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IBM, SIBM AND IBS

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We construct a super iterated Brownian motion (SIBM) from a historical version of iterated Brownian motion (IBM) using an iterated Brownian snake (IBS). It is shown that the range of super iterated Brownian motion is qualitatively quite different from that of super Brownian motion in that there are points with explosions in the branching. However, at a fixed time the support of SIBM has an exact Hausdorff measure function that is the same (up to a constant) as that of super Brownian motion at a fixed time.

0. Introduction. In this paper we construct a superprocess based on iterated Brownian motion naturally called super iterated Brownian motion. The construction given uses an iterated Brownian snake (an iterated version of the Brownian snake of Le Gall) and a historical version of iterated Brownian motion.

Our main interest may be motivated by a particle picture. Iterated Brownian motion is the process formed by taking a d-dimensional Brownian motion on the real line and running in the time variable a one-dimensional Brownian motion. It was noticed by Burdzy and Khoshnevisan [3] that iterated Brownian motion may be viewed as the limit of a Brownian motion in a crack as the crack converges to the outside Brownian motion. We shall adopt the view that we have a two-sided Brownian crack, $x(\cdot)$, which is a function from $\mathbb{R}$ to $\mathbb{R}^d$, and in the "time" variable we run an independent one-dimensional Brownian particle, w_t, to yield iterated Brownian motion, $x(w_t)$. To make a Markov process, we adopt a historical view. We keep track of the amount of crack explored by the particle and the history of the particle $\{(x((\min_{[0,s]} w \vee s') \wedge \max_{[0,s]} w),\ w_{s\wedge t})\}_{s'\in\mathbb{R}, s\geq 0}$. When the particle reaches an end of the crack, the crack is seen as extending itself in an independent manner to accommodate the motion of the particle.

We create a branching process by allowing this process to branch critically; a particle dies or splits in two with equal probability. The offspring particles immediately move off in independent manners, but the cracks they live in will agree until one of the particles reaches an end of this common crack. If two offspring do not make it to an end of the crack before dying, then they spend their lives moving along the same crack. However, if the two offspring do manage to get to an end, then they move off in new, independent cracks. That is,

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the crack branches at one of its ends. If we view the process of crack formation, then the branching at the particle level is suppressed until a particle reaches an end of an existing crack. In the high-density limit, at the heuristic level, there could possibly be infinitely many "particles" moving along a common crack. If infinitely many of the particles reach the end of this crack, this end would have that many branches emanating from it. We see then that the range of the super iterated Brownian motion (heuristically, the d-dimensional set of positions occupied by some particle at some time) may contain points where there are explosions in the branching. Qualitatively, this would be unlike the range of a super Brownian motion which has binary branching on the microscopic scale. In Theorem 2.1 this is shown to be the case at the end of a typical crack.

The picture for the support at a fixed time, however, is somewhat similar to that of super Brownian motion. Heuristically, the support is the set of positions occupied by the particles at a fixed time. It is clear from the construction that a number of particles never leave a parent crack. In fact, there are enough of them to create mass along the entire crack. Thus the support of super iterated Brownian motion will contain a segment of a Brownian path. In contrast, in Theorem 2.3 it is shown that for any fixed time almost surely the support of the super iterated Brownian motion will be contained in a finite union of Brownian cracks. The exact Hausdorff measure function for the support of super iterated Brownian motion will then be within a constant of that for d-dimensional Brownian motion. By results of Dawson and Perkins [5] and Le Gall and Perkins [12], this is the same as the exact Hausdorff measure function for super Brownian motion in dimensions two or greater. The proof of the theorem rests on Lemma 2.2. This shows that almost surely the support of historical super Brownian motion at a fixed time does not contain a path which has a maximum at its end.

We actually construct the process in terms of a Markov snake instead of the particle system described above. The two approaches create identical measure valued processes, although the snake approach gives a better description for our present needs. This paper is arranged as follows: Section 1 contains the construction of the process and Section 2 contains the result on the Hausdorff measure function.

1. Construction of IBM, SIBM and IBS.

1.1. *IBM as a Markov process.* To define IBM and SIBM, we start by defining some notation used to "paste" functions together. Let w and w' be two functions in $\mathscr{C}([0,\infty),\mathbb{R}^d)$. Then for $0 \le s \le t$, set

$$(w/s/w'/t)(u) = \begin{cases} w(u), & 0 \le u \le s, \\ w'(u-s) - w'(0) + w(s), & s \le u \le t, \\ w'(t-s) - w'(0) + w(s), & t < u. \end{cases}$$

As well, for two-sided pasting, let w_1, and w_2 be elements of $\mathscr{C}([0,\infty),\mathbb{R}^d)$, w be an element of $\mathscr{C}((-\infty,\infty),\mathbb{R}^d)$, and define $w_1'(s) = w_1(-s)$. Then, in an ordering that will remain consistent throughout this paper, for $a \le b \le 0 \le c \le d$, set

$$(a/w_1/b/w/c/w_2/d)(u) = \begin{cases} w_1'(a-b) - w_1'(0) + w(b), & u < a, \\ w_1'(u-b) - w_1'(0) + w(b), & a \le u < b, \\ w(u), & b \le u < c, \\ w_2(u-c) - w_2(0) + w(c), & c \le u < d, \\ w_2(d-c) - w_2(0) + w(c), & d \le u. \end{cases}$$

Our function space on which we define iterated Brownian motion will be the space of paired, stopped paths satisfying $E = \{(x, w) \in \mathscr{C}((-\infty, \infty), \mathbb{R}^d) \otimes \mathscr{C}([0, \infty), \mathbb{R}) : \text{there exists } \zeta \ge 0 \text{ such that } w(s) = w(s \wedge \zeta),\ x(s) = x((\min_{[0,\zeta]} w \vee s) \wedge \max_{[0,\zeta]} w)\}$. We call ζ the lifetime of the pair (x, w). On the space E, define a metric as follows:

$$d((x, w), (x', w')) = \sup_{(-\infty,\infty)} \|x(\cdot) - x'(\cdot)\| + \sup_{[0,\infty)} \|w(\cdot) - w'(\cdot)\|.$$

Next, define two probability transition functions based on the pasting functions:

$$P^1_{s,t}(w, dw') = P((w/s/\tilde{w}/t) \in dw'),$$

where under P, $\tilde{w}$ is distributed as a one-dimensional Brownian path and

$$P^{d\otimes d}_{(b,c),(a,d)}(x, dx') = P((a/x_1/b/x/c/x_2/d) \in dx'),$$

where under P, x_1 and x_2 are distributed as two independent Brownian paths in $\mathbb{R}^d$.

We require one more vector-valued Markov process before defining iterated Brownian motion. Let $y_s = (x_s, w_s) \in E$, where $\zeta = \zeta_s = s$ [so that $w_s(t) = w(t \wedge s)$] and define the transition density

$$\begin{aligned} &\mathbb{P}(y_t \in dy \mid y_s = (x_s, w_s)) \\ &\quad = \int P^1_{s,t}(w_s, dw) P^{d\otimes d}_{(\min_{[0,s]} w, \max_{[0,s]} w),(\min_{[0,t]} w, \max_{[0,t]} w)} \\ &\qquad \times (x_s, dx) \mathbf{1}(dx \otimes dw \in dy). \end{aligned} \tag{1.1}$$

REMARK 1.1. The composition $z_t = x_t(w_t(\zeta_t))$ is now well defined, and is an iterated Brownian motion in the sense of [2] and [3]. As defined, the "inside" process w is a one-dimensional Brownian motion and the "outside" process (or crack) is a two-sided Brownian motion in $\mathbb{R}^d$, but only the amount exposed by $w_t(\cdot)$ is known at time t. Note also that while iterated Brownian motion itself is not a Brownian motion, the "inside" and "outside" processes, properly formulated, are Markovian, with transition density (1.1).

LEMMA 1.2. *The process y_s under $\mathbb{P}$ is a Markov process satisfying the following continuity condition: for every $p > 0$ there exists C_p a nontrivial constant such that for $0 < t - s < 1$,*

$$\mathbb{P}(d(y_s, y_t)^p \mid y_s) \le C_p |t - s|^{p/4}.$$

PROOF. The easily verified Chapman–Kolmogorov equations ensure us of the Markov property. Essentially, the continuity condition comes down to fact that both the inside and outside Brownian motions satisfy a similar condition with exponent $p/2$ (cf. [8]).

Note first that $w_s(\cdot)$ and $w_t(\cdot)$ are equal on the interval $[0, s]$, and w_t is constant after t. This means that

$$\sup_{[0,\infty)} \|w_t(\cdot) - w_s(\cdot)\| = \sup_{[s,t]} \|w_t(\cdot) - w_t(s)\|.$$

In a similar manner, as x_s and x_t agree on the interval $[\min_{[0,s]} w_s(\cdot), \max_{[0,s]} w_s(\cdot)]$,

$$\begin{aligned}\sup_{(-\infty,\infty)} \|x_s(\cdot) - x_t(\cdot)\| \le & \sup_{[\min_{[0,t]} w_t(\cdot), \min_{[0,s]} w_t(\cdot)]} \|x_t(\cdot) - x_t(\min_{[0,s]} w_t(\cdot))\| \\ & + \sup_{[\max_{[0,s]} w_t(\cdot), \max_{[0,t]} w_t(\cdot)]} \|x_t(\cdot) - x_t(\max_{[0,s]} w_t(\cdot))\|.\end{aligned}$$

It follows that

$$\begin{aligned}&\mathbb{P}\left(\sup_{r\in[s,t]} d(y_s, y_r)^p \mid y_s\right) \\ &\quad \le 2^p \mathbb{P}\left(\sup_{r\in[s,t]} \sup_{[0,\infty)} \|w_s(\cdot) - w_r(\cdot)\|^p + \sup_{r\in[s,t]} \sup_{(-\infty,\infty)} \|x_s(\cdot) - x_r(\cdot)\|^p \mid y_s\right) \\ &\quad \le 2^p P_0\left(\sup_{[0,t-s]} \|\xi^1\|^p + \sup_{[0,-\inf_{[0,t-s]} \xi^1]} \|\xi^2\|^p + \sup_{[0,\sup_{[0,t-s]} \xi^1]} \|\xi^3\|^p\right),\end{aligned}$$

where ξ^1 is a one-dimensional Brownian motion, and ξ^2, ξ^3 are independent, d-dimensional Brownian motions started from 0 under P_0. We then have

$$\begin{aligned}\mathbb{P}\left(\sup_{r\in[s,t]} d(y_s, y_r)^p \mid y_s\right) &\le 2^p \left(C'_p|t-s|^{p/2} + 2P_0\left(C'_p|\sup_{[0,t-s]} \xi^1|^{p/2}\right)\right) \\ &\le C'_p|t-s|^{p/2} + 2C'_pC'_{p/2}|t-s|^{p/4} \le C_p|t-s|^{p/4}. \qquad \square\end{aligned}$$

1.2. *Brownian snake.* We will construct SIBM through an iterated Brownian snake. Before doing so, we recall some facts about the Brownian snake that can be found, for example, in [10].

A Brownian snake is a Markov snake based on Brownian motion. That is, a process $W_s = \{w_s, \zeta_s\}$ in the space of stopped paths $E = \{(w, \zeta) \in \mathscr{C}([0,\infty), \mathbb{R}^d) \times [0,\infty),\ w(t) = w(t \wedge \zeta)\}$. The process ζ_s is called the lifetime process. Heuristically, the snake evolves between the time s and $t > s$ by erasing from the path $w_s(\cdot)$ for some amount of time and then extending the path by an independent path of the Markov process. We will often abbreviate notation by writing $W_s(t)$ for the path $w_s(t \wedge \zeta_s)$. To be precise, the transition kernel for $s < t$ is given by

$$P_{s,t}((w_s, \zeta_s), dw \otimes da) = \int P^1(\zeta_t \in da, \inf_{[s,t]} \zeta \in db \mid \zeta_s) P^2((w_s/b/\tilde{w}/a) \in dw),$$

where P^1 is a measure under which ζ_t is a reflecting Brownian motion and, under P^2, $\tilde{w}$ is a d-dimensional Brownian motion. There is a continuous, strong Markov process with this transition kernel. If the point $x \in \mathbb{R}^d$ is associated to the trivial stopped path (w, ζ) such that $w(0) = x$ and $\zeta = 0$, then x is a recurrent point, and one can define an excursion measure from x. This proposition characterizes this excursion measure (cf. Proposition 2.2 [11]).

PROPOSITION 1.3. *The excursion measure*, $\mathbb{N}_x$, *is characterized by the following properties* (*up to a multiplicative constant*).

(i) *The lifetime process* (ζ_s) *is distributed under* $\mathbb{N}_x$ *according to the Itô measure of positive excursions of linear Brownian motion.*

(ii) $w_0 = x$, $\mathbb{N}_x$ *a.e.*

(iii) *The conditional distribution of* (W_s), *knowing* (ζ_s), *is an inhomogeneous Markov process whose transition kernel between* u *and* v *is*

$$P\Big(\Big(w_u / \inf_{[u,v]} \zeta / \tilde{w} / \zeta_v\Big) \in dw\Big),$$

where $\tilde{w}$ *is an independent Brownian motion under* P.

Super Brownian motion defined under $\mathbb{N}_x$ is the measure-valued process given by its action on a Borel measurable function on E as

$$\langle \mathscr{W}_a, \phi \rangle = \int_0^\sigma dL_s^a(\zeta) \phi(w_s(\zeta_s)),$$

where $L^a(\zeta)$ is the local time at level a for the Brownian excursion ζ ($\mathbb{N}_x$ is normalized to make this well defined) and σ is the lifetime of the excursion.

REMARK 1.4. We use a nonstandard notation for the superprocess. The same letter but a different typeface is used for the measure-valued process and its related Markov snake.

The following is a consequence of the critical branching,

$$\mathbb{N}_x(\langle \mathscr{W}_a, \phi \rangle) = E_x(\phi(B_a)), \tag{1.2}$$

where B_a is a Brownian motion under E_x.

In this paper, we make use of different excursion decompositions of the Brownian snake. Let $a > 0$ and set $I_a = \{(\alpha_i, \beta_i)\}$ to be an enumeration of the excursion intervals of the Brownian excursion from 0, ζ, above a height a. We set $W_s^{(\alpha,\beta)}(t) = W_{\alpha+s}(t + \zeta_\alpha) - W_\alpha(\zeta_\alpha)$, $0 \le s \le \beta - \alpha$ to be the excursion of the snake. A general excursion formula (cf. [1], Chapter VII) allows us to make computations,

$$\mathbb{N}_x\left(\sum_{(\alpha_i,\beta_i)\in I_a} \exp(-Z_{\alpha_i}) f\circ\Theta_{\alpha_i}\right) = \mathbb{N}_x\left(\int_0^\sigma dL_s^a(\zeta)\exp(-Z_s)\mathbb{N}_{w_s(\zeta_s)}(f)\right)$$

Here Z is a process adapted to $\mathscr{F}_s = \sigma((w_u, \zeta_u),\ u \le s)$ and f is a measurable function on E which is 0 on trivial paths.

We actually make use of this later on the historical version of the snake. The historical snake gives rise to historical super Brownian motion, denoted below as $\bar{\mathscr{W}}_a$. The minor differences are explained in [9]. In particular the excursions keep track of their common ancestor path. The excursion $W_s^{(\alpha,\beta)}$ under $\mathbb{N}_x$ would correspond to the path $W_{\alpha+s}(t)$, with $t \ge \zeta_s = a$ under an excursion measure $\mathbb{N}_{(w_s,\zeta_s)}$.

For a specific case of the excursion formula, which will be used later, we would have for measurable functions f which are 0 on stopped paths which have $\zeta = a$,

$$\mathbb{N}_x\left(\sum_{(\alpha_i,\beta_i)\in I_a} f(W\circ\Theta_{\alpha_i})\right) = \mathbb{N}_x\left(\int_0^\sigma dL_s^a \mathbb{N}_{(w_s,\zeta_s)}(f(W))\right) \tag{1.3}$$

$$= \mathbb{N}_x\left(\int_0^\sigma dL_s^a \mathbb{N}_{w_s(\zeta_s)}(f((w_s/\zeta_s/w/\zeta_s+\zeta)\right) \tag{1.4}$$

$$= \mathbb{N}_x\left(\langle \bar{\mathscr{W}}_a, \mathbb{N}_\cdot(f((\bar{w}/\bar{\zeta}/w/\bar{\zeta}+\zeta)))\rangle\right) \tag{1.5}$$

$$= E_x\left(\mathbb{N}_{B_a}(f((B/a/w/a+\zeta)))\right).$$

Line 1.4 follows as W under $\mathbb{N}_{(w_s,\zeta_s)}$ is an excursion, (w,ζ), with base path $w_s(\cdot\wedge\zeta_s)$ evolving under $\mathbb{N}_{w_s(\zeta_s)}$. In 1.5 the pair $(\bar{w},\bar{\zeta})$ is an excursion under the exterior excursion measure $\mathbb{N}_x$, and the pair (w,ζ) is an excursion under the interior excursion measure $\mathbb{N}_\cdot$. The last line follows from 1.2 suitably extended for the historical super Brownian motion $\bar{\mathscr{W}}_a$.

Another excursion decomposition that we use is related to the process as it evolves from a fixed path. The probability measure $\mathbb{P}_W^*$ is used to describe the evolution starting from a fixed path $W = (w,\zeta)$, killed when ζ hits 0. Let I be the excursion intervals, (α,β), of excursions of ζ above the minimum process $\inf_{u\in[0,\sigma]}\zeta_u$ up until σ, the hitting time of 0. Let $W^{(\alpha,\beta)}$ be the corresponding excursion of the snake, $W_u^{(\alpha,\beta)}(t) = W_{\alpha+u}(\zeta_\alpha+t) - W_\alpha(\zeta_\alpha)$, and let $\mathbb{N}_x(d\kappa)$ be a measure on $\mathscr{W}$, the space of excursions of Brownian snake. That is, under $\mathbb{N}_x(d\kappa)$, κ is an excursion of Brownian snake started from x.

This proposition gives the probabilistic structure of the excursions.

PROPOSITION 1.5 (Proposition 2.5 [11]). *Under $\mathbb{P}_W^*$ the random measure*

$$\sum_{(\alpha,\beta)\in I} \delta_{\zeta_\alpha, W^{(\alpha,\beta)}}. \tag{1.6}$$

is a Poisson point measure with intensity $2dt\mathbf{1}(t \leq \zeta)\mathbb{N}_{W(t)}(d\kappa)$ *on the space* $[0, \infty) \times \mathscr{W}$.

1.3. *Iterated Brownian snakes.* We define now a Markov snake based on the historical iterated Brownian motion which is a generalization of the Brownian snake of Le Gall [10] and Dynkin and Kuznetsov [6]. By "snake" here we mean a stopped process (z, ζ), with ζ a process in $\mathbb{R}_+$ related to z by $z_s(t + \zeta_s) = z_s(\zeta_s)$ for all $t \geq 0$ and, given the entire process ζ, the conditional transition functions between times s and t for z is given by

$$P(z_s, dz_t \mid \zeta) = P\Big(\Big(z_s / \inf_{[s,t]} \zeta / \tilde{z} / \zeta_t\Big) \in dz_t\Big).$$

That is, the path z erases from its head an amount determined by a minimum of ζ over the time interval and attaches an independent piece of the underlying diffusion of a length determined by ζ. The snake is a Markov snake if in addition ζ is a strong Markov process.

The processes may be defined under a probability measure, but we prefer to use σ-finite excursion measures. If we identify $y \in \mathbb{R}^d$ with the stopped paths (x, w) with $\zeta = 0$, and $x(w(0)) = y$, then the points $y \in \mathbb{R}^d$ are regular points for the Markov snake (if the lifetime process is reflecting Brownian motion). The excursion measure will be the usual excursion measure associated to excursions from a regular point of a nice Markov process. The existence is a consequence of this theorem.

THEOREM 1.6. *There exists a Markov snake process* $(Y_s, \zeta_s) = ((x_s, w_s), \zeta_s)$ *based on the iterated Brownian motion with excursion measure* $\mathscr{N}_y$. *Under* $\mathscr{N}_y$, ζ_s *is distributed like a positive excursion of Brownian motion. Furthermore, under* $\mathscr{N}_y$ *there exists a measure-valued process* $\mathscr{Y}_a$ *defined by its action on bounded measurable functions*

$$\langle \mathscr{Y}_a, \phi \rangle = \int_0^\sigma ds L_s^a(\zeta) \phi(Y_s),$$

where L_s^a *is the local time at level* a *for the lifetime process* ζ *and* σ *is the lifetime of the excursion* ζ.

PROOF. This follows immediately from Theorem 1.1 of Le Gall [10], where it is shown that a Markov snake exists for any nonhomogeneous Markov processes satisfying the continuity assumption that there exists $k \geq 3$ and constants C, ε such that for every $x \in E$, $r \geq 0$, $t \geq r$,

$$P_{r,x}\left(\sup_{r \leq s \leq t} d(x, y_s)^k\right) \leq C|t - r|^{2+\varepsilon}.$$

By Lemma 1.2 this is true with $k = p > 4$. □

REMARK 1.7. As well, we could have followed the general construction of a super process of Fitzsimmons [7], using the historical version of iterated

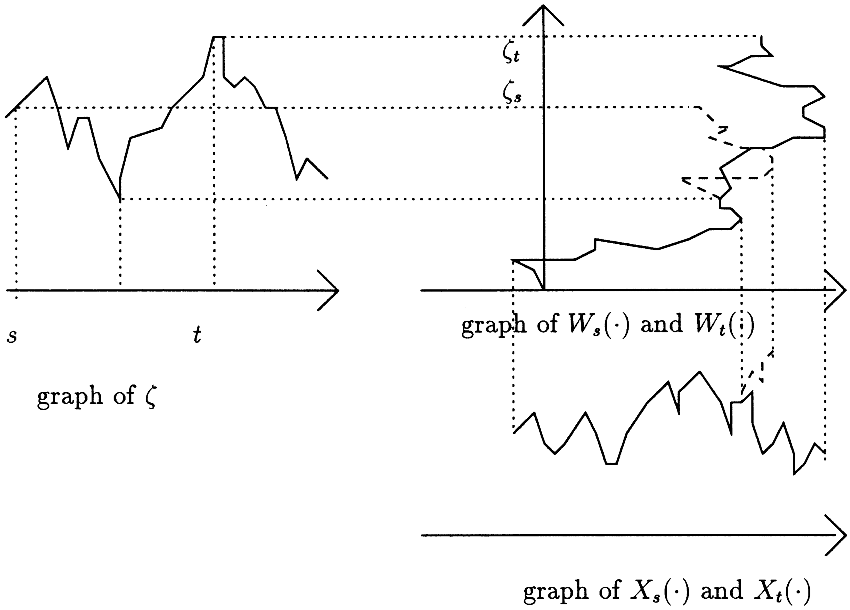

FIG. 1. *Definition of* (X, W, ζ).

Brownian motion given by y_s, to find an equivalent definition of $\mathscr{Y}_a$. With this approach it is easy to see that the Laplace functional of $\mathscr{Y}_a$ will be related to a family of operators $V_{s,t}$ satisfying

$$V_{s,t}\phi(x, w) = \mathbb{P}_{s,t}\phi(x, w) - \int_s^t \mathbb{P}_{s,r}(V_{r,t}(\phi)^2)(x, w)\, dr,$$

where $\mathbb{P}_{s,t}((x, w), dx \otimes dw) = \mathbb{P}(y_t \in dx \otimes dw \mid y_s = (x, w))$.

We collect some further properties of the process Y_s in the following corollary.

In a direct generalization of a Markov snake, we call a process $X = (X_s, l_s, r_s)$ a two-headed snake if $l_s \leq 0 \leq r_s$ and the left and right sides are snakes. That is, $(X_s(-t), l_s),\ t \geq 0$ and $(X_s(t), r_s),\ t \geq 0$ are snakes. The two-headed snake is a Markov snake, if the process (l_s, r_s) is strong Markov.

COROLLARY 1.8. *The processes X_s and W_s, where $Y_s = (X_s, W_s)$, are snakes satisfying*:

(i) *Under $\mathscr{N}_y$ the process (W_s, ζ_s) is a Brownian snake.*

(ii) *Under $\mathscr{N}_y$, conditionally given the history of the Brownian snake (W_s, ζ_s), the process X_s is a two-headed Brownian snake with lifetime process $(\min_{[0,\zeta_s]} W_s(\cdot), \max_{[0,\zeta_s]} W_s(\cdot))$. The conditional distribution of X given*

W and ζ is given by the transition kernel

$$P(X_t \in dx \mid X_s, \sigma(W,\zeta)) = P^{d\otimes d}_{(\min_{[0,\min_{[s,t]}\zeta]} W_t, \max_{[0,\min_{[s,t]}\zeta]} W_t)(\min_{[0,\zeta_t]} W_t, \max_{[0,\zeta_t]} W_t)}(X_s, dx).$$

Here $\sigma(W,\zeta)$ is the σ-field generated by the one-dimensional Brownian snake (W,ζ).

(iii) *For a fixed time s, under $\mathcal{N}_y$, W_s is distributed like a Brownian motion from* 0 *stopped at ζ_s and X_s is distributed like a two-headed Brownian motion with $X_s(0) = y$ stopped at* $(\min_{[0,\zeta_s]} W_s, \max_{[0,\zeta_s]} W_s)$.

PROOF. The corollary is a consequence of the construction of Y. From the proof of Theorem 1.1 in [10] the conditional distribution of Y given ζ is given by

$$P(Y_t \mid Y_s, \sigma(\zeta)) = Q_{\min_{[s,t]}\zeta, \zeta_t}(Y_s, dy),$$

where under Q, Y evolves by erasing from ζ_s to $\min_{[s,t]}\zeta$, and then adding an independent piece of Y under the measure $\mathbb{P}$. By 1.1 this translates into

$$\begin{aligned} P(Y_t &\in dx \otimes dw \mid Y_s, \sigma(\zeta)) \\ &= \int P^1_{\min_{[s,t]}\zeta, \zeta_t}(W_s(\cdot), dw) \\ &\qquad \times P^{d\otimes d}_{(\min_{[0,\min_{[s,t]}\zeta]} W_t, \max_{[0,\min_{[s,t]}\zeta]} W_t)(\min_{[0,\zeta_t]} W_t, \max_{[0,\zeta_t]} W_t)}(X_s, dx). \end{aligned}$$

By conditioning on $\sigma(W)$ we see that parts (i) and (ii) follow immediately. Part (iii) then follows from from parts (i) and (ii). □

REMARK 1.9. Let $Z_t(\cdot) = X_t(W_t(\cdot))$; then (Z_t, ζ_t) will be called "iterated Brownian snake" as it is the composition of two Brownian snakes. The measure-valued process $\mathcal{Z}$ defined on bounded measurable functions on $\mathbb{R}^d$ by

$$\langle \mathcal{Z}_a, \phi \rangle = \int_0^\sigma dL^a_s(\zeta)\phi(Z_s(\zeta_s)),$$

is then called super iterated Brownian motion.

REMARK 1.10. The process $\mathcal{Z}_a$ is well defined because under the measure dL^a_s one has $\zeta = a$. Consequently, $\mathcal{Z}_a$ is supported on the points corresponding to times s when $\zeta = a$.

2. Hausdorff dimension of the support of super iterated Brownian motion. First we define the *range* of SIBM to be

$$\begin{aligned} \mathcal{R} &= \{Z_s(\zeta_s) \in \mathbb{R}^d : 0 \le s \le \sigma\} \\ &= \{X_s(t) \in \mathbb{R}^d : 0 \le s \le \sigma, t \in \mathbb{R}\}, \end{aligned}$$

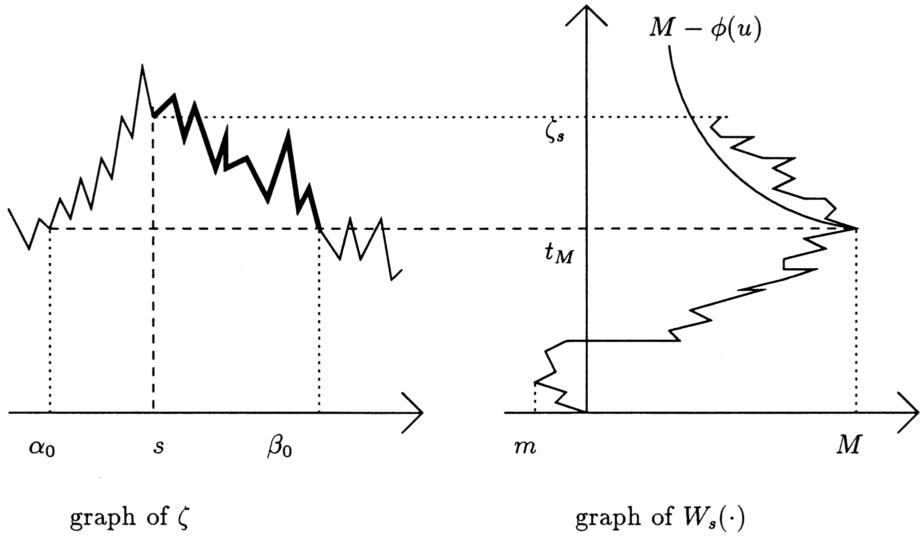

FIG. 2. *Definition of M and t_M.*

and the support of SIBM, $\mathscr{S}_a$ to be the support of the measure $\mathscr{Q}_a$. It is clear from the definition of SIBM that almost surely for all s,

$$\mathscr{S}_a \subseteq \text{closure}\{Z_s(\zeta_s) : 0 \leq s \leq \sigma, \zeta_s = a\}.$$

For a fixed s, the reverse inequality is true almost surely (cf. Proposition 2.2 in [10] for a similar statement for the Brownian snake).

We show in this section that the range of super iterated Brownian motion is unlike that of super Brownian motion, but the supports at a fixed time are similar with respect to an exact Hausdorff measure function.

2.1. *The range of $\mathscr{Q}$.* In the introduction we used the image of a particle moving in a crack traced out by X. Typically the particle will not be close to an end of the crack, so that most particles do not contribute to branching of the crack generated by X. However, the particles are branching at a rate independent of X, and the next lemma shows that when a particle gets close to an end of the crack, actually infinitely many of its offspring will reach the end. This causes explosions in the branching of the range of X.

THEOREM 2.1. *For fixed s, $\mathbb{N}$-almost surely, the range of $\mathscr{Q}$ contains infinitely many branches at the edges of $X_s(\cdot)$.*

PROOF. First we need to define our notation. The reader is advised to refer to Figure 2 for help along the way. Let s be a fixed time with $\zeta_s > 0$, set $M = \sup_{t\in[0,\zeta_s]} W_s(t)$, $m = \inf_{t\in[0,\zeta_s]} W_s(t)$, and let t_M be the time at which this maximum is achieved. Although all of these are s dependent, we continue to follow our previous convention of dropping the s unless there is possibility of confusion. The time t_M is uniquely defined, almost surely, as $W_s(\cdot)$ is distributed as a Brownian path stopped at ζ_s. Furthermore, let $B^1(u) = M - W(t_M - u)$ for $0 \le u \le t_M$ and set $B^2(u) = M - W(t_M + u)$ for $0 \le u \le \zeta_s - t_M$. By the well known decomposition of a Brownian path at its maximum the two processes B^1 and B^2 are distributed as Bessel(3) processes almost surely. As such, if $\phi(u) = \sqrt{10u\log(1/u)}$ there exists an $\varepsilon > 0$ for which $\phi(u) \ge B^2(u)$ for $0 \le u \le \varepsilon$ (random) as the modulus of continuity of a Bessel process is the same as for a Brownian motion. Let (α_0, β_0) be the excursion of ζ above t_M which straddles s. Furthermore, let I' be the set of excursions $\{(\alpha, \beta)\}$ of ζ_u above the minimum process $\zeta_u^* = \inf_{v\in[s,u]} \zeta_v$ with $u \le \beta_0$.

For all $u \in (\alpha_0, \beta_0)$ the path $X_u(\cdot)$ is contained in the path $X_s(\cdot)$ on the interval $[m, M]$. However, if during an excursion $(\alpha, \beta) \in I'$ it happens that there exists a time u and a level $t > t_M$ such that $W_u(t) > M$, then the path $X_u(\cdot)$ branches from $X_s(\cdot)$ at its end $X_s(M)$. There is an independent path for each different excursion for which this happens. We show that, basically due to the clumping of small excursions near β_0, this happens infinitely often, thereby causing an explosion in the branching at the point $X_s(M)$. [A similar argument will apply to explosions at $X_s(m)$, but there is no need to consider both points to prove the theorem.]

Let A_n and B_n be the events

$$A_n = \Big\{\exists(\alpha, \beta) \in I' \colon \zeta_\alpha \in [t_M + 2^{-n-1}, t_M + 2^{-n}),\ \sup_{u\in(\alpha,\beta)} \sup_{t>t_M} W_u(t) > M\Big\}$$

and

$$\begin{aligned} B_n &= \Big\{\exists(\alpha, \beta) \in I' \colon \zeta_\alpha \in [t_M + 2^{-n-1},\ t_M + 2^{-n}),\\ &\qquad \sup_{u\in(\alpha,\beta)} \sup_{t>t_M} W_u(t) - W_u(\zeta_\alpha) > \phi(2^{-n})\Big\}\\ &= \Big\{\exists(\alpha, \beta) \in I' \colon \zeta_\alpha \in [t_M + 2^{-n-1},\ t_M + 2^{-n}),\\ &\qquad \sup_{u\in(0,\beta-\alpha)} \sup_{t>0} W_u^{(\alpha,\beta)}(t) > \phi(2^{-n})\Big\}. \end{aligned}$$

Here $W^{(\alpha,\beta)}$ is the excursion of the snake over (α, β), namely, $W_u^{(\alpha,\beta)}(t) = W_{\alpha+u}(\zeta_\alpha + t) - W_\alpha(\zeta_\alpha)$ for $0 \le u \le \beta - \alpha$. Clearly if $2^{-n} < \varepsilon$ and the event B_n happens, then A_n happens.

We show the events B_n occur infinitely often under the measure $\mathbb{P}^*_{W_s}$. For this measure, $W_s(\cdot)$ is known and so t_M is a deterministic time and $\varepsilon > 0$ is fixed. As the excursions in I' are independent it is enough to show by the Borel–Cantelli lemmas that $\sum_n \mathbb{E}^*_{W_s}(\mathbf{1}(B_n)) = \infty$.

Let $\Delta_n = [t_M + 2^{-n-1}, t_M + 2^{-n})$ and $\Gamma_n = \{w : \sup_{u\in(0,\sigma)} \sup_{t>0} w_u(t) > \phi(2^{-n})\}$. Set $\chi = \sum_{(\alpha,\beta)\in I'} \delta_{(\zeta_\alpha, W^{(\alpha,\beta)})}$. Then, by Proposition 1.5, χ is a Poisson random measure with intensity $2dt\mathbf{1}_{\zeta=t}\mathbb{N}_{W_s(t)}(\cdot)$. Consequently, we have that

$$\begin{aligned}
\mathbb{P}^*_{W_s}(B_n) &= \mathbb{P}^*_{W_s}\Big(\exists(\alpha,\beta)\in I'\colon \mathbf{1}(\zeta_\alpha\in\Delta_n)\mathbf{1}(W^{(\alpha,\beta)}\in\Gamma_n)>0\Big)\\
&= \mathbb{P}^*_{W_s}\left(\sum_{(\alpha,\beta)\in I'}\mathbf{1}(\zeta_\alpha\in\Delta_n)\mathbf{1}(W^{(\alpha,\beta)}\in\Gamma_n)>0\right)\\
&= \lim_{\lambda\to\infty}(1-\mathbb{P}^*_{W_s}(\exp(-\langle\chi,\lambda\mathbf{1}(\zeta\in\Delta_n)\mathbf{1}(W\in\Gamma_n)\rangle)))\\
&= \lim_{\lambda\to\infty}\left(1-\exp\left(-\int_0^{\zeta_s}2\,dt\mathbb{N}_{W_s(t)}(1-\exp(-\lambda\mathbf{1}(t\in\Delta_n)\mathbf{1}(W\in\Gamma_n)))\right)\right)\\
&= 1-\exp(-\int_{\Delta_n}2\,dt\mathbb{N}_{W_s}(t)(W\in\Gamma_n)).
\end{aligned}$$

Now,

$$\begin{aligned}
\mathbb{N}_{W_s}(t)(W\in\Gamma_n)) &= \mathbb{N}_0(\mathscr{R}(W)\cap[\phi(2^{-n}),\infty))\\
&= c\phi(2^{-n})^{-2}.
\end{aligned}$$

The last line follows by Proposition 2.3 of [11] which derives it via a scaling argument.

This gives the estimate

$$\begin{aligned}
\mathbb{P}^*_{W_s}(B_n) &= 1-\exp(-c2^{-n}\phi(2^{-n})^{-2})\\
&\geq c_1/n - c_2/n^2.
\end{aligned}$$

The sum in n clearly diverges and the theorem follows. □

2.2. *The support of $\mathscr{Z}_a$.* The range of X, which corresponds to the range of $\mathscr{Z}$ looks quite different qualitatively than the range of a super Brownian motion. However, we will see in this section that, for a fixed a, the support of $\mathscr{Z}_a$ is exactly that given by a finite number of Brownian motions, and so in terms of Hausdorff measure looks like that of the super Brownian motion (cf. [4] or [12]).

Observe that the set of values $W_s(\zeta_s)$ for which $\zeta_s = a$ is simply the support of a one-dimensional super Brownian motion. Since the support of $\mathscr{Z}_a$ contains these points evaluated into the functions X_s, they will determine the support of $\mathscr{Z}_a$. As a one-dimensional super Brownian motion is absolutely continuous with respect to Lebesgue measure, to understand the support of $\mathscr{Z}_a$ we need to investigate the range of X_s for all s with $\zeta_s = a$.

Recall that X_s is a two-headed snake with a lifetime process given by $(\min_{[0,\zeta_s]} W_s(\cdot), \max_{[0,\zeta_s]} W_s(\cdot))$. At a typical time s, the tip of the path $W_s(\cdot)$ will not be near the global maximum or minimum along the entire path, a consequence of the fact that, at its tip, $W_s(\cdot)$ is typically Brownian motion stopped at a time which is independent of the path properties of W_s. Thus X_s will typically remain constant for some period of time. It only changes when the snake process W moves forward to a new global maximum (global minimum), or backtracks past an old global maximum (global minimum). When this happens, then X adds or subtracts from itself.

This lemma shows that for a typical lifetime value, a one-dimensional Brownian snake will not have any paths which are at a maximum or a minimum at their tip. In terms of historical measures, this can, alternatively, be interpreted by saying that for a fixed a the support of historical super Brownian motion almost surely does not contain a path whose maximum or minimum occurs at a.

LEMMA 2.2. *Let (W_s, ζ_s) be a Brownian snake under $\mathbb{N}_x$. For fixed $a > 0$,*

$$\mathbb{N}_x(\exists s : \zeta_s = a, \sup_{[0,\zeta_s]} W_s(\cdot) = W_s(\zeta_s) \quad or \quad \inf_{[0,\zeta_s]} W_s(\cdot) = W_s(\zeta_s)) = 0$$

PROOF. We use the excursion notation of [9]. That is, for a and ε, with $a > \varepsilon$, let $(\alpha_i, \beta_i) \in I_{a-\varepsilon}$ index the excursion intervals for ζ from level $a - \varepsilon$. Let ζ^i and W^i be the corresponding excursions of ζ and W. ($W^i = W^{(\alpha_i,\beta_i)}$ from before.)

We will work with the Brownian broom defined for each excursion ζ^i in $I_{a-\varepsilon}$ by its handle $W_{\alpha_i}(\cdot)$ and its fan $\{W^i_u(\cdot) : \ \zeta^i_u = \varepsilon\}$. Let $A_\varepsilon(i)$ be the indicator of the event

$$\Bigg\{ \sup_{[0,\beta^i-\alpha^i]} \zeta^i \ge \varepsilon, \ \sup_{[0,a-\varepsilon]} W_{\alpha_i}(\cdot) \le \sup_{\substack{u\in[0,\beta^i-\alpha^i] \\ \zeta^i_u=\varepsilon}} \sup_{[0,\varepsilon]} (W^i_u(\cdot) + W_{\alpha_i}(a-\varepsilon)) \quad \text{or}$$
$$\inf_{[0,a-\varepsilon]} W_{\alpha_i}(\cdot) \ge \inf_{\substack{u\in[0,\beta^i-\alpha^i] \\ \zeta^i_u=\varepsilon}} \sup_{[0,\varepsilon]} (W^i_u(\cdot) + W_{\alpha_i}(a-\varepsilon)) \Bigg\}.$$

That is, $A_\varepsilon(i)$ is 1 each time the fan of the Brownian broom speads out further, on at least one of its two sides, than does the handle.

We wish to show that for some sequence $\varepsilon_k \to 0$, the events

$$B_k = \bigcup_{I_{a-\varepsilon_k}} \{A_{\varepsilon_k}(i) \ne 0\}$$

do not happen almost surely.

It is enough to show that for some sequence we have

$$\mathbb{N}_x\Bigg(\sum_{I_{a-\varepsilon_k}} A_\varepsilon(i) > 0 \Bigg) = a_k \to 0.$$

Note that Brownian brooms have an extinction property: if $\{\bigcup_{I_{a-\varepsilon}} A_\varepsilon(i) = 0\}$ for some ε, it holds for all smaller values. Thus if the above limit is 0, we can find a subsequence on which the a_k sum and by the Borel–Cantelli argument applied to this subsequence of brooms there is an ε_0 for which $\sum_{I_{a-\varepsilon}} A_\varepsilon(i) = 0$ for all $0 < \varepsilon < \varepsilon_0$.

Let $C_\varepsilon(i)$ be the indicator of the event

$$\left\{ \sup_{[0,\beta^i-\alpha^i]} \zeta^i \ge \varepsilon,\ \sup_{[0,a-\varepsilon]} W_{\alpha_i}(\cdot) \le \sup_{\substack{u\in[0,\beta^i-\alpha^i]\\ \zeta^i_u=\varepsilon}} \sup_{[0,\varepsilon]}(W^i_u(\cdot) + W_{\alpha_i}(a-\varepsilon) \right\}.$$

By the symmetry of the one-dimensional Brownian snake we have

$$\begin{aligned}
&\mathbb{N}_x\left(\sum_{I_{a-\varepsilon}} A_\varepsilon(i) > 0\right)\\
&\quad \le \mathbb{N}_x\left(\sum_{I_{a-\varepsilon}} A_\varepsilon(i)\right) \le c_0 \mathbb{N}_x\left(\sum_{I_{a-\varepsilon}} C_\varepsilon(i)\right)\\
&\quad = \mathbb{N}_x\left(\int_0^\sigma dL_s^{a-\varepsilon} \mathbb{N}_{W_s(\zeta_s)}\left(\sup\tilde{\zeta} \ge \varepsilon,\ \sup_{[0,a-\varepsilon]} W_s(\cdot)\right.\right.\\
&\qquad\qquad \left.\left. \le \sup\nolimits_{\substack{u\in[0,\tilde{\sigma}]\\ \tilde{\zeta}_u=\varepsilon}} \sup\nolimits_{[0,\varepsilon]} \tilde{W}(\cdot) + W_s(a-\varepsilon)\right)\right),
\end{aligned} \tag{2.1}$$

where the variables $\tilde{W}$, $\tilde{\zeta}$ and $\tilde{\sigma}$ refer to a Brownian snake under the interior excursion measure. (2.1) follows from (1.3).

Now, for all $\gamma > 0$, we have

$$\mathbb{N}_0(\sup\tilde{\zeta} \ge \varepsilon,\ \gamma \le \sup_{\substack{u\in[0,\tilde{\sigma}]\\ \tilde{\zeta}_u=\varepsilon}} \sup_{[0,\varepsilon]} \tilde{W}_u(\cdot)) \le \mathbb{N}_0(\mathscr{W}_\varepsilon(B(0,\gamma)^c) > 0) \tag{2.2}$$

$$= \mathbb{N}_0(\mathscr{W}_1(B(0,\gamma\varepsilon^{-1/2})^2) > 0) \qquad \text{(scaling)}$$

$$\le \begin{cases} c_1 \dfrac{\varepsilon^{1/2}}{\gamma} \exp\left(-c_2 \dfrac{\gamma^2}{\varepsilon}\right), & \gamma\varepsilon^{-1/2} > 1\\ c_3, & \gamma\varepsilon^{-1/2} \le 1. \end{cases} \tag{2.3}$$

(2.2) comes by translating the problem into one for super Brownian motion. (2.3) follows by Theorem 3.3(b) in [4] on the probability that super Brownian motion charges the exterior of a ball, and the fact that when $\gamma\varepsilon^{-1/2} \le 1$ one has

$$\mathbb{N}_0(\mathscr{W}_1(B(0,\gamma\varepsilon^{-1/2})^c) > 0) \le \mathbb{N}_0(\langle \mathscr{W}_1, 1\rangle > 0) = 1/2.$$

Combining (2.3) and (1.3) into (2.1), one gets

$$\begin{aligned}
\mathbb{N}_x\left(\sum_{I_{a-\varepsilon}} A_\varepsilon(i)\right) &\le \int_0^{\varepsilon^{1/2}} c_3 P\left(\sup_{s\in[0,a-\varepsilon]} B_s \in d\gamma\right)\\
&\quad + \int_{\varepsilon^{1/2}}^\infty c_1 \frac{\varepsilon^{1/2}}{\gamma} \exp\left(-c_2\frac{\gamma^2}{\varepsilon}\right) P\left(\sup_{s\in[0,a-\varepsilon]} B_s \in d\gamma\right)\\
&\le c_3\varepsilon^{1/2} + c_4\varepsilon^{1/2}\ \log(1/\varepsilon) \to 0.
\end{aligned}$$

□

THEOREM 2.3. *Fix $a > 0$. Let ϕ be the exact Hausdorff measure function for d-dimensional Brownian motionand ψ be the exact Hausdorff measure function for* $\operatorname{supp}(\mathscr{Z}_a)$. *Then there exists nonzero finite, random constants for which $\mathscr{N}_y$-almost surely,*

$$c_1(\omega)\phi \le \psi \le c_2(\omega)\phi. \tag{2.4}$$

PROOF. The basic idea is to show that the support of $\mathscr{Z}_a$ is contained in a finite union of paths, $\bigcup_{i=1}^{n(\omega)} X_{s_i}$, each of which is distributed like a d-dimensional Brownian motion. Hence the exact measure function of $\mathscr{Z}_a$ is less than that of d-dimensional Brownian motion. The lower bound will follow from showing that along a piece of one of these paths, there are enough particles to put mass over the entire piece.

First, we find a finite number of times $\{s_i\}$ which yield the range of $\{X_s(\cdot) : \zeta_s = a\}$. By the proof of Lemma 2.2, almost surely, for a fixed ω, there exists an $\varepsilon = \varepsilon(\omega)$ for which all maxima and minima in the set of paths $\{W_s(\cdot) :\ \zeta_s = a\}$ occur before $a - \varepsilon$. Let $I \subset I_{a-\varepsilon}$ consist of those excursion intervals from $a - \varepsilon$ which hit a. This is a finite set almost surely, say of size $n(\omega)$. For each excursion interval, pick a rational point s_i contained in the interval, then $\{\bigcup_{i=1}^{n(\omega)} X_{s_i}(\cdot)\}$ contains all points in the support of $\mathscr{Z}_a$.

Since the s_i are rational, and $\mathscr{N}_y$-almost surely X_{s_i} is distributed as a stopped, two-sided Brownian motion we see that the support of $\mathscr{Z}_a$ is contained in a range of a finite union of stopped Brownian paths. Thus the exact Hausdorff measure function for $\mathscr{Z}_a$ will be less than a constant multiple of that for a d-dimensional Brownian motion.

To see the other inequality in (2.4), since (W_s, ζ_s) is a one-dimensional Brownian snake, for a fixed a almost surely on the event $\mathscr{W}_a \neq 0$, the measure $\mathscr{W}_a$ is absolutely continuous with respect to Lebesgue measure. Thus there exists an excursion interval, (α_i, β_i) in $I_{a-\varepsilon}$ corresponding to one of the s_i for which the measure given by

$$A \to \int_{\alpha_i}^{\beta_i} dL_s^a(\zeta)\mathbf{1}_A(W_s(\zeta_s))$$

is absolutely continuous with respect to Lebesgue measure. The support of $\mathscr{Z}_a$ corresponding to this is simply the support of

$$\begin{aligned}\int_{\alpha_i}^{\beta_i} dL_s^a(\zeta)\mathbf{1}_A(X_s(W_s(\zeta_s))) &= \int_{\alpha_i}^{\beta_i} dL_s^a(\zeta)\mathbf{1}_A(X_{s_i}(W_s(\zeta_s)))\\ &= \mathscr{W}_a^{(\alpha_i,\beta_i)} \circ X_{s_i}^{-1}(A)\end{aligned}$$

which shows that the support is found by pushing an absolutely continuous measure through the Brownian path X_{s_i}. Hence just this part has exact Hausdorff measure function greater than or equal to that of a d-dimensional Brownian motion. From this we conclude the theorem. □

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