c + 3e, \qquad c + a < d + e.$$

We create a suitable supermartingale by extending the method used by Holley and Liggett in their treatment of the contact process.

1. Survival of interacting particle systems. In this paper we shall be concerned with processes ξ defined on the state-space $\{0, 1\}^Z$. We say a site x is occupied if $\xi(x) = 1$, unoccupied when $\xi(x) = 0$. The processes will evolve by means of five types of pairwise interaction between neighbors where, on Z, x and y are neighbors if $|x - y| = 1$. Neighboring sites interact with each other independently of all other pairs of sites. In this paper we shall not allow "spontaneous birth" and assume that two unoccupied neighbors do not interact with each other. The only change we shall consider which is not pairwise (though it can be represented as such) is "single death," when occupied sites become unoccupied at a fixed rate independently of all other sites. We represent this transition $1 \to 0$ at site x at rate 1 by the two transitions $11 \to 01$ and $10 \to 00$, both at rate $1/N_x$, where N_x is the number of neighbours of x. $N_x = 2$ on Z.

In the following representation, $10 \to 11$ means that a pair of neighboring sites, one of which is occupied and the other not, flip to the state of both being occupied, and so on. Directional symmetry is assumed, so that if $10 \to 11$ at a certain rate, then $01 \to 11$ at the same rate. The possible changes are:

Annihilation	$11 \to 00$	at rate a
Birth	$10 \to 11$	at rate b
Coalescence	$11 \to 10$	at rate c
Death	$10 \to 00$	at rate d
Exclusion	$10 \to 01$	at rate e

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The Interacting Particle Systems (IPS) most frequently considered are special cases in which only some of a, b, c, d, e are positive. In general they correspond to more or less plausible models of physical systems.

The only processes for which survival is an issue are those with $b > 0$ and one of $a, c, d > 0$. If there is spontaneous birth then survival is certain. In general these problems are hard to solve; see, for example, Bramson and Gray (1985), Bezuidenhout and Grimmett (1990), Sudbury (1990). One of the most celebrated treatments is "The survival of contact processes" by Holley and Liggett (1978). It is their ingenious method that we intend to follow here.

2. The Holley–Liggett method. The Holley–Liggett method was introduced in "The survival of contact processes" (1978). Although very effective at providing bounds for critical values of the contact process, the method has not been used very often. Examples include Liggett (1995a), in which the method is enhanced to provide a better bound for the 1-dimensional contact process, Liggett (1995b) where the technique is applied to discrete time models and in Chapter 5 of Konno (1994) where the θ-contact process is treated. The treatment in Liggett (1985) is probably easier to follow than in the 1978 paper. Two interacting particle systems ξ, ζ are said to be dual with parameter μ when

$$E\{\mu^{|\xi_t^A \cap B|}\} = E\{\mu^{|\zeta_t^B \cap A|}\}, \tag{1}$$

for all finite A, B. It is possible to have one of either A or B infinite or a measure as $|\xi_t^A \cap B|$ and $|\zeta_t^B \cap A|$ are a.s. finite. Sudbury (1999) shows that when $b > 0$, all IPS which have pairwise interactions and do not have spontaneous birth are self-dual with self-duality parameter $\mu = (d - a - c)/b$, $\mu \geq 0$ for attractive processes and $\mu < 0$ for nonattractive processes. When $\mu > 1$, Sudbury (2000) shows that the process will die out.

We consider a process ξ with rates a, b, c, d, e and infinitesimal generator Ω_ξ. The Holley–Liggett method essentially consists of differentiating both sides of the duality equation and finding conditions under which the derivative is always negative. They replace A by a renewal measure η, and then

$$\begin{aligned} \frac{d}{dt} E\{\mu^{|\xi_t^\eta \cap B|}\} &= \frac{d}{dt} E\{\mu^{|\xi_t^B \cap \eta|}\} \\ &= \sum_A P(\xi_t^B = A)\Omega_\xi E_\eta \mu^{|A \cap \eta|}. \end{aligned} \tag{2}$$

The method is then to determine conditions under which $\Omega_\xi E_\eta \mu^{|A\cap\eta|} < 0$ for all finite sets A. However, this is also the condition for $E_\eta \mu^{|\xi_t \cap \eta|}$ to be a supermartingale for the IPS ξ. We shall adopt this way of looking at the problem.

Equation (2) was also used by Holley and Liggett to show the convergence of ξ_t^η, where η is the renewal measure mentioned above. When $|\mu| < 1$, this convergence is ensured by Theorem 7 of Sudbury (2000) since η is an essentially infinite measure, that is, a measure whose intersection with any infinite set of sites is a.s. infinite.

3. Properties of the renewal measure. Suppose η is a renewal measure on Z defined by a probability density $\{f(k)\colon k = 1, 2, \ldots\}$ with

$$m^{-1} = \sum kf(k) < \infty,$$

so that $P(\eta(0) = 1) = m$. (Note: in the original paper k starts at 0.)

In what follows, we wish to use the term s in a flexible way: $s = s_1 s_2 \ldots, s_n$, $s_i = 0$ or 1 is a finite string of 0's and 1's, but it may also be used to define the set of occupied sites when s is put directly to the right of the origin. In this case the set of occupied sites on Z is $\{i\colon s_i = 1\}$. The string $0s$ is the string produced by adding a 0 to the left end of s; s^T is to mean s in reverse order.

Define

$$K(s) = E_\eta\{\mu^{|\eta\cap s|}\}. \tag{3}$$

Since η is translation invariant, the location of s is not important. Let s be placed to the right of 0. Define

$$G(s) = E_\eta\{\mu^{|\eta\cap s|}|\eta(0) = 1\}. \tag{4}$$

We note $G(0s) \neq G(s)$ although $K(0s) = K(s)$.

The following lemma demonstrates the crucial simplification obtained by requiring that η be a renewal measure.

LEMMA 1. *If* s, t *are finite strings of* 0*'s and* 1*'s*,

$$K(s0t) - K(s1t) = m' G(s^T)G(t). \tag{5}$$

where $m' = m(1-\mu)$.

PROOF. The only way in which $K(s0t)$ and $K(s1t)$ can have different values is when the site at which $s0t$, $s1t$ differ is occupied in the renewal measure. That site does not belong to the set $s0t$ so there is no contribution to $K(s0t)$, but it belongs to $s1t$, and thus there is a multiplicative contribution of μ. Given that the site is occupied in the renewal process, the distributions to the right and left of the site are independent. Since the renewal measure is symmetric, the contribution to the left is $G(s^T)$.

In the special case when the occupied set is $\{1, 2, \ldots, r\}$ we put

$$G_r = E\{\mu^{|\eta\cap\{1,2,\ldots,r\}|} \mid \eta(0) = 1\}. \tag{6}$$

Then, $G_0 = 1$, $G_1 = 1 - (1-\mu)f_1$ and conditioning on the first jump of the renewal process

$$G_r = \mu \sum_{l=1}^{r} f_l G_{r-l} + 1 - \sum_{l=1}^{r} f_l. \tag{7}$$

At this point we should like to find the optimal choice of $\{G_n\}$; that is, the choice which will allow for a minimum value of b. Holley and Liggett (1978) correctly surmised that the worst possible case for the growth of a contact process would be when there were no gaps between the particles. Once the IPS is nonattractive, it is not clear that a continuous group of particles is the worst possible case. Nevertheless, it was tried, but the resulting recurrence relation gave no simple solution for the $\{G_n\}$. Thus we shall use for G, their function F, even though $\mu = 0$ for the contact process. This has the property that

$$\beta G_n = \sum_{k=1}^{n} G_{k-1} G_{n-k}, \qquad n \geq 1,\ G_0 = 1, \tag{8}$$

where the exact choice of β can be made later. The solution to this equation is $G_n = (2n)!/(n!(n+1)!)\beta^{-n}$. $G_1 = 1/\beta$. If $\beta > 4$ then $G_n \downarrow 0$, and we show with $\mu \leq 0$ that this is a sufficient condition for there to exist a renewal measure $\{f_i\}$. After Lemma 4 it is pointed out that the cases with $\mu > 0$ can also be treated.

LEMMA 2. *If* $\{G_n\}$ *satisfy* (8), $\beta > 4$ *and* $\mu \leq 0$, *then there exists a probability mass function* $\{f_i\}$ *which satisfies* (7).

PROOF. First we note that since $\mu \leq 0$, $G_1 = 1 - (1-\mu)f_1$ implies that $f_1 < 1$. Next we show that the $\{f_i\}$ are positive. Subtracting the equation (7) for r from that for $r-1$, we obtain

$$G_{r-1} - G_r - \mu \sum_{l=1}^{r-1} (G_{r-1-l} - G_{r-l}) f_l = (1-\mu) f_r.$$

Since $\{G_n\}$ are monotonic decreasing and $\mu \leq 0$, f_r is positive as long as $f_l \geq 0$, $l < r$, and thus each f_r is positive by induction. To show that they sum to 1, we put $\gamma(u) = \sum_0^\infty G_r u^r$ and $\phi(u) = \sum_1^\infty f_r u^r$. Equation (7) implies

$$\gamma(u) = \frac{1-\phi(u)}{1-u} + \mu\phi(u)\gamma(u). \tag{9}$$

From (7) we see that $\sum_{l=1}^r f_l < 1$, so that $\phi(u)$ is bounded. Further, $\gamma(1)$ exists for $\beta > 4$. Since both $\gamma(u)$ and $\phi(u)$ are bounded for $u \leq 1$, it follows that $\phi(1) = 1$. $\square$

LEMMA 3. *Let* $s = s_1 s_2 \cdots s_n$ *be a string of* 0*'s and* 1*'s. Define* $s(i) = \{s_{i+1}, s_{i+2}, \dots, s_n\}$, $i < n$. *Let* $s^c = \{i\colon s_i = 0\}$. *Then*

$$G(s) = \sum_{i \in s^c} g_i G(s(i)) + G_n,$$

where $g_i = G_{i-1} - G_i$.

PROOF. Define s^i, $i = 1, \dots, n+1$ to be the string s.t. $s^i_j = 1$, $j < i$, $s^i_j = s_j$, $n \geq j \geq i$. $s^1 = s$. $G(s^{n+1}) = G_n$. We shall proceed by induction going backward through the members of s^c.

Assume the theorem is true for $s = s^k$. Let l be the largest value of i s.t. $i < k$ and $s_i = 0$. The theorem is still true for $l < j < k$, trivially, since then $s^j = s^k$. The strings s_i, s_{i+1} differ only at the point i so

$$\begin{aligned}(10)\qquad & K(0s^i) - K(1s^i) - (K(0s^{i+1}) - K(1s^{i+1})) \\ &\quad = K(0s^i) - K(0s^{i+1}) - (K(1s^i) - K(1s^{i+1}) \\ &\quad = m'[G(s(i))G_{i-1} - G(s(i))G_i] \\ &\quad = m'G(s(i))g_i.\end{aligned}$$

since s^i consists of 1's for $j < i$. Summing $K(0s^i) - K(1s^i) - (K(0s^{i+1}) - K(1s^{i+1}))$ from $i = 1, \dots, n-1$ we obtain

$$K(0s^1) - K(1s^1) - (K(0s^{n+1}) - K(1s^{n+1})) = m' \sum_{i \in s^c} G(s(i))g_i,$$

from which the lemma follows since $K(0s^{n+1}) - K(1s^{n+1}) = m'G_n$ and $K(0s^1) - K(1s^1) = m'G(s)$. $\square$

COROLLARY. *If s is a string of 0's and 1's then* $G(0s) - G(1s) = (1 - G_1)G(s)$.

With Lemma 3 we have virtually returned to the situation in Holley and Liggett (1978). In that paper, the duality was coalescing and in place of G_n was $F(n) = \sum_{i=n}^{\infty} f_i$. This can be thought of as being the case $\mu = 0$ in (3), because $F(s)$ is only nonzero when the renewal measure does not coincide anywhere with the set s. Conditioning on the first jump of the renewal process it is obvious that if s is a finite string, $F(s) = \sum_{i \in s^c} f_i F(s(i))$, since, if the first jump lands on an occupied site, the term inside the expectation in (4) is 0. Thus, when $\mu \neq 0$, g_i plays the role that f_i did when the dual was coalescing.

In fact, this observation allows us to decouple the whole argument from duality and renewal measures. We can define G_n by (8) and $G(s)$ for all finite strings by Lemma 3. It is obvious $K(\{0\}) = 1 - m(1-\mu)$. Lemma 1 then defines $K(s)$ for all finite strings s using a value of m' which ensures K is bounded [equation (3) shows any value with $0 \geq \mu > -1$ will do]. What we are now going to find is values of a, b, c, d, e which will make $K(\xi_t)$ a supermartingale. Since K is bounded, K will converge, and since its expectation will be less that its initial value, the probability the IPS dies out will be < 1. We shall not abandon the friends that have brought us this far, but, as we shall see, a value of μ will not actually be needed in our calculations. For IPS with self-duality parameter in $(0, 1)$ we simply adopt a renewal measure defined by (7) with $\mu \in (-1, 0]$. This causes no problems, as we are longer relying on the duality equation.

It is in a way unnecessary to prove the next two lemmas, as once we have established such a close parallel between our G, g and Holley and Liggett's F, f, we can plunge into the Holley–Liggett stream and it will take us much of the rest of the way. However, they do have some interest in their own right.

LEMMA 4. *Define* $B_n = E\{\mu^{\eta(n)} | \eta(0) = 1\}$, *then*

$$B_n = \sum_{r=0}^{n-1} G_r/\beta. \tag{11}$$

PROOF. $B_n = 1 - (1-\mu)P(\eta(n) = 1 | \eta(0) = 1)$, so

$$B(u) = \sum_{n=1}^{\infty} B_n u^n = \frac{u}{1-u} - (1-\mu)\frac{\phi(u)}{1-\phi(u)} = \frac{1}{1-u}\left[1 - \frac{1}{\gamma(u)}\right]$$

from (9). However, $\beta G_n = \sum_{k=1}^{n} G_{k-1} G_{n-k}$, so that

$$\beta\left[\frac{\gamma(u) - 1}{u}\right] = \gamma^2(u),$$

giving $1 - \gamma^{-1}(u) = u\gamma(u)/\beta$ and

$$B(u) = \frac{u\gamma(u)}{\beta(1-u)},$$

from which the lemma follows. □

COROLLARY. $P(\eta(n) = 1 | \eta(0) = 1)$ *is decreasing in* n.

LEMMA 5. *If* s *is a finite string, and* 0_r *a string or* r0's, *then* $G(0_r 1s)$ *is an increasing function of* r.

PROOF. Lemma 4 has shown that the lemma is true when s has no 1's Assume the lemma is true when s contains j 1's, $j < k$. Then

$$\begin{aligned} G(0_r 0s) - G(0_r 1s) &= E\left[\mu^{\sum_i \eta(r+1+i)s_i} - \mu^{\eta(r+1) + \sum_i \eta(r+1+i)s_i} \,|\, \eta(0) = 1\right] \\ &= E\left[\mu^{\sum_i \eta(r+1+i)s_i} - \mu^{\eta(r+1) + \sum_i \eta(r+1+i)s_i},\ \eta(r+1) = 1 | \eta(0) = 1\right] \\ &= (1-\mu)P(\eta(r+1) = 1 | \eta(0) = 1)G(s). \end{aligned} \tag{12}$$

If s has $< k$ 1's, then $G(0_r 0s)$ is increasing in r by the inductive hypothesis and $P(\eta(r+1) = 1 | \eta(0) = 1)$ is decreasing in r by Lemma 4. □

LEMMA 6.

$$(2 - G_1)G(s) \geq G(0s) \geq G(s) > 0, \qquad G(1s) \geq G_1 G(s). \tag{13}$$

PROOF. Suppose s is of length n. Lemma 3 gives $G(s) = \sum_{i \in s^c} g_i G(s(i)) + G_n$. The equation $G_n = (2_n)!/(n!(n+1)!)\beta^{-n}$ implies $g_i > 0$. The inequality $G(s) > 0$ then follows by induction on the length of s. Lemma 5 shows $G(0s) \geq G(s)$. We may then use $G(s) = \sum_{i \in s^c} g_i G(s(i)) + G_n$ to give

$$G(0s) = (1 - G_1)G(s) + \sum_{i \in s^c} g_{i+1} G(s(i)) + G_{n+1}.$$

$G_{n+1} < G_n$ and $g_{i+1} < g_i$ for all i, so $(2 - G_1)G(s) \geq G(0s)$. Since $G(0s) - G(1s) = (1 - G_1)G(s)$, $G(0s) \geq G(s)$ implies $G(1s) \geq G_1 G(s)$.

Following Holley and Liggett (1978) we now fix a finite subset $A = \bigcup_{i=1}^k A_i$ of Z where $A_1, \dots, A_k$ are the ordered maximal connected subsets of A, so that there are integers l_i and r_i such that $A_i = [l_i + 1, r_i - 1]$ and $r_i \leq l_{i+1} < r_{i+1}$ for all i. Define

$$\rho(x) = E\{\mu^{|\eta \cap \{A \cap (x,\infty)\}|} | \eta(x) = 1\} \tag{14}$$

and

$$\lambda(x) = E\{\mu^{|\eta \cap \{A \cap (-\infty,x)\}|} | \eta(x) = 1\}. \tag{15}$$

In the lemmas that follow we should note that $G_r = 0$ for $r < 0$ and $g_r = 0$ for $r \leq 0$. In what follows, if a term such as g_{z-x} appears, it is to be assumed that $z > x$.

LEMMA 7. $\rho(x) = \sum_{x \in A^c} g_{z-x} \rho(z)$,

$$\rho(x) \leq \sum_{j \geq i} G_{r_j - x - 1} \rho(r_j), \qquad \lambda(x) \leq \sum_{j \leq i} G_{x - l_j - 1} \lambda(l_j), \ \ x \in A_i. \tag{16}$$

PROOF. Lemma 3 implies

$$\begin{aligned} \rho(x) &= \sum_{z \in A^c,\, z < r_k} g_{z-x} \rho(z) + G_{r_k - x - 1} \\ &= \sum_{z \in A^c,\, z < r_k} + \sum_{z \in A^c,\, z \geq r_k} g_{z-x} \rho(z), \end{aligned} \tag{17}$$

since $\rho(z) = 1$ for $z \geq r_k$. The first proposition follows. Lemma 5 shows that $\rho(z) \leq \rho(r_j)$ for $r_j \leq z \leq l_{j+1}$. Thus

$$\sum_{r_j \leq z \leq l_{j+1}} g_{z-x} \rho(z) < \rho(r_j) \sum_{r_j \leq z \leq l_{j+1}} g_{z-x} < G_{r_j - x - 1} \rho(r_j).$$

Similarly for the inequality in $\lambda(x)$. □

4. The conditions for *K* to be a supermartingale. We now consider the rate of change of $E_\eta\{\mu^{|\eta \cap A|}\}$ for an IPS with its pairwise interactions defined as in the introduction by a, b, c, d, e. The terms produced by annihilation a are of the form

$$\begin{aligned} K(s00t) - K(s11t) &= K(s00t) - K(s01t) + K(s01t) - K(s11t) \\ &= m'[G(0s^T)G(t) + G(s^T)G(1t)] \\ &= m'[G(t)\{(1 - G_1)G(s^T) + G(1s^T)\} + G(s^T)G(1t)]. \end{aligned} \tag{18}$$

The contribution from $x \in A_i$ is

$$
\begin{aligned}
&m'\left[\sum_{x\in A_i/\{r_i-1\}} \rho(x+1)[(1-G_1)\lambda(x)+\lambda(x+1)]+\lambda(x)\rho(x)\right]\\
&\quad = m'\left\{\sum_{x\in A_i\setminus\{r_i-1\}}(1-G_1)\lambda(x)\rho(x+1)+2\sum_{x\in A_i}\rho(x)\lambda(x)\right.\\
&\qquad\qquad \left. -\rho(l_i+1)\lambda(l_i+1)-\rho(r_i-1)\lambda(r_i-1)\right\}.
\end{aligned}\tag{19}
$$

The contribution from the effect of $c+d$ at the left end of a group of 1's is $K(s00t)-K(s01t)$ or, with the right end as well,

$$
m'[\rho(l_i+1)\lambda(l_i+1)+\rho(r_i-1)\lambda(r_i-1)].\tag{20}
$$

Each particle within a group of 1's disappears at rate $2c$ giving a contribution of

$$
m'\left\{\sum_{x\in A_i\setminus\{l_i+1,\, r_i-1\}}\rho(x)\lambda(x)\right\}.\tag{21}
$$

The contribution from b is

$$
-m'\big[\rho(l_i)\lambda(l_i)+\rho(r_i)\lambda(r_i)\big].\tag{22}
$$

Terms in e are of the form

$$
\begin{aligned}
K(s01t)-K(s10t) &= K(s01t)-K(s11t)+K(s11t)-K(s10t)\\
&= m'\big[G(s^T)G(1t)-G(1s^T)G(t)\big].
\end{aligned}\tag{23}
$$

We have seen in Lemma 6 that $G(0t)-G(1t)=(1-G_1)G(t)$. Thus if the 0 in $s10t$ is at r_i, contributions of this kind are of the form

$$
m'\big[\lambda(r_i-1)\{\rho(r_i-1)-(1-G_1)\rho(r_i)\}-\lambda(r_i)\rho(r_i)\big].\tag{24}
$$

Call the rate of change of $E_\eta\{\mu^{|\eta\cap A|}\}$, S, then, collecting the terms associated with $x\in A_i$ and dividing by $-m'$ we obtain from equations (18)–(23)

$$
\begin{aligned}
S = \sum_i \Bigg[&(b+e)\{\rho(l_i)\lambda(l_i)+\rho(r_i)\lambda(r_i)\}\\
&+e(1-G_1)\{\lambda(l_i)\rho(l_i+1)+\lambda(r_i-1)\rho(r_i)\}\\
&+(c+a-d-e)\{\lambda(l_i+1)\rho(l_i+1)+\lambda(r_i-1)\rho(r_i-1)\}\\
&-2(a+c)\sum_{x\in A_i}\lambda(x)\rho(x)-a(1-G_1)\sum_{x\in A_i\setminus\{r_i-1\}}\lambda(x)\rho(x+1)\Bigg].
\end{aligned}\tag{25}
$$

We aim to collect the terms in $\lambda(l_i)$, using the left-hand half of the terms in $b+e, e, c+a-d-e$ in the above equation and half the terms in $a+c$ and a alone. The conditions under which these terms are positive will, by symmetry, be conditions under which the right-hand terms are also positive. We shall use the inequalities of Lemma 7 to replace terms in $\lambda(x)$:

$$\rho(x) \le \sum_{j\ge i} G_{r_j-x-1}\rho(r_j), \qquad \lambda(x) \le \sum_{j\le i} G_{x-l_j-1}\lambda(l_j),\ \ x \in A_i. \tag{26}$$

Contributions to terms in $\lambda(l_i)$ come from

$$\begin{aligned} -\sum_{x\in A}\lambda(x)\rho(x) &> -\sum_i \sum_{x\in A} G_{x-l_i-1}\rho(x)\lambda(l_i) \\ &= -\sum_i \sum_{j\ge i} \sum_{x\in A_j} G_{x-l_i-1}\rho(x)\lambda(l_i). \end{aligned}$$

The left-hand term in $c+a-d-e$ is $\lambda(l_i+1)$ not $\lambda(l_i)$. Since $G(s) < G(0s) < (2-G_1)G(s)$, we have $\lambda(l_i) < \lambda(l_i+1) < (2-G_1)\lambda(l_i)$. Which side of the inequality we use depends on the sign of $c+a-d-e$. Thus we put

$$\begin{aligned} C &= (c+a-d-e)+e(1-G_1) = c+a-d-G_1 e,\ c+a \ge d+e \\ &= (c+a-d-e)(2-G_1)+e(1-G_1) \\ &= (c+a-d)(2-G_1)-e,\ c+a < d+e. \end{aligned} \tag{27}$$

Define

$$\begin{aligned} S_i^l &= (b+e)\rho(l_i) + C\rho(l_i+1) - (a+c)\sum_{j\ge i}\sum_{x\in A_j} G_{x-l_i-1}\rho(x) \\ &\quad -\frac{a}{2}(1-G_1)\sum_{j\ge i}\sum_{x\in A_j\setminus\{r_j-1\}} G_{x-l_i-1}\rho(x+1). \end{aligned} \tag{28}$$

S_i^r is defined similarly in terms of the right-hand halves of the brackets in (24), and the argument above has shown that $S > \sum_i (S_i^l + S_i^r)$. The rest of the proof will be devoted to determining conditions under which $S_i^l > 0$.

LEMMA 8.

$$\beta\rho(l_i) = \sum_{x\in A_j,\ j\ge i} G_{x-l_i-1}\rho(x).$$

PROOF. It follows by subtracting the equation for n from that for $n-1$ in (8) that

$$\beta g_n = \sum_{k=1}^{n-1} G_{k-1} g_{n-k} - G_{n-1}, \qquad n \ge 2. \tag{29}$$

Lemma 3 says that $\rho(l_i) = \sum_{z>l_i,\ z\in A^c} g_{z-l_i}\rho(z)$, so using (29),

$$\beta\rho(l_i) = \sum_{z>x>l_i,\ z\in A^c} \rho(z)\big[g_{z-x}G_{x-l_i-1} - G_{z-l_i-1}\big]. \tag{30}$$

But

$$\sum_{z\in A^c}\rho(z)G_{z-l_i-1}=\sum_{x\in A^c}\rho(x)G_{x-l_i-1}=\sum_{z,\,x\in A^c,\,z>x}\rho(z)G_{x-l_i-1}g_{z-x}.$$

Substituting this into (29) subtracts out the terms in $x\in A^c$ leaving those in $x\in A$ and the lemma follows once again using Lemma 7.

Similarly, when $l_i+2\in A$,

$$\beta\rho(l_i+1)=\sum_{x\in A\setminus\{l_i+1\}}G_{x-l_i-2}\rho(x).$$

However, when $l_i+2\in A^c$, the analysis is not so straightforward, because equation (29) requires $n\geq 2$. In that case we have

$$\rho(l_i+1)=g_1\rho(l_i+2)+\sum_{z>l_i+2}g_{z-l_1-1}\rho(z).$$

The argument for summation proceeds as above, except that the term in $x=l_1+2$ is not subtracted out. We obtain

$$\beta\rho(l_i+1)=\beta g_1\rho(l_i+2)+\sum_{x\in A\cup\{l_i+2\}}G_{x-l_i-2}\rho(x)>\sum_{x\in A}G_{x-l_i-2}\rho(x).$$

Changing variables so that $x\to x+1$, the sum is over the sets $\{A_j\}$ with the $\{r_j-1\}$ removed and $\{l_j,\,j>i\}$ added in. Thus

$$\beta\rho(l_i+1)>\sum_{j\geq i}\sum_{x\in A_j\setminus\{r_j-1\}}G_{x-l_i-1}\rho(x+1). \tag{31}$$

Note that when $l_i+2\in A^c$, $A_i\setminus\{r_i-1\}$ is empty.

Substituting Lemma 8 and (30) into (27) we obtain

$$S^l>[b+e-\beta(a+c)]\rho(l_i)+\Big[C-\beta\frac{a}{2}(1-G_1)\Big]\rho(l_i+1). \tag{32}$$

We have seen in Lemma 6 that $\rho(l_i)\geq G_1\rho(l_i+1)$. Also $G_1=1/\beta$. Thus a sufficient condition for $S^l>0$ and thus $S>0$ is that

$$\frac{b+e}{\beta}-(a+c)+C-\frac{a}{2}(\beta-1)>0. \tag{33}$$

When $c+a\geq d+e$ we need

$$\frac{b+e}{\beta}-d-\frac{e}{\beta}-\frac{a}{2}(\beta-1)>0, \tag{34}$$

or, since G_n is finite for $\beta>4$,

$$b>4d+6a,\qquad c+a\geq d+e. \tag{35}$$

When $c+a<d+e$, we need

$$\frac{b+e}{\beta}-(a+c)+\left(2-\frac{1}{\beta}\right)(c+a-d)-e-\frac{a}{2}(\beta-1)>0,$$

giving

$$b>7d+3a-3c+3e,\qquad c+a<d+e.$$

THEOREM 1. *An IPS with rates a, b, c, d, e has a positive probability of survival if*

$$b > 4d + 6a, \qquad c + a \geq d + e$$

or

$$b > 7d + 3a - 3c + 3e, \qquad c + a < d + e.$$

5. Bounds on the survival probabilities. Since K is a bounded supermartingale, $K(\xi_t)$ tends to a limit a.s. The only possible limits are 0 and 1. Suppose $0 < |\xi_t| < n$. If the change that occurs in $K(\xi_t)$ is of the form $K(s1t)$ to $K(s0t)$ or vice versa, there is a change of $m'G(s^T)G(t)$. If the change is of the form $K(s11t)$ to $K(s00t)$ then (18) shows the change is greater than $m'G(s^T)(1 - G_1)G(t)$. It is clear from Lemma 3 that $G(s^T), G(t) > G_n$. The other possible change is from $K(s10t)$ to $K(s01t)$ or vice versa, but for any finite configuration, this is the next change with probability less than $e/(b+e)$. If $0 < |\xi_t| < n$ i.o., then changes of size $> m'(1-G_1)G_n^2$ occur i.o. contradicting the convergence of $K(\xi_t)$. Thus $|\xi_t| \to \infty \Rightarrow K(\xi_t) \to 0$ for $|\mu| < 1$. We have the following theorem.

THEOREM 2. $K(\xi_t) \to \{0, 1\}$ *a.s.*

Because $K(\xi_t)$ is a supermartingale, $E\{K(\xi_t)\} < K(\xi_0)$. But Theorem 2 shows that from initial occupied set A, $E\{K(\xi_t^A)\} \to 1 - s_A$ where s_A is the probability of survival starting from A. We have the lemma.

LEMMA 9. $s_A > 1 - K(A)$.

From the definition in (5) it is obvious that $K(\emptyset) = 1$ and $K(\{0\}) = 1 - m(1-\mu)$, where m^{-1} is the mean of the renewal measure. Equation (9) can be written

$$(1-u)(1-\mu\phi(u))\gamma(u) = 1 - \phi(u).$$

Differentiating this gives

$$-(1-\mu)\gamma(1) = -\phi'(1) = -m^{-1},$$

so that

$$m(1-\mu) = \frac{1}{\gamma(1)}.$$

$K(\{0\}) = 1 - m(1-\mu)$ and Lemma 9 imply that $s_{\{0\}} > 1/\gamma(1) = 1/\sum_{i=0}^{\infty} G_i$.

Now Lemma 1 gives

$$K(\{0, 1, \ldots, n\}) - K(\{0, 1, \ldots, n-1\}) = -m(1-\mu)G_n,$$

implying that $K(\{0, 1, \ldots, n\}) = 1 - m(1-\mu)\sum_{i=0}^{n} G_i$. From Lemma 9 we may thus deduce the theorem.

THEOREM 3.

$$s_{\{0,1,\dots,n\}} > \frac{\sum_{i=0}^{n} G_i}{\sum_{i=0}^{\infty} G_i}.$$

The series $\{G_i\}$ defined in (8) converges faster as β increases; thus the best bound for s above will be using $\{G_i\}$ with largest possible β. The β to be used should be the largest that will satisfy the inequalities in (33) and (36).

The so-called correlation identities follow directly from the self-duality equation. A particular case is

$$\mu P(\xi_\infty(0) = 1) + 1 - P(\xi_\infty(0) = 1) = 1 - s_{\{0\}} < 1 - \gamma(1).$$

We have

$$P(\xi_\infty(0) = 1) > \frac{1}{(1-\mu)\gamma(1)}. \tag{36}$$

6. Conclusion. It has been possible to show that given values of a, c, d, e a b can be found which will ensure that extinction is not certain. However, the G_i used in this paper were chosen for their convenience rather than being tailored to a particular interacting particle system. It is unlikely that they cannot be improved upon.

One of the most unsatisfactory aspects of the inequalities given in Theorem 1 is that as e increases to the point that $c + a < d + e$, b must then increase rapidly, yet intuition would suggest that e should assist the spread of an IPS, not hinder it. This is confirmed for the contact process with exclusion, that is, $b > 0$, $e > 0$, $c = d = 1/2$. Durrett and Neuhauser (1994) showed that the critical value of $b \downarrow 1/2$ as $e \to \infty$. The bound given in Theorem 1 would be $2 + 3e$. It suggests that the form of G_n chosen may not be very suitable for exclusion.

Another problem is that is does not provide any bound for the branching annihilating random walk of Bramson and Gray (1985), because in that process $b = c$ always.

An example may suggest how good the inequalities are. Sudbury (1998) finds a lower bound for the critical value of b for a model which is essentially an annihilating random walk counteracted by birth; that is, with $a = 2$, b, $e = 1$. The bound given is 0.30, but because the rates are inverted in the treatment in that paper, this translates to extinction for $b < 1/0.3 = 3.33$. Theorem 1 gives nonextinction possible for $b > 12$.

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