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SMOOTHNESS OF HARMONIC MAPS FOR HYPOELLIPTIC DIFFUSIONS

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Harmonic maps are viewed as maps sending a fixed diffusion to manifold-valued martingales. Under a convexity condition, we prove that the continuity of real-valued harmonic functions implies the continuity of harmonic maps. Then we prove with a probabilistic method that continuous harmonic maps are smooth under Hörmander's condition; the proof relies on the study of martingales with values in the tangent bundle.

1. Introduction. Harmonic maps between two Riemannian manifolds M and N are maps satisfying some partial differential equation linked with the Laplacian on M and the Riemannian metric on N. This notion can be extended to the case where the Laplacian is replaced by the generator of a diffusion X_t on M, and the Riemannian metric on N is replaced by a connection; then harmonic maps can be defined as maps sending X_t to N-valued martingales.

The simplest problem involving harmonic maps is the Dirichlet problem. Suppose that M is a manifold with boundary ∂M, and let g be a N-valued map defined on ∂M; then the Dirichlet problem consists in finding a map h which is harmonic on the interior of M and which converges to g on ∂M; in the stochastic framework, we consider the diffusion X_t on M stopped when it hits the boundary, and we look for a map h such that $h(X_t)$ is a martingale and $h = g$ on ∂M. In the classical Riemannian case, solutions of the Dirichlet problem are generally obtained by solving a parabolic equation (the heat equation), or by looking for energy minimizing maps (harmonic maps are the critical points of an energy functional); see for instance [8, 14], and [15] for the sub-Riemannian case. It appears that these methods have probabilistic counterparts. The analogue of the heat equation is the following problem: if G is an N-valued random variable, we want to know whether there exists an N-valued martingale converging to G; if this problem has a solution, we can apply it to the variable $G = g(X_\infty)$ and obtain a solution of the Dirichlet problem. Actually, the existence can be proved under convexity assumptions on the image of g; see [16, 24, 2]. On the other hand, if the diffusion X_t is symmetric on the interior of M and if N is Riemannian, we can associate to X_t a reflected Dirichlet space, we define an energy functional for maps h: $M \to N$ in this space and the Dirichlet problem can be solved by looking for an energy minimizing map as in the classical case; see [26]. This approach

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does not require convexity conditions, but it can be worked out only in the symmetric case.

The aim of this work is to study the smoothness of harmonic maps. First we consider their continuity. In the classical Riemannian case, it is known that weakly harmonic maps which are the harmonic maps with finite energy (see [26]) are continuous when M has dimension 2 [13], but there may exist noncontinuous harmonic maps with infinite energy; when the dimension of M is greater, harmonic maps can be nowhere continuous [27] and one only has estimates on the Haussdorff dimension of singular points for energy minimizing maps [28]. However, the continuity can be obtained under convexity assumptions on N; see [16] for a stochastic proof. Here, under our more general framework, we consider a weaker convexity assumption on N and check that it is also sufficient; our condition on the diffusion is that harmonic real-valued functions are continuous (this is a nondegeneracy condition replacing the ellipticity of the classical Laplacian).

Then we study the C^∞ smoothness of harmonic maps under Hörmander's condition. In the symmetric case, the answer is well known in the elliptic case, and is extended to the hypoelliptic case in [15] by using the sub-Riemannian geometry and the results of [29] concerning quasilinear partial differential equations; it appears that h is C^∞ provided that it is continuous. In the elliptic case, probabilistic proofs have been given in [19] (coupling method) and in [3] (change of probability), but these proofs seem difficult to extend to the hypoelliptic case. Here we use another method based on a calculus in a local chart and which works under the general Hörmander condition; if we compare with [15], we notice that the operator is not necessarily symmetric (there is a drift which is involved in the Hörmander condition). Our method relies on a calculus on the tangent bundle $T(N)$, and the basic tool consists of estimates on real-valued solutions of the heat equation; these estimates can be obtained from Malliavin's calculus; see [22, 5, 20, 6].

By combining our smoothness results with previous existence results, we can deduce the existence of a C^∞ solution to the Dirichlet problem under Hörmander's condition and under a convexity assumption on the subset of N in which the boundary condition g takes its values; however this assumption is weaker when the diffusion is symmetric and N is Riemannian. For instance, if $N = S^d \subset \mathbb{R}^{d+1}$, $d \geq 2$, is the d-dimensional sphere and if g takes its values in

$$S^d_\varepsilon = \left\{ z = (z_1, \dots, z_{d+1}) \in S^d; z_{d+1} > \varepsilon \right\}, \tag{1}$$

then, by applying [16, 24, 2], we obtain the existence of a smooth S^d_ε-valued solution when $0 < \varepsilon < 1$. In the symmetric case, by applying [26], this existence also holds for $\varepsilon = 0$ and under some regularity on g (when g is continuous and $\varepsilon > 0$, the result is also obtained with the analytical method of [15]).

We first introduce the framework and give the main definitions in Section 2. The continuity of harmonic maps is studied in Section 3, and the result concerning their C^∞ smoothness is stated in Section 4. To prove this result, we work out in Section 5 a study of the martingales with values in the tangent bundle $T(N)$, and we complete the proof in Section 6.

2. The framework. Let N be a d-dimensional C^∞ manifold endowed with a C^∞ connection. Then a real-valued function defined on N is said to be convex on a subset of N if it is convex on the geodesic curves of this subset. We will use the following convexity conditions on the subsets of N which are taken from [2].

DEFINITION 1. Let K be a compact subset of N.

(i) We say that K is convex if two points of K are joined by one and only one geodesic in K, and if this geodesic depends smoothly on the two points.

(ii) We say that K has convex geometry if there exist an open neighborhood N_0 of K and a convex function γ on $N_0 \times N_0$ (endowed with the product connection) which is zero on the diagonal and positive outside the diagonal; we say that K has 2-convex geometry if, moreover, $c\delta^2 \le \gamma \le C\delta^2$ on K for some (or equivalently any) Riemannian distance δ.

The connection also enables us to define the notion of N-valued continuous martingales (we will rely on [10] for the stochastic calculus on manifolds); they are the continuous semimartingales Y_t such that

$$F(Y_t) - \frac{1}{2}\int_0^t \operatorname{Hess} F(Y_s)(dY_s, dY_s)$$

is a local martingale for any C^2 real-valued function F; here Hess F denotes the Hessian of F relative to the connection, and $\operatorname{Hess} F(y)$ is therefore a bilinear form on $T_y(N) \times T_y(N)$. The N-valued continuous martingales can also be characterized locally as the continuous adapted processes which are transformed into submartingales by real-valued convex C^2 functions; actually, convex functions transform martingales into local submartingales even when they are not C^2 [12]. Subsequently, since we do not use noncontinuous martingales, we will omit the word "continuous." The definition of martingales can also be extended to processes $(Y_t)_{t>0}$ indexed by positive times; such a process is a martingale if $(Y_{t+\varepsilon})$ is a martingale for any $\varepsilon > 0$; if, moreover, $(Y_t)_{t>0}$ converges almost surely as $t \downarrow 0$, then its extension $(Y_t)_{t\ge 0}$ is a martingale (apply the characterization with convex functions).

Since the Hessian only uses the torsion free part of the connection, it will not be a restriction to suppose that the connection is torsion free.

When N is not compact, we denote by $\overline{N} = N \cup \{\infty\}$ the one-point compactification of N (otherwise $\overline{N} = N$).

Now let us give the assumptions about the state space M and the operator L. The space M is assumed to be a metric separable space which can be decomposed as $M = M_0 \cup \partial M_0$ for a dense open subset M_0. Let Ω be the space of continuous functions from $\mathbb{R}_+$ into M stopped when they quit M_0; let X_t be the canonical process with filtration $\mathcal{F}_t$, and let $(\mathbb{P}^x; x \in M)$ be a family of probabilities on Ω such that $X_0 = x$ almost surely under $\mathbb{P}^x$. We suppose that $(\Omega, \mathcal{F}_t, X_t; \mathbb{P}^x, x \in M)$ is a diffusion with generator L, with infinite lifetime.

DEFINITION 2. Consider a map h: $M \to N$.

(i) The map h is said to be harmonic (with respect to L) if $(h(X_t);\ t \geq 0)$ is an N-valued (continuous) martingale under $\mathbb{P}^x$, for any x in M_0.

(ii) The map h is said to be quasi-harmonic if $(h(X_t);\ t > 0)$ is a $\mathbb{P}^x$-martingale, for any x in M_0.

(iii) The map h is said to be finely continuous at x if $h(X_t)$ converges $\mathbb{P}^x$ almost surely to $h(x)$ as $t \downarrow 0$.

(iv) A map $f : [0, r] \times M \to N$ is said to be a solution of the heat equation if the process $(f(r-t, X_t); 0 \leq t \leq r)$ is a $\mathbb{P}^x$-martingale, for any x in M_0.

Our notion of harmonic map corresponds to the notion of finely harmonic map introduced by [16]; these maps can be obtained by solving the Dirichlet problem with values in a convex compact subset of N with convex geometry; see [16, 24, 2]. The motivation for the more general notion of quasi-harmonic map is [26], where we have checked that the weakly harmonic maps of the classical theory are the quasi-harmonic maps with finite energy; thus quasi-harmonic maps are obtained when one looks for energy minimizing maps. Notice that quasi-harmonic maps can be modified on a polar set. Of course, the distinction between the two definitions disappears for continuous functions; actually, h is harmonic if and only if it is quasi-harmonic and finely continuous on M_0 (apply the above discussion about the extension at $t = 0$ of martingales indexed by $t > 0$).

3. The continuity of harmonic maps. The aim of this section is to obtain a continuity result for harmonic maps on M; the same technique will be applied for the continuity on the interior M_0 (Theorem 1) and on the boundary ∂M_0 (Theorem 2). In the classical case of maps between Riemannian manifolds, one knows that weakly harmonic maps are not necessarily continuous when the dimension m of M is greater than 2. The Hausdorff dimension of the set of singularities of energy minimizing maps is at most $m-3$; see [28]. The theory of [28] also shows that energy minimizing maps are continuous under some conditions on their image, for instance, when they support a strictly convex function. On the other hand, weakly harmonic maps which are nowhere continuous have been found in [27]. If $m = 2$, then all the weakly harmonic maps are continuous [13]; notice, however, that quasi-harmonic maps with infinite energy are not necessarily continuous; consider for instance the function $h(x) = x/|x|$ from $\mathbb{R}^2$ to the circle S^1 (with any value at 0). If X_t is the Brownian motion on $\mathbb{R}^2$, then $(h(X_t))_{t>0}$ is a $\mathbb{P}^x$-martingale on the circle because 0 is polar, so h is quasi-harmonic but not continuous. Here, the continuity will be obtained under a convexity assumption on N. If the harmonic map takes its values in a compact convex subset with convex geometry (for instance, in a regular geodesic ball) and if the diffusion is elliptic, the continuity on M_0 has been proved with a stochastic method (coupling of diffusions) in [16]. Here, we extend the result to nonelliptic diffusions, and the convexity assumption of the following theorem is weaker than the convex geometry (see Proposition 1 below); the method will be applied in Theorem 2 to study the continuity on the boundary ∂M_0.

THEOREM 1. *Suppose that bounded real-valued functions which are harmonic on an open subset of* M_0 *are continuous on this subset. Suppose that there exists a bounded negative convex function* γ_0 *on* N *which converges to* 0 *at infinity*; *suppose moreover that*, *for any points* $y \neq z$ *in* N, *there exists a bounded convex function* γ_{yz} *on* N *satisfying* $\gamma_{yz}(y) < \gamma_{yx}(z)$. *Let* h: $M \to N$ *be a quasi-harmonic map. Then there exists a function which is continuous on* M_0 *and which is equal to* h *except on a polar set*; *in particular*, *if* h *is harmonic*, *then* h *is continuous on* M_0.

Let us first discuss the assumptions. The continuity of bounded real-valued harmonic functions holds, for instance, under the Hörmander condition (defined in Section 4). The existence of the functions γ_{yz} implies that N is not geodesically complete (an infinite geodesic curve cannot support a bounded nonconstant convex function); thus N is generally defined as an open subset of a larger manifold. If γ_0 is a continuous function on a compact manifold, and if N is defined by $N = \{\gamma_0 < 0\}$, then γ_0 satisfies the required assumptions provided it is convex on N. For instance, N can be the subset S^d_ε, $0 \leq \varepsilon < 1$, of the sphere defined in (1); in this case, one can choose for γ_0 the function

$$\gamma_0(z) = \delta(z) + \arcsin \varepsilon - \pi/2 = \arcsin \varepsilon - \arcsin z_d,$$

where δ is the distance to the north pole. In the terminology of [11], the assumption about the existence of γ_{yz} can also be stated by saying that the convex barycentre of a Dirac mass at a point y is reduced to the singleton $\{y\}$; it has been used in [1] in order to prove the continuity of $\mathscr{C}$-martingales (processes which are transformed by convex functions into submartingales). This assumption is satisfied if N has convex geometry; however, Theorem 1 can also be applied to sets which do not have convex geometry, as can be shown from Proposition 1 below. For instance, the manifolds $N = S^d_\varepsilon$, $\varepsilon > 0$, have convex geometry and satisfy therefore the assumption; on the other hand, it is clear that there exist noncontinuous harmonic maps with values in the closed hemisphere $\overline{S}^d_0$; here we consider the intermediate case of the open hemisphere S^d_0 which does not have convex geometry (see [17]), and we can deduce from the following proposition that the condition of Theorem 1 holds in this case.

PROPOSITION 1. *Suppose that* N *supports a function* γ_0 *satisfying the conditions of Theorem* 1. *If the subsets* $\{\gamma_0 < a\}$, $a < 0$, *satisfy the assumption of Theorem* 1 *about the existence of* γ_{yz}, *then* N *also satisfies it.*

REMARK. We are going to construct the convex functions γ_{yz} of Theorem 1 by a probabilistic method; this type of construction has been used in [18] to study the probabilistic interpretation of convex geometry, and we have taken our inspiration from this work.

PROOF OF PROPOSITION 1. Let us fix y in N; we are going to prove the existence of a convex function γ on N such that $\gamma(y) = 0$ and $\gamma(z) > 0$ for $z \neq y$; then the assumption of the theorem is clearly satisfied by putting $\gamma_{yz} = \gamma$ for any z. Let us define

$$\gamma(z) = \inf \mathbb{P}^W[\sigma_y = \infty], \tag{2}$$

where the infimum is taken on N-valued martingales Z_t on the Wiener space $W = C(\mathbb{R}_+, \mathbb{R})$ such that $Z_0 = z$, and

$$\sigma_y = \inf\{t \geq 0;\ Z_t = y\}.$$

Let $(g(v);\ 0 \leq v \leq 1)$ be a geodesic curve in N, denote by w_t the canonical process of W, and for $v \in [0, 1]$, consider the process $g(v + w_t)$ stopped at time

$$\sigma = \inf\{t \geq 0;\ v + w_t \notin [0, 1]\}.$$

This is a martingale and it hits the two points $g(0)$ or $g(1)$ with respective probabilities $1 - v$ and v; then, in (2), by restricting the infimum to martingales coinciding with $g(v + w_t)$ up to σ, we obtain

$$\begin{aligned}\gamma(g(v)) &\leq \inf\{\mathbb{P}^W[\sigma_y = \infty];\ Z_t = g(v + w_t) \text{ for } t \leq \sigma\}\\ &\leq \inf\{\mathbb{P}^W[\inf\{t \geq \sigma;\ Z_t = y\} = \infty];\ Z_t = g(v + w_t) \text{ for } t \leq \sigma\}\\ &= (1 - v)\gamma(g(0)) + v\gamma(g(1)).\end{aligned}$$

Thus γ is convex. It is clear that $\gamma(y) = 0$. Let $z \neq y$ be another point such that $\gamma(z) = 0$, and let us look for a contradiction. The condition $\gamma(z) = 0$ means that we can find martingales Z_t^k such that $Z_0^k = z$ and $\mathbb{P}^W[\sigma_y^k < \infty]$ converges to 1; we can suppose that these martingales are stopped at σ_y^k. Let us use the convex function γ_0; the process $\gamma_0(Z_t^k)$ is a uniformly bounded negative submartingale, and its limit as $t \to \infty$ is equal to $\gamma_0(y)$ with a probability converging to 1 as $k \to \infty$; we deduce that Z_t^k quits $N_1 = \{\gamma_0 < \gamma_0(y)/2\}$ with a probability converging to 0. Thus, by stopping Z_t^k at the exit time of N_1, we obtain $\overline{N}_1$-valued martingales which hit y with probability converging to 1, and which will be again denoted by Z_t^k. But γ_0 is convex and therefore continuous on N, so $\overline{N}_1$ is included in $N_2 = \{\gamma_0 < \gamma_0(y)/3\}$ which satisfies the condition of Theorem 1, and we have a convex function $\overline{\gamma}_{yx}$ on this subset; then $U_t^k = \overline{\gamma}_{yz}(Z_t^k)$ are uniformly bounded submartingales such that

$$\lim_{k\to\infty} \mathbb{P}\Big[U_\infty^k = \overline{\gamma}_{yz}(y) < \overline{\gamma}_{yz}(z) = U_0^k\Big] = 1.$$

This is not possible. □

Theorem 1 will be proved in two steps. First, we will find a finely continuous modification of h; in this step, we will only need the existence of γ_{yz} or γ_{zy} (the existence of a strictly convex function on N is therefore sufficient); thus we will obtain an harmonic modification of h. Second, we will prove that harmonic maps are continuous; in this step, we will need the full assumption on γ_{yz}.

LEMMA 1. *There exists a function which is finely continuous on M_0 and which is equal to h except on a polar set.*

PROOF. We have to prove that $h(X_t)$ has a $\mathbb{P}^x$-almost sure limit as $t \downarrow 0$; then this limit will satisfy the required properties. Recall that $\overline{N}$ is the one-point compactification of N, and let H be the set of accumulation points of $h(X_t)$ as $t \to 0$; this is a random subset of $\overline{N}$. If δ is a distance compatible with the topology of $\overline{N}$, then

$$\delta_H(y) = \delta(y, H) = \lim_{t\to 0} \inf_{0<s\le t} \delta(y, h(X_s))$$

uniformly in $y \in \overline{N}$, so δ_H is a measurable random variable with values in continuous functions on $\overline{N}$ (this means that H is measurable when one puts the Hausdorff topology on closed sets); it is therefore $\mathbb{P}^x$-almost surely deterministic from the zero–one law. The closed set H (the points where this function is 0) is also deterministic; it is not empty (from the compactness of $\overline{N}$). The process $(\gamma_0 \circ h)(X_t)$ is a bounded negative submartingale, so its limit as $t \downarrow 0$ is also negative, and therefore H is included in N. Suppose that H contains two different points y and z; then $(\gamma_{yz} \circ h)(X_t)$ is a bounded submartingale admitting two limit points as $t \downarrow 0$ (the function γ_{yz} is convex and therefore continuous on N); this is not possible, so H has exactly one point, and we can conclude. □

PROOF OF THEOREM 1. We can suppose from Lemma 1 that h is finely continuous, so h is harmonic; we want to prove that h is continuous. Consider a sequence x_k converging to x in M_0, and let us prove that $h(x_k)$ converges to $h(x)$. Let γ be a bounded convex function on N, let $\varepsilon > 0$, let B_r be a family of open neighborhoods of x decreasing to $\{x\}$ as $r \downarrow 0$, and let τ_r be the first exit time of B_r. Either $\tau_r = \infty$ for any r (this means that x is a trap), or $\tau_r \downarrow 0$ as $r \downarrow 0$. The continuity of γ on N and the fine continuity of h imply that

$$\mathbb{E}^x\big[(\gamma \circ h)\big(X_{\tau_r}\big)\big] \le (\gamma \circ h)(x) + \varepsilon$$

in both cases for r small enough, where the variable in the expectation should be understood as the limit of $(\gamma \circ h)(X_t)$ on $\{\tau_r = \infty\}$. The function

$$G\colon z \mapsto \mathbb{E}^z\big[(\gamma \circ h)\big(X_{\tau_r}\big)\big] \tag{3}$$

is real-valued, bounded, and harmonic on B_r, so we deduce from our assumption that G is continuous at x, and therefore

$$\mathbb{E}^{x_k}\big[(\gamma \circ h)\big(X_{\tau_r}\big)\big] \le (\gamma \circ h)(x) + 2\varepsilon$$

for k large enough. Since γ is convex,

$$(\gamma \circ h)(x_k) \le (\gamma \circ h)(x) + 2\varepsilon.$$

Thus

$$\limsup(\gamma \circ h)(x_k) \le (\gamma \circ h)(x)$$

for any bounded convex function γ on N. If we apply this property to the function $\gamma = \gamma_0$, we see that the accumulation points in $\overline{N}$ of $h(x_k)$ are in N; on the other hand, if we apply it to $\gamma = \gamma_{yz}$ for $y = h(x)$ and $z \neq h(x)$, we see that z cannot be an accumulation point; thus $h(x_k)$ converges to $h(x)$. $\square$

The method of Theorem 1 can also be applied to the behavior at the boundary; for instance, one can obtain nontangential convergence as in [16]. In the following result, we study the continuity on M when the value on ∂M_0 is continuous.

THEOREM 2. *Suppose that N satisfies the conditions of Theorem* 1. *Suppose also that for any open subset B of M, if a bounded real-valued function is harmonic on B and if its restriction to $B \cap \partial M_0$ is continuous, then it is continuous on B. Let h: $M \to N$ be a harmonic map, the restriction of which to ∂M_0 is continuous*; *then h is continuous on M.*

PROOF. Let x be a point of the boundary ∂M_0, and let x_k be a sequence of points of M_0 which converges to x; we have to prove that $h(x_k)$ converges to $h(x)$. This is done with the method of Theorem 1 by considering again a family of open neighborhoods B_r decreasing to $\{x\}$; notice that $\tau_r = 0$ under $\mathbb{P}^x$. The function G of (3) is harmonic on B_r, coincides on $B_r \cap \partial M_0$ with the restriction of $\gamma \circ h$ which is continuous, so it is continuous at x. Thus the proof can be worked out similarly. $\square$

4. The smoothness of harmonic maps. We now want to study the C^∞ smoothness on M_0 of continuous harmonic maps; to this end, we have to suppose that the space M and the generator L are smooth. Thus we suppose that $M = M_0 \cup \partial M_0$ is a C^∞ manifold with boundary, and that L is written on M_0 in Hörmander's form

$$L = \Xi_0 + \tfrac{1}{2} \sum_{i=1}^{q} \Xi_i^2 \tag{4}$$

for C^∞ vector fields $\Xi_0, \Xi_1, \dots, \Xi_q$ on M; in this formula, the vector field Ξ_i is identified with the Lie derivative in the direction Ξ_i. The law of the process (X_t) under $\mathbb{P}^x$ can be realized on the Wiener space as the solution of the Stratonovich equation

$$dX_t^x = \Xi_0(X_t^x)\,dt + \sum_{i=1}^{q} \Xi_i(X_t^x) \circ dW_t^i, \qquad X_0^x = x. \tag{5}$$

Let h: $M \to N$ be a map which is C^2 on a neighborhood of a point x of M_0; we can use on N the exponential map and its inverse, and the map

$$h_x\colon z \mapsto \exp_{h(x)}^{-1} h(z)$$

is defined and C^2 on a neighborhood of x; it takes its values in the tangent space $T_{h(x)}(N)$, so L acts on it and we can define

$$L_N h(x) = L h_x(x) \in T_{h(x)}(N). \tag{6}$$

This map is classically called the tension field of h. For instance, if N is a submanifold of $\mathbb{R}^n$, it can be endowed with the induced Riemannian metric, and in this case $L_N h(x)$ is the orthogonal projection of the vector $Lh(x) \in \mathbb{R}^n$ on the tangent space $T_{h(x)}(N)$. Itô's stochastic calculus shows that

$$(F \circ h)(X_t) - \frac{1}{2}\int_0^t \text{Hess } F \circ h(X_s)(dh(X_s), dh(X_s)) - \int_0^t F' \circ h(X_s) L_N h(X_s)\, ds$$

is a local martingale for any C^2 function F, so h is harmonic if and only if $L_N h = 0$ on M_0. Thus a consequence of the C^2 smoothness of harmonic maps will be that these maps are classical solutions of the partial differential equation $L_N h = 0$ on M_0.

If L is symmetric and elliptic, the C^∞ smoothness of continuous weakly harmonic maps is classical; see, for instance, Theorem 8.5.1 of [14]; the result has been extended to the symmetric hypoelliptic case in [15] by using the sub-Riemannian metric associated with L and estimates of [29]. In our probabilistic framework and in the elliptic case, the Lipschitz continuity has been obtained in [19] by a coupling method, and in [3] by a change of probability; moreover, the C^∞ smoothness is deduced in [19] by an analytical argument and in [3] by a probabilistic argument. Here, we consider the general Hörmander condition defined as follows.

DEFINITION 3. Let $\mathscr{L}$ be the Lie algebra of vector fields on M_0 generated by $(\Xi_i;\ 0 \le i \le q)$. We say that the Hörmander condition, or condition (H), holds at $x \in M_0$ if the space of vector fields of $\mathscr{L}$ taken at x is the whole tangent space $T_x(M_0)$; we say that the restricted Hörmander condition, or condition (H'), holds if the same property holds for the Lie algebra generated by $(\Xi_i, [\Xi_0, \Xi_i],\ 1 \le i \le q)$.

The set of points x where (H) or (H') holds is an open subset of M_0. The restricted condition (H') means that the space–time operator $L + \partial/\partial t$ on $\mathbb{R} \times M$ satisfies (H). We are now ready to state the main result.

THEOREM 3. *Suppose that M is a smooth manifold and that L can be written in a Hörmander form* (4) *satisfying (H) on M_0. If h is a continuous harmonic map, then h is C^∞ on M_0.*

As a corollary, continuous solutions of the heat equation are smooth under (H').

Theorem 3 will be proved in several steps. The idea is first to localize the problem, so that one can suppose that h takes its values in a small subset of N. Then we approximate h by differentiable solutions of the heat equation, and we obtain an a priori bound on the first-order derivatives of these approximations; an induction argument shows that higher order derivatives also satisfy a priori bounds, so we can deduce that h is smooth. The derivative of h involves the tangent bundle of N, so the aim of next section is to study martingales on this manifold; results for these martingales can also be found in [23, 2, 4].

5. Martingales on the tangent bundle. We will have to use the existence of N-valued martingales with prescribed limit in a small subset of N. When the probability space is the Wiener space, this problem has been studied in [16, 24] (at least when N is a Riemannian manifold; see also [25] for more general connections and martingales with jumps on the Wiener–Poisson space). It appears that the existence holds on more general probability spaces; it is sufficient to suppose that the real martingales are continuous; this result is much more difficult and is due to [2]; let us recall it. The notions of convexity have been introduced in Definition 1.

THEOREM 4. *Suppose that all the real martingales are continuous. Let K be a compact convex subset of N with convex geometry. Let Y_∞ be a random variable with values in K. Then there exists a unique martingale Y_t with values in K and converging to Y_∞.*

If now K is not compact but is the increasing limit of compact sets satisfying the conditions of Theorem 4, then the existence can be obtained under some conditions by the results of [7]; for instance, if $K = N$ is a Cartan–Hadamard manifold with uniformly negative curvature, the condition is the integrability on $\{\tau < \infty\}$ of the distance of Y_∞ to a fixed point of N (as in the Euclidean case).

The noncompact manifold which is needed in our study is the tangent bundle $T(N)$ endowed with the complete lift of the connection of N, and this manifold apparently does not satisfy the conditions of [7]; thus we work out another proof for the existence of martingales. Let N' be a finite-dimensional vector bundle over the manifold N [we will choose $N' = T(N)$ later], and denote by π the projection of N' onto N; we suppose that the manifolds N and N' are endowed with connections so that π is affine (it maps geodesic curves of N' to geodesic curves of N); in particular,

$$N'_{(2)} = \{(z_1, z_2) \in N' \times N';\ \pi(z_1) = \pi(z_2)\}$$

is a totally geodesic submanifold of $N' \times N'$ (geodesic curves which are tangent to $N'_{(2)}$ lie in $N'_{(2)}$), and each fiber $\pi^{-1}(y)$ is totally geodesic in N; we also suppose that the maps

$$\Lambda_{\alpha,\beta}\colon N'_{(2)} \to N', \qquad (z_1, z_2) \mapsto \alpha z_1 + \beta z_2$$

are affine; in particular, the set of zero vectors of N' (which can be identified with N) is totally geodesic in N', and $\pi^{-1}(y)$ is endowed with its canonical flat connection. Let K be a compact subset of N, and let $K' = \pi^{-1}(K)$.

Consider a continuous map $|\cdot|$ from N' into $\mathbb{R}_+$ which is a norm on each $\pi^{-1}(y)$; one has several choices for this map, but all these norms are equivalent on K'. A K'-valued variable Z is said to be in L^r if $|Z|^r$ is integrable; this notion does not depend on the choice of the norm, nor does the following one.

DEFINITION 4. A K'-valued martingale Z_t is said to be in $\mathscr{U}^r$, $r \geq 1$, if the set of variables $|Z_\sigma|^r$, as σ ranges over all optional times, is uniformly integrable.

We now prove an existence result on the vector bundle N' under some convexity assumptions.

LEMMA 2. *Suppose that all the real martingales are continuous* (*as in Theorem* 4). *Suppose that there exists a choice of the norm* $|\cdot|$ *such that* $|\cdot|^r$ *is convex on* K' *for some* $r \geq 1$. *Suppose also that the subsets* $\{z \in K'; |z| \leq C\}$ *of* N' *are convex and have convex geometry. If* Z_∞ *is a variable in* L^r *with values in* K', *then there exists a unique* K'-*valued martingale* Z_t *of* $\mathscr{U}^r$ *converging to* Z_∞.

REMARK. The map $|\cdot|^p$ is also convex for $p > r$. Thus, if Z_∞ is in L^p for some $p > r$, then the martingale Z_t is in $\mathscr{U}^p$; by applying the Doob inequality to the submartingale $|Z_t|^r$, we also deduce that $\sup_t |Z_t|$ is in L^p.

PROOF OF LEMMA 2. Let $Y_\infty = \pi(Z_\infty)$; from Theorem 4, there exists a unique K-valued martingale Y_t converging to Y_∞, and Z_t has to be above Y_t. If Z_∞ is bounded, then the existence of a bounded Z_t also follows from our assumptions and Theorem 4. If now Z_∞ is only in L^r, one considers bounded variables Z_∞^k above Y_∞ such that Z_∞^k converges in L^r to Z_∞; then the bounded martingales Z_t^k are above Y_t, the process $Z_t^k - Z_t^j$ is also a martingale, so $|Z_t^k - Z_i^j|^r$ is a submartingale and

$$\mathbb{P}\Big[\sup_t |Z_t^k - Z_t^j|^r \geq \lambda\Big] \leq \frac{1}{\lambda}\mathbb{E}|Z_\infty^k - Z_\infty^j|^r.$$

Thus $(Z_t^k;\ 0 \leq t \leq \infty)$ is a Cauchy sequence for the uniform convergence in probability, and has a limit $(Z_t;\ 0 \leq t \leq \infty)$; this process is a martingale (Theorem (4.43) of [10]) with limit Z_∞. On the other hand, for t fixed in $[0, \infty]$, the variables Z_t^k are a Cauchy sequence in L^r, so Z_t^k converges to Z_t in L^r, and therefore $|Z_t^k|^r$ converges to $|Z_t|^r$ in L^1; this implies that the process $|Z_t|^r;\ 0 \leq t \leq \infty$ is a nonnegative submartingale, so the variables $|Z_\sigma|^r$, σ optional time, are uniformly integrable. Thus Z_t is a martingale of $\mathscr{U}^r$. The uniqueness of this martingale follows from the uniqueness of $Y_t = \pi(Z_t)$ and from the convexity of $|\cdot|^r$; if indeed Z_t' is another solution, then $|Z_t - Z_t'|^r$ should be a uniformly integrable submartingale converging to 0. □

We now consider the particular case where N' is the tangent bundle $T(N)$. It will be sufficient for us to consider the case where N is $\mathbb{R}^d$ is endowed with a non-Euclidean connection (this means that N can be described with only one chart); then N' can be identified with $\mathbb{R}^d \times \mathbb{R}^d$. On the manifold $T(N)$ one can define the complete lift of the connection of N (described in [23, 2, 4]. It is called the geodesic connection in [23]); the geodesic curves for this connection are given by the Jacobi fields along geodesic curves of N. If K is a compact subset of N, we consider as previously $K' = T(K) = \pi^{-1}(K)$ which is identified with $K \times \mathbb{R}^d$. If ϕ is a C^2 function on $T(K)$, its Hessian at a point can be viewed as a symmetric bilinear form on $(\mathbb{R}^d \times \mathbb{R}^d)^2$, and ϕ is said to be uniformly convex if its Hessian is uniformly elliptic; this notion does not depend on the choice of the coordinates (since K is compact). If y_0 is a point of N, we now obtain some geometrical properties of the small neighborhoods of y_0.

LEMMA 3. *Fix y_0 in $N = \mathbb{R}^d$. There exists a smooth function ψ on N such that $\psi(y_0) = 0$, $\psi(y) > 0$ for $y \neq y_0$, and for η small enough, $K = \{\psi \leq \eta\}$ is a compact neighborhood of y_0 satisfying the following properties*:

(i) *The set K is convex and has a 2-convex geometry given by a C^2 function γ.*

(ii) *Compact subsets of $T(K)$ have 2-convex geometry.*

(iii) *There exist nonnegative quadratic forms Q_y on $T_y(N)$ such that*

$$\phi(z) = Q_{\pi(z)}(z), \qquad \phi_1(z) = \phi(z) + (\psi \circ \pi)(z)$$

are, respectively, convex and uniformly convex C^∞ functions on $T(K)$.

REMARK. The uniform convexity of ϕ_1 implies that ψ is strictly convex and that Q_y is uniformly elliptic.

PROOF. By using coordinates $(y, y') \in \mathbb{R}^d \times \mathbb{R}^d$ for $T(N)$ such that $y_0 = 0$, we introduce the functions

$$\psi(y) = |y|^2/2, \qquad \phi(y, y') = (\varepsilon + |y|^2)|y'|^2/2, \qquad \phi_1(y, y') = \psi(y) + \phi(y, y')$$

(the definition of ϕ is adapted from the function used in Lemma 4.59 of [10]). We are going to prove that $K = \{\psi \leq \eta\}$ satisfies the conditions of the lemma for $\eta \leq \varepsilon^2/2$ and ε small enough. The convexity of K will follow from the strict convexity of ψ (for ε small); the 2-convexity for ε small is a consequence of Lemma 4.59 of [10]. The zero vector of $Ty_0(N)$ has a neighborhood with 2-convex geometry, so if ε is small, the set of zero vectors of $T(K)$ also has a neighborhood with 2-convex geometry; since all the compact subsets of $T(K)$ can be embedded in this neighborhood with an affine map $(y, y') \mapsto (y, \rho y')$ (for ρ small), we deduce that these compact subsets also have 2-convex geometry. Thus we only have to study the convexity of ϕ and ϕ_1. The connection on $T(N)$ can be defined by Christoffel symbols Γ^i_{jk}, $1 \leq i, j, k \leq d$, which are symmetric in (j, k) (the connection is torsion free). On $T(N) = \mathbb{R}^d \times \mathbb{R}^d$, we index the first d components by letters $i, j, \ldots$, and the subsequent corresponding d components by the same letter with a bar (so that for instance $\bar{i} = i + d$). Let us recall the Christoffel symbols on the tangent space (the connection is also torsion free) at a point (y, y') (see for instance [2]). They are obtained by writing the equation satisfied by the Jacobi fields, which are the geodesic curves for this connection. The symbols $\Gamma^i_{jk}(y, y')$ are the original symbols $\Gamma^i_{jk}(y)$, and the other symbols are given by

$$\Gamma^{\bar{i}}_{jk} = \sum_l y'_l \partial_l \Gamma^i_{jk}, \qquad \Gamma^{\bar{i}}_{\bar{j}k} = \Gamma^i_{jk}, \qquad \Gamma^i_{\bar{j}\,k} = \Gamma^i_{\bar{j}\,\bar{k}} = \Gamma^{\bar{i}}_{\bar{j}\,\bar{k}} = 0, \tag{7}$$

where ∂_l denotes the derivative with respect to the lth component of y. If now we consider a C^2 function on the tangent bundle, its Hessian can be expressed in terms of its first- and second-order derivatives D_i and D_{ij} computed in the Euclidean space; we have

$$\begin{aligned}
\mathrm{Hess}_{jk} &= D_{jk} - \sum_i \Gamma^i_{jk} D_i - \sum_i \Gamma^{\bar{i}}_{jk} D_{\bar{i}} \\
&= D_{jk} - \sum_i \Gamma^i_{jk} D_i - \sum_{i,l} y'_l \partial_l \Gamma^i_{jk} D_{\bar{i}}, \\
\mathrm{Hess}_{\bar{j}k} &= D_{\bar{j}k} - \sum_i \Gamma^{\bar{i}}_{\bar{j}k} D_{\bar{i}} = D_{\bar{j}k} - \sum_i \Gamma^i_{jk} D_{\bar{i}}, \\
\mathrm{Hess}_{\bar{j}\,\bar{k}} &= D_{\bar{j}\,\bar{k}}.
\end{aligned}$$

Thus easy calculations yield

$$\begin{aligned}
\mathrm{Hess}_{jk}\, \phi(y, y') &= |y'|^2\Big(\delta_{jk} - \sum_i y_i \Gamma^i_{jk}(y)\Big) - \sum_{i,l} y'_l y'_i (\varepsilon + |y|^2) \partial_l \Gamma^i_{jk}(y), \\
\mathrm{Hess}_{\bar{j}k}\, \phi(y, y') &= 2 y_k y'_j - \sum_i \Gamma^i_{jk}(y) y'_i (\varepsilon + |y|^2), \\
\mathrm{Hess}_{\bar{j}\,\bar{k}}\, \phi(y, y') &= (\varepsilon + |y|^2) \delta_{jk}.
\end{aligned}$$

We deduce on $T(K)$ the estimates on the symmetric submatrices

$$\begin{aligned}
\Big(\mathrm{Hess}_{jk}\, \phi(y, y')\Big) &\geq (1 - C\varepsilon)|y'|^2 I, \\
\Big(\mathrm{Hess}_{\bar{j}\,\bar{k}}\, \phi(y, y')\Big) &\geq \varepsilon I
\end{aligned}$$

and the other terms of the matrix are estimated by

$$\Big|\mathrm{Hess}_{\bar{j}k}\, \phi(y, y')\Big| \leq C\varepsilon |y'|.$$

If $U = (y, y', u, u')$ is a vector of $T(T(K))$, we have

$$\begin{aligned}
\Big(\mathrm{Hess}\, \phi(y, y')\Big)(U, U) &\geq (1 - C\varepsilon)|y'|^2 |u|^2 + \varepsilon |u'|^2 - C\varepsilon |y'|\, |u|\, |u'| \\
&\geq (1 - C\varepsilon - C^2\varepsilon/2)|y'|^2 |u|^2 + \varepsilon |u'|^2/2.
\end{aligned}$$

If we choose ε small enough, this Hessian is positive semidefinite, so ϕ is convex. The Hessian of ϕ_1 is obtained by adding

$$\big(\mathrm{Hess}_{jk}\,\psi(y)\big) = \big(\delta_{jk} - \sum_i y_i \Gamma^i_{jk}(y)\big) \geq (1 - C\varepsilon)I,$$

so

$$\begin{aligned}\big(\mathrm{Hess}\,\phi_1(y, y')\big)(U, U) &\geq \big(\mathrm{Hess}\,\phi(y, y')\big)(U, U) + (1 - C\varepsilon)|u|^2 \\ &\geq (1 - C\varepsilon)|u|^2 + \varepsilon|u'|^2/2.\end{aligned}$$

We deduce that $\mathrm{Hess}\,\phi_1$ is uniformly elliptic. □

In particular, the subset $K' = T(K)$ of $T(N)$ and the norm $|z| = \phi(z)^{1/2}$ satisfy the assumptions of Lemma 2 for $r \geq 2$, so we deduce the existence of martingales of $\mathscr{U}^r$ (Definition 4) converging to a variable of L^r.

LEMMA 4. *Let $K = \{\psi \leq \eta\}$ be a set satisfying the properties of Lemma* 3.

(i) *There exist positive constants c and C such that for any K-valued martingale $(Y_t; 0 \leq t \leq \infty)$,*

$$\mathbb{E} \exp c\langle Y\rangle_\infty \leq C,$$

where $\langle Y\rangle$ is the quadratic variation of Y computed in $\mathbb{R}^d$ (or for another Riemannian metric on K). In particular, the moments of $\langle Y\rangle_\infty$ are uniformly dominated.

(ii) *Let Z_∞ be a $T(K)$-valued variable of L^r, $r \geq 2$. Then there exists a unique $T(K)$-valued martingale $(Z_t, 0 \leq t \leq \infty)$ of $\mathscr{U}^r$ with limit Z_∞. Moreover, the quadratic variation $\langle Z\rangle_\infty$ computed in $\mathbb{R}^d \times \mathbb{R}^d$ is integrable.*

PROOF. Suppose $\eta \leq 1/2$ so that $1 - \psi$ takes values in $[1/2, 1]$. It follows from the strict convexity of ψ that the process $(1 - \psi(Y)) \exp c\langle Y\rangle$ is for c small a positive supermartingale; thus its expectation is bounded and we obtain the first part of the lemma (this proof is taken from Proposition 2.1.2 of [24]). In the second part, the existence and uniqueness of Z_t are a consequence of Lemma 2. Finally, it follows from the strict convexity of ϕ_1 that the expectation of $\langle Z\rangle_\infty$ is dominated by the expectation of $\phi_1(Z_\infty)$. □

REMARK. We have proved that if a martingale is in $\mathscr{U}^2$, then its quadratic variation is integrable. The converse is probably false, and the integrability of the quadratic variation depends on the choice of the coordinates on K.

Let $T^m(N)$ be the vector bundle with fibers $T^m_y(N)$ which are the spaces of linear maps from $\mathbb{R}^m$ into $T_y(N)$. By considering the canonical basis of $\mathbb{R}^m$, the space $T^m_y(N)$ becomes isomorphic to the product $(T_y(N))^m$, so $T^m(N)$ can be identified with the totally geodesic submanifold of $(T(N))^m$ consisting of $(z^1, \dots, z^m)$ having the same projection on N. Since K is a part of $N = \mathbb{R}^d$, elements of $T^m(K)$ are represented by a point of K and a $d \times m$ matrix. The properties of $T(K)$ can be extended to $T^m(K)$. The functions

$$\phi^m(z^1, \dots, z^m) = \sum_{i=1}^m \phi(z^i), \qquad \phi_1^m(z^1, \dots, z^m) = \sum_{i=1}^m \phi_1(z^i) \tag{8}$$

are, respectively, convex and uniformly convex on $T^m(K)$; the compact subsets of $T^m(K)$ have 2-convex geometry, and one can construct $T^m(K)$-valued martingales of $\mathscr{U}^r$ as in Lemma 4. The linear group $GL(m)$ acts on the right on each fiber $T_y^m(N)$. If f is a C^1 map from $\mathbb{R}^m$ into N, then its derivative f' can be viewed as a $T^m(N)$-valued function.

6. Proof of Theorem 3. Since we want to prove a local condition and h is continuous, we can suppose that M_0 is a small open subset of $\mathbb{R}^m$, that the vector fields Ξ_i are C^∞ with compact support in $\mathbb{R}^m$ and that $N = \mathbb{R}^d$ endowed with a non-Euclidean connection. We now give two preliminary results linked with the Hörmander and Malliavin calculus. We first verify that the Hörmander condition (H) can be reduced to the restricted condition (H'), so that we can assume (H') subsequently; this result is due to [5], and we adopt here the presentation of [6].

LEMMA 5. *Let ρ be a smooth positive function on $\mathbb{R}$, and define on $\mathbb{R}^m \times \mathbb{R}$ the operator*

$$\widetilde{L}f(x,u) = \rho^2(u)Lf(x,u) + \frac{1}{2}\rho^2(u)\frac{\partial^2 f}{\partial u^2}(x,u) + \frac{1}{2}\rho\rho'(u)\frac{\partial f}{\partial u}(x,u),$$

where L acts on the x component; it is associated with the vector fields

$$\widetilde{\Xi}_0(x,u) = \rho^2(u)\Xi_0(x), \quad \widetilde{\Xi}_i(x,u) = \rho(u)\Xi_i(x), \quad \widetilde{\Xi}_{q+1}(x,u) = \rho(u)\partial/\partial u.$$

If L satisfies (H) at x and if $\rho'(u) \neq 0$, then $\widetilde{L}$ satisfies (H') at (x,u). Moreover, the function h is harmonic for L if and only if $(x,u) \mapsto h(x)$ is harmonic for $\widetilde{L}$.

The first part of this result is simply proved by computing the Lie brackets of the vector fields $\widetilde{\Xi}_i$; the second part comes from the probabilistic interpretation of $L \to \widetilde{L}$ as a change of time on the diffusion X_t. We will also use the following gradient estimate for real-valued solutions of the heat equation.

LEMMA 6. *Suppose that the condition (H') holds on an open subset M_1 of $\mathbb{R}^m$ and let $\overline{M}_2$ be a compact subset of M_1. Consider a real-valued bounded function f which is a solution of the heat equation with initial value f_0. Then $f(t,\cdot)$ is C^∞ on M_1, and there exist a C and an l which do not depend on $0 < t \le 1$, f and x in $\overline{M}_2$ such that*

$$|f'(t,x)| \le \frac{C}{t^l} \sup\{|f_0(z)|; z \in M_1\}.$$

This lemma is of course the fundamental smoothness result on which Theorem 3 is based. The smoothness of f is classical from Hörmander's theorem, and a probabilistic proof can be worked out from Malliavin's calculus (see [5, 20, 6]); the estimate on the derivative of f can be obtained from estimates on the derivative of the probability transition density as they can also be found in [20] (see Theorem 2.18 of [21] when (H') holds uniformly on $M_1 = \mathbb{R}^m$).

Let us now enter the proof of Theorem 3. Fix x_0 in $M_0 \subset \mathbb{R}^m$ and $y_0 = h(x_0)$ in $N = \mathbb{R}^d$; for any J, we want to prove that h is C^J in a small neighborhood of x_0. Since h is continuous and M_0 can be taken small, we can suppose that the image $h(M_0)$ is in a small compact neighborhood K of y_0.

Let $X_t = X_t^x$ be the solution of the Stratonovich equation (5) (we will often omit the superscript x). One can choose a version which is a C^∞ diffeomorphism with respect to x, and we denote by X'_t the Jacobian matrix of X_t^x with respect to x; then X'_t takes its values in the linear group $GL(m)$, and the derivative of $f(X_t)$ is $f'(X_t)X'_t$. We also know that the moments of X'_t and of higher order derivatives are bounded for $t \leq 1$.

LEMMA 7. *Choose $K = \{\psi \leq \eta\}$ so that $K_0 = \{\psi \leq 2\eta\}$ satisfies the conditions of Lemma 3. Let f_1 be a K-valued C^∞ function on $\mathbb{R}^m$ which is constant outside a compact set. Then there exist a unique K-valued continuous martingale $Y_t = f(t, X_t)$, $0 \leq t \leq 1$, with terminal value $f_1(X_1)$, so $f(1-t, x)$ is solution of the heat equation with initial value f_1. Moreover, the map f is differentiable with respect to x and its derivative f' is bounded. For any $r \geq 2$, the process $f'(t, X_t)X'_t$ is the unique $T^m(K)$-valued martingale of $\mathscr{U}^r$ with final value $f'_1(X_1)X'_1$.*

PROOF. The assumptions which were required in Lemma 3 imply that any K-valued variable is the limit of a unique K-valued martingale (see Theorem 4). Thus we obtain Y_t and we deduce from a standard argument using the Markov property (see for instance [16]) that it is of the form $Y_t = f(t, X_t)$ [let $f(t, x)$ be the initial value of the martingale with terminal value $f_1(X_{1-t})$]. Moreover, if γ is a function describing the 2-convex geometry of K and by using the coordinates of $N = \mathbb{R}^d$,

$$
\begin{aligned}
|f(0, x) - f(0, \bar{x})| &\leq C\gamma(f(0, x), f(0, \bar{x}))^{1/2} \leq C\big(\mathbb{E}\gamma(f_1(X_1^x), f_1(X_1^{\bar{x}}))\big)^{1/2} \\
&\leq C'\|f_1(X_1^x) - f_1(X_1^{\bar{x}})\|_2 \leq C''|\bar{x} - x|,
\end{aligned}
$$

so $f(0, \cdot)$ is Lipschitz; by considering the diffusion $(X_s; s \geq t)$ with initial value x at time t, one proves similarly that $f(t, x)$ is uniformly Lipschitz with respect to x. On the other hand, from Lemma 4, there also exists a unique martingale Z_t of $\mathscr{U}^r$ above Y_t with final value $Z_1 = f'_1(X_1)X'_1$. Suppose that we have proved that $Y_0 = Y_0^x$ is differentiable with respect to x with derivative

$$
\partial Y_0^x / \partial x = Z_0^x \tag{9}
$$

Then we obtain the differentiability of $f(0,\cdot)$; by considering the diffusion $(X_s; s \geq t)$, one proves similarly that $f(t,\cdot)$ is differentiable, and the boundedness of $f'(t,\cdot)$ follows from the Lipschitz continuity of $f(t,\cdot)$. By working conditionally on $\mathscr{F}_t$, one also checks from (9) that the derivative of Y_t^x is Z_t^x; thus the derivative of $Y_t^x = f(t, X_t^x)$ is Z_t for the left-hand side and $f'(t, X_t)X_t'$ for the right-hand side, so these two processes are equal and the lemma is proved. So we only have to differentiate Y_0^x at a fixed x. The calculus has been worked out in [2] in the case when Z is bounded and x real; we propose here another method. By using the coordinates on $T^m(K)$, the process Z_t is written as $(Y_t, \overline{Y}_t)$, and $\overline{Y}_t$ is a matrix; by considering the columns $\overline{Y}_t^p, 1 \leq p \leq m$, each process $(Y_t, \overline{Y}_t^p)$ is a $T(K)$-valued martingale, and this property can be characterized by means of the coordinates; from the form (7) of the Christoffel symbols and by using the convention of summation over repeated indices, we obtain the relations

$$
\begin{aligned}
dY_t^i &\overset{(\mathrm{m})}{=} -\tfrac{1}{2}\Gamma^i_{jk}(Y_t)d\langle Y^j, Y^k\rangle_t,\\
d\overline{Y}_t^{ip} &\overset{(\mathrm{m})}{=} -\tfrac{1}{2}\partial_l\Gamma^i_{jk}(Y_t)\overline{Y}_t^{lp}d\langle Y^j, y^k\rangle_t - \Gamma^i_{jk}(Y_t)d\langle Y^j, \overline{Y}^{kp}\rangle_t,
\end{aligned}
\tag{10}
$$

where $\overset{(\mathrm{m})}{=}$ means equality modulo a local martingale. Let Λ be a smooth map from $T(K) \subset \mathbb{R}^d \times \mathbb{R}^d$ into $K_0 \subset \mathbb{R}^d$ such that $\Lambda(y, y') = y + y'$ when $|y'| \leq c$ (c small enough). We consider the process $Y_t^\rho = \Lambda(Y_t, \overline{Y}_t\rho)$ where ρ is a small vector of $\mathbb{R}^m$; let

$$
\sigma = \sigma(x,\rho) = \inf\Big\{t \geq 0; |\overline{Y}_t| \geq |\rho|^{-1/4} \quad \text{or} \quad \big|X_t^{x+\rho} - X_t^x\big| \geq |\rho|^{3/4}\Big\} \wedge 1.
$$

The martingale Z is in $\mathscr{U}^2$, so $\overline{Y}_\sigma$ is bounded in L^2 uniformly in ρ, and since

$$
\Big\|\sup_{t\leq 1}|X_t^{x+\rho} - X_t^x|\Big\|_2 = O(|\rho|),
$$

we deduce

$$
\mathbb{P}[\sigma < 1] = O\big(|\rho|^{1/2}\big).
\tag{11}
$$

In particular, one has $\sigma > 0$ for ρ small enough. Up to σ, one has $Y_t^\rho = Y_t + \overline{Y}_t\rho$, so

$$
\begin{aligned}
dY_t^{\rho,i} &\overset{(\mathrm{m})}{=} -\tfrac{1}{2}\Gamma^i_{jk}(Y_t)d\langle Y^j, Y^k\rangle_t - \tfrac{1}{2}\rho_p\partial_l\Gamma^i_{jk}(Y_t)\overline{Y}_t^{lp}d\langle Y^j, Y^k\rangle_t\\
&\qquad - \rho_p\Gamma^i_{jk}(Y_t)d\langle Y^j, \overline{Y}^{kp}\rangle_t\\
&\overset{(\mathrm{m})}{=} -\tfrac{1}{2}\Gamma^i_{jk}\big(Y_t^\rho\big)d\langle Y^{\rho,j}, Y^{\rho,k}\rangle_t + dV_t^i
\end{aligned}
\tag{12}
$$

with

$$
\begin{aligned}
\int_0^\sigma |dV_t| &\leq C\Big(|\rho|^2\sup_{t\leq\sigma}|\overline{Y}_t|^2\langle Y\rangle_1 + |\rho|^2\sup_{t\leq\sigma}|\overline{Y}_t|\langle Y\rangle_1^{1/2}\langle\overline{Y}\rangle_1^{1/2} + |\rho|^2\langle\overline{Y}\rangle_1\Big)\\
&\leq C\Big(|\rho|^{3/2}\langle Y\rangle_1 + |\rho|^{7/4}\langle Y\rangle_1^{1/2}\langle\overline{Y}\rangle_1^{1/2} + |\rho|^2\langle\overline{Y}\rangle_1\Big).
\end{aligned}
$$

Since $\langle Y\rangle_1$ and $\langle\overline{Y}\rangle_1$ are integrable (Lemma 4),

$$\mathbb{E}\int_0^\sigma |dV_t| = O(|\rho|^{3/2}). \tag{13}$$

The interpretation of the decomposition (12) is that the infinitesimal increment dV_t can be viewed as the finite variation part of the K_0-valued semimartingale Y_t^ρ; it is the term which prevents Y_t^ρ from being a martingale. Now we compare this process $Y_t^\rho = Y_t^{x,\rho}$ with the martingale $Y_t^{x+\rho}$; the process $(Y_t^{x,\rho}, Y_t^{x+\rho})$ is a semimartingale on $N \times N$; if $\gamma(y_1, y_2)$ is a C^2 function describing the 2-convex geometry of K_0 and if γ_1' is its derivative with respect to y_1, then

$$\gamma\big(Y_t^{x,\rho}, Y_t^{x+\rho}\big) = \int_0^t \gamma_1'\big(Y_t^{x,\rho}, Y_t^{x+\rho}\big)dV_s + \text{a submartingale}$$

up to σ. On $\{t \le \sigma\}$, one has

$$\begin{aligned}\big|\gamma_1'\big(Y_t^{x,\rho}, Y_t^{x+\rho}\big)\big| &\le C\big|Y_t^{x,\rho} - Y_t^{x+\rho}\big| \le C\Big(\big|Y_t^{x,\rho} - Y_t^x\big| + \big|Y_t^{x+\rho} - Y_t^x\big|\Big) \\ &\le C\Big(|\overline{Y}_t^x|\,|\rho| + |f(t, X_t^{x+\rho}) - f(t, X_t^x)|\Big) \le C'|\rho|^{3/4}\end{aligned}$$

from the definition of σ and the Lipschitz continuity of f. Thus, from (13) and (11),

$$\begin{aligned}\gamma\big(Y_0^{x,\rho}, Y_0^{x+\rho}\big) &\le \mathbb{E}\Big[\gamma\big(Y_\sigma^{x,\rho}, Y_\sigma^{x+\rho}\big)\Big] + O(|\rho|^{9/4}) \\ &\le \mathbb{E}\Big[\gamma\big(Y_\sigma^{x,\rho}, Y_\sigma^{x+\rho}\big)^2\Big]^{1/2}\mathbb{P}[\sigma < 1]^{1/2} \\ &\quad + \mathbb{E}\Big[\gamma\big(Y_1^{x,\rho}, Y_1^{x+\rho}\big)1_{\{\sigma=1\}}\Big] + O(|\rho|^{9/4}) \\ &\le \mathbb{E}\big[\big|Y_\sigma^{x+\rho} - Y_\sigma^x - \overline{Y}_\sigma^x\rho\big|^4\big]^{1/2}O(|\rho|^{1/4}) \\ &\quad + \mathbb{E}\big|f_1(X_1^{x+\rho}) - f_1(X_1^x) - f_1'(X_1^x)(X_1^x)'\rho\big|^2 + O(|\rho|^{9/4}) \\ &\le \mathbb{E}\big[|f(\sigma, X_\sigma^{x+\rho}) - f(\sigma, X_\sigma^x)\big|^4\big]^{1/2}O(|\rho|^{1/4}) \\ &\quad + \mathbb{E}\big[|\overline{Y}_\sigma^x|^4\big]^{1/2}O(|\rho|^{9/4}) \\ &\quad + \mathbb{E}\big|f_1(X_1^{x+\rho}) - f_1(X_1^x) - f_1'(X_1^x)(X_1^x)'\rho\big|^2 + O(|\rho|^{9/4}).\end{aligned}$$

The function $f(t, x)$ is Lipschitz continuous with respect to x, and the process (X_t^x) is also Lipschitz continuous for the norm $\|\sup_t|\cdot|\|_4$, so the first expectation of the last expression is $O(|\rho|^4)$; the derivatives of f_1 are bounded, and the derivatives of $x \mapsto X_1^x$ have bounded moments, so the last expectation is also $O(|\rho|^4)$; finally, the martingale (Z_t) is in $\mathscr{U}^4$, so the expectation of $|\overline{Y}_\sigma^x|^4$ is bounded. Thus

$$\big|f(0, x+\rho) - f(0, x) - \overline{Y}_0^x\rho\big| \le C\gamma\big(Y_0^{x,\rho}, Y_0^{x+\rho}\big)^{1/2} = O(|\rho|^{9/8}).$$

This proves that $\overline{Y}_0^x$ is the derivative of $f(0, \cdot)$ at x. □

LEMMA 8. *Consider the maps f of Lemma 7 with values in $K = \{\psi \leq \eta\}$. Suppose that (H') holds at any x of an open M_1 and let $\overline{M}_2$ be a compact subset of M_1. Then, for $\eta \leq \eta_0$ small enough, the derivative satisfies for any f and $x \in \overline{M}_2$ an a priori bound $|f'(0,x)| \leq C$; the bound η_0 depends only on the geometry of N.*

PROOF. We know from Lemma 7 that f' is a bounded function and we are looking for an upper bound on $\overline{M}_2$ which does not depend on f. Let $\xi \in \overline{M}_2$ be a point where we want to estimate f'. We consider the process X_t^x with initial value $x = \xi$, and the martingale

$$(Y_t, \overline{Y}_t) = \big(f(t, X_t), f'(t, X_t)X_t'\big)$$

with values in $T^m(K)$. By writing $\overline{Y}_t$ as a matrix, we obtain as in (10) the real local martingales

$$\Lambda_t^{ip} = \overline{Y}_t^{ip} + \tfrac{1}{2}\int_0^t \partial_l \Gamma^i_{jk}(Y_s)\overline{Y}_s^{lp}\, d\langle Y^j, Y^k\rangle_s + \int_0^t \Gamma^i_{jk}(Y_s)\, d\big\langle Y^j, \overline{Y}^{kp}\big\rangle_s \tag{14}$$

which are actually martingales. The interesting fact is that the right-hand side is linear in $\overline{Y}$. Consider the time

$$\sigma = \sigma_r = \inf\big\{t \geq 0; |X_t - \xi| \geq r^{1/4}/2\big\} \wedge r$$

for r small. We deduce from the martingale property of Λ_t in (14) that

$$\overline{Y}_0 = f'(0, X_0)X_0' = f'(0, \xi)$$

is the linear expectation of Λ_σ. We choose a localization function ϕ; this is a C^∞ function from $\mathbb{R}^m$ into $[0,1]$ such that $\phi(x) = 1$ for $|x| \leq 1/2$ and $\phi(x) = 0$ for $|x| \geq 1$; we define $\phi_r(x) = \phi((x - \xi)/r^{1/4})$. We have

$$\begin{aligned}
|\mathbb{E}\overline{Y}_\sigma| &\leq \big|\mathbb{E}\big[\phi_r(X_r)\overline{Y}_r \mathbf{1}_{\{\sigma = r\}}\big]\big| + \big|\mathbb{E}\big[\overline{Y}_\sigma 1_{\{\sigma < r\}}\big]\big| \\
&\leq \big|\mathbb{E}\big[\phi_r(X_r)\overline{Y}_r\big]\big| + \mathbb{E}\big[\big(\big|\overline{Y}_\sigma\big| + |\phi_r(X_r)\overline{Y}_r|\big)1_{\{\sigma<r\}}\big] \\
&\leq \left|\frac{\partial}{\partial x}\mathbb{E}\big[\phi_r(X_r^x)f(r, X_r^x)\big]\right| + \mathbb{E}\big[|Y_r|\,|\phi_r'(X_r)X_r'|\big] \\
&\qquad + \Big\|\sup_{t \leq r}|X_t'|\Big\|_2 \mathbb{P}[\sigma < r]^{1/2} \sup\big\{|f'(t,x)|; t \leq r, |x - \xi| \leq r^{1/4}\big\},
\end{aligned}$$

where the first derivative is taken at $x = \xi$. This is also the derivative at $(t, x) = (r, \xi)$ of the function

$$(t, x) \mapsto \mathbb{E}\big[\phi_r\big(X_t^x\big)f\big(t, X_t^x\big)\big].$$

The components of this function are real-valued solutions of the heat equation, so Lemma 6 shows that its derivative is bounded by C/r^l for some l. The second term involves the derivative of ϕ_r which is $O(r^{-1/4})$. The probability of $\{\sigma < r\}$ is

$$\mathbb{P}[\sigma < r] = \mathbb{P}\left[\sup_{t<r}|X_t^x - \xi| \geq r^{1/4}/2\right] \leq C\exp(-cr^{-1/2})$$

at $x = \xi$, so

$$\left|\mathbb{E}\overline{Y}_\sigma\right| \le \frac{C}{r^l} + \frac{1}{4}\sup\left\{|f'(t,x)|; t \le r, |x-\xi| \le r^{1/4}\right\} \tag{15}$$

for r small. By applying also the boundedness of Γ and its derivative, we deduce from (14) that

$$\begin{aligned} |f'(0,\xi)| &\le |\mathbb{E}\overline{Y}_\sigma| + C\mathbb{E}\Big[\sup_{t\le\sigma}|\overline{Y}_t|\langle Y\rangle_\sigma\Big] + C\mathbb{E}\big[\langle Y\rangle_\sigma^{1/2}\langle\overline{Y}\rangle_\sigma^{1/2}\big] \\ &\le |\mathbb{E}\overline{Y}_\sigma| + C\mathbb{E}\Big[\sup_{t\le\sigma}|\overline{Y}_t|^2\Big]^{1/2}\mathbb{E}\big[\langle Y\rangle_\sigma^2\big]^{1/2} \\ &\quad + C\big(\mathbb{E}\langle Y\rangle_\sigma\big)^{1/2}\big(\mathbb{E}\langle\overline{Y}\rangle_\sigma\big)^{1/2}, \end{aligned} \tag{16}$$

where $\langle\cdot\rangle$ is the quadratic variation computed in $\mathbb{R}^d$ or $\mathbb{R}^m\otimes\mathbb{R}^d$. The quadratic variation of Y is estimated by using the strictly convex function ψ of Lemma 3 on K; we have

$$\mathbb{E}\langle Y\rangle_\sigma \le C\mathbb{E}\Big[\sup_{t\le\sigma}\psi(Y_t)\Big] \le C\eta,$$

and the moments of $\langle Y\rangle_\sigma$ are uniformly dominated from Lemma 4, so

$$\mathbb{E}\langle Y\rangle_\sigma^2 \le \big(\mathbb{E}\langle Y\rangle_\sigma\big)^{1/2}\big(\mathbb{E}\langle Y\rangle_\sigma^3\big)^{1/2} \le C\sqrt{\eta}.$$

On the other hand, from the strict convexity of ϕ_1^m and since it is quadratic [see (8) and Lemma 3],

$$\big(\mathbb{E}\langle\overline{Y}\rangle_\sigma\big)^{1/2} \le C\mathbb{E}\Big[\sup_{t\le\sigma}\phi_1^m(Y_t,\overline{Y}_t)\Big]^{1/2} \le C'\left(1+\mathbb{E}\Big[\sup_{t\le\sigma}|\overline{Y}_t|^2\Big]^{1/2}\right)$$

and since $\overline{Y}_t = f'(t,X_t)X_t'$ and X_t' has bounded moments,

$$\mathbb{E}\Big[\sup_{t\le\sigma}|\overline{Y}_t|^2\Big]^{1/2} \le C\sup\left\{|f'(t,x)|; t\le r, |x-\xi|\le r^{1/4}\right\}.$$

Thus by applying these estimates, (16) and (15) become

$$|f'(0,\xi)| \le \frac{C}{r^l} + \frac{1}{2}\sup\left\{|f'(t,x)|; t\le r, |x-\xi|\le r^{1/4}\right\}$$

if η is chosen small enough. Then we iterate this procedure to estimate the right-hand side; we estimate the derivative $f'(t,x)$ like $f'(0,\xi)$, but we replace r by λr, for some $2^{-1/l} < \lambda < 1$; thus

$$\begin{aligned} |f'(0,\xi)| &\le \frac{C}{r^l}\left(1+\frac{1}{2\lambda^l}\right) \\ &\quad + \frac{1}{4}\sup\Big\{|f'(t,x)|; t\le r(1+\lambda), |x-\xi|\le r^{1/4}(1+\lambda^{1/4})\Big\}. \end{aligned}$$

Similarly, at the jth step, we replace r by $\lambda^j r$, and we obtain at the limit

$$|f'(0,\xi)| \le \frac{C}{r^l}\sum_{j=0}^{\infty}(1/2\lambda^l)^j$$

if r is small enough so that $r/(1-\lambda)\le 1$ and $r^{1/4}/(1-\lambda^{1/4})$ is less than the distance between $\overline{M}_2$ and M_1^c. □

We have proved that the solution $f(1-t,x)$ of the heat equation is differentiable with respect to x, that its derivative satisfies an a priori estimate and that if

$$F\colon [0,1]\times\mathbb{R}^m\times GL(m)\to T^m(K), \qquad (t,x,x')\mapsto F(t,x,x')=f'(t,x)x',$$

then $F(1-t,x,x')$ is a solution of the heat equation for the diffusion (X_t, X'_t). We now want to consider derivatives of higher order by an induction argument; for instance, in order to study the second-order derivatives of f, we want to apply the previous study to F instead of f. The problem is that the diffusion (X_t, X'_t) does not satisfy the Hörmander condition. Thus we are going to modify it by modifying the derivative X'_t. In the literature, modifications of X'_t have been used to study the stability of the stochastic flow; in some cases, one can filter out the redundant noise [9]. Here, on the contrary, we add extra noise; another induction argument is given for the elliptic case in [3].

LEMMA 9. *Under the framework of Lemma* 7, *the map* $F(1-t,x,x')$ *is a solution of the heat equation for a diffusion* $(X_t,\overline{X}_t)$ *satisfying* (H') *at* (x,x') *as soon as* X_t *satisfies it at* x.

PROOF. We introduce an independent auxiliary process U_t which is a left invariant Brownian motion on the linear group $GL(m)$; if $\overline{W}_t=(\overline{W}_t^{ij})$ is the standard Brownian motion on the set $\mathbb{R}^m\otimes\mathbb{R}^m$ of matrices (independent from W), we let U_t be the solution of

$$dU_t = U_t\,d\overline{W}_t = U_t\circ d\overline{W}_t - U_t\,dt/2. \tag{17}$$

Then U_t can be viewed as an elliptic diffusion on $GL(m)$; if we denote by $\widetilde{\Xi}_0(u)$ and $\widetilde{\Xi}_{ij}(u)$ the m^2+1 vector fields associated with (17), then $\widetilde{\Xi}_{ij}$ are the left invariant vector fields associated with the canonical basis of the Lie algebra $\mathbb{R}^m\otimes\mathbb{R}^m$ of $GL(m)$. If we define $\overline{X}_t = X'_tU_t$, we verify now that $(X_t,\overline{X}_t)$ is a diffusion satisfying the requirements of the lemma. It is the solution of a stochastic differential equation driven by the $(q+m^2)$-dimensional Wiener process $(W_t,\overline{W}_t)$; the corresponding vector fields on $\mathbb{R}^m\times GL(m)$ are $q+1$ vector fields of the form $(\Xi_i(x),\overline{\Xi}_i(x,x'))$ and the m^2 vector fields $(0,\widetilde{\Xi}_{ij}(x'))$. Suppose that X_t satisfies (H') at x, and let x' be a point of $GL(m)$; at (x,x'), the Lie algebra corresponding to the fields $(\Xi_i,\overline{\Xi}_i)$ generate a subspace of the tangent space $\mathbb{R}^m\times T_{x'}(GL(m))$ which, when projected on $\mathbb{R}^m$, has full dimension m. On the other hand, the fields $\widetilde{\Xi}_{ij}$ generate the $T_{x'}(GL(m))$ part of the tangent space, so we can deduce the Hörmander condition at (x, x'). In order to conclude the proof, we have to check that $F(t, X_t, \overline{X}_t)$ is a martingale. This relies on the following property: if z_t is a $T(K)$-valued integrable martingale, and if u_t is an independent real-valued martingale, then $u_t z_t$ is a $T(K)$-valued martingale (this can be viewed with the characterization of martingales in local coordinates). If $Z_t = f'(t, X_t)X'_t = (Z_t^1, \dots, Z_t^m)$, we can apply this property to Z_t^i (which is a martingale from Lemma 7) and the components U_t^{ij} of U_t; thus $\sum_i Z_t^i U_t^{ij}$ is a $T(K)$-valued martingale, and therefore $F(t, X_t, \overline{X}_t) = Z_t U_t$ is a $T^m(K)$-valued martingale. □

LEMMA 10. *Consider the maps f of Lemma 7, and subsets M_1 and M_2 satisfying the conditions of Lemma 8. Let J be a positive integer. If η is chosen small enough, then $f(0, \cdot)$ is J times differentiable on $\mathbb{R}^m$ and the derivatives satisfy a priori estimates on $\overline{M}_2$.*

PROOF. This is proved by induction on J. We have checked in Lemma 8 the case $J = 1$, so suppose that the lemma holds at rank J. The derivative f' is given by $f'(t, x) = F(t, x, I)$, and $F(1-t, x, x')$ is a locally bounded map which is a solution of the heat equation for an operator satisfying the Hörmander condition on $M_1 \times GL(m)$ (Lemma 9); moreover, its initial value $F(1, x, x') = f_1'(x)x'$ is smooth. We consider the domain $M_1 \times G_1$, where G_1 is a relatively compact neighborhood of I; then F is bounded on this domain. Consider the function $\eta' F$ and the process $(X_t, \overline{X}_t)$; if η and η' are small enough, then the image of this function is small, so it satisfies the conditions of Lemma 8. Thus we can apply the induction assumption to it; we deduce estimates on the derivatives of F up to order J, and therefore on the derivatives of f up to order $J + 1$. □

PROOF OF THEOREM 3. For $x_0 \in M_0$ and any integer J, we have found a neighborhood $K = \{\psi \leq \eta\}$ of $y_0 = h(x_0)$ satisfying the property of Lemma 10. As has been said previously, we can choose M_0 small so that $h(M_0) \subset K$; let M_1 and M_2 be other open neighborhoods such that

$$\overline{M}_2 \subset M_1 \subset \overline{M}_1 \subset M_0.$$

By a change of time, we transform the diffusion X_t into a diffusion $\widetilde{X}_t$ which lives in M_0 and which satisfies the original equation (5) on M_1; the generator of $\widetilde{X}_t$ can be written in a Hörmander form satisfying (H') on M_1, and h is harmonic for $\widetilde{X}_t$. Let $h_k(1, \cdot)$ be C^∞ functions on M_0 with values in K, which converge to h on M_0; we solve the heat equation and obtain solutions $h_k(t, x)$, $t \leq 1$, with values in K. The 2-convex geometry of K implies the convergence of these martingales to $h(\widetilde{X}_t)$, so $h_k(0, \cdot)$ converges to h on M_0. If we apply Lemma 10 to the diffusion $\widetilde{X}_t$, we obtain on $\overline{M}_2$ a priori estimates on the derivatives of $h_k(0, \cdot)$ up to order J. These estimates do not depend on k, so h is C^{J-1} on $\overline{M}_2$ with Lipschitz derivatives. Thus we have found a neighborhood of x_0 on which h is C^{J-1}, and J is arbitrary. □

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