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ON THE SPEED OF CONVERGENCE FOR TWO-DIMENSIONAL FIRST PASSAGE ISING PERCOLATION

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Consider first passage Ising percolation on Z^2. Let β denote the reciprocal temperature and let h denote an external magnetic field. Denote by β_c the critical temperature and, for $\beta < \beta_c$, let

$$h_c(\beta) = h_c = \sup\{h: \theta(\beta, h) = 0\},$$

where $\theta(\beta, h)$ is the probability that the origin is connected by an infinite $(+)$-cluster. With these definitions let us consider first passage Ising percolation on Z^2. Let $a_{0,n}$ denote the first passage time from $(0, 0)$ to $(n, 0)$. It follows from a subadditive argument that

$$\lim_{n\to\infty} \frac{a_{0,n}}{n} = \nu \text{ a.s and in } L_1.$$

It is known that $\nu > 0$ if $\beta < \beta_c$ and $|h| < h_c(\beta)$. Here we will estimate the speed of the convergence,

$$\nu n \le Ea_{0,n} \le \nu n + C(n \log^5 n)^{1/2}$$

for some constant C. Define $\mu_{\beta,h}$ to be the unique Gibbs measure for $\beta < \beta_c$. We also prove that there exist $\tilde{C}, \tilde{\alpha} > 0$ such that

$$\mu_{\beta,h}(|a_{0,n} - Ea_{0,n}| \ge x) \le \tilde{C} \exp\left(-\tilde{\alpha} \frac{x^2}{n \log^4 n}\right).$$

In addition to $a_{0,n}$, we shall also discuss other passage times.

1. Introduction to Ising first passage percolation. Consider the Z^2 lattice and the sample space $\Omega = \{+1, -1\}^{Z^2}$ with spin configurations on Z^2. Given a sample $w \in \Omega$ and $x \in Z^2$, $w(x)$ denotes the spin value at x in the configuration w. For any set $V \subset Z^2$, denote by $\mathscr{F}_V$ the σ-algebra generated by $\{w(x): x \in V\}$, and we simply write $\mathscr{F}$ for $\mathscr{F}_{Z^2}$. For any finite V, let the Hamiltonian in V be

$$H_V^w(\sigma) = -\tfrac{1}{2} \sum_{x,\, y\in V,\, \|x-y\|=1} \sigma(x)\sigma(y) - \sum_{x\in V} \left[h + \sum_{y\notin V,\, \|x-y\|=1} w(y)\right]\sigma(x),$$

for $\sigma \in \Omega_V = \{+1, -1\}^V$, where h is a real number called the external field, and $\|\cdot\|$ is the L_1 norm. We then define the finite Gibbs measure on V by

$$q^w_{V,\beta,h}(\sigma) = \left[\sum_{\sigma'\in\Omega_V} \exp\{-\beta H_V^w(\sigma')\}\right]^{-1} \exp\{-\beta H_V^w(\sigma)\}.$$

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Here β is a positive number called the inverse temperature. For each $\beta > 0$ and $h \in R^1$, a Gibbs measure is a probability measure $\mu_{\beta, h}$ on Ω in the sense of the following DLR equation:

$$\mu_{\beta, h}(\cdot|\mathscr{F}_{V^C})(w) = q^w_{V, \beta, h},$$

where $V^C = Z^2 \setminus V$. Let β_c be the critical value such that if $\beta < \beta_c$ or $h \neq 0$, the Gibbs measure is unique for (β, h). Let $E_{\beta, h}$ and $E^w_{V, \beta, h}$ denote the expectations with respect to $\mu_{\beta, h}$ and $q^w_{V, \beta, h}$, respectively. We say that a probability measure μ on $(\Omega, \mathscr{F})$ possesses a mixing property if there exist constants $C > 0$ and $\alpha > 0$ such that for every pair of finite subsets V and W of Z^2 with $V \subset W$,

$$\sup_{\substack{\omega\in\Omega \\ A\in\mathscr{F}_V}} \left|\mu(A) - \mu(A \mid \mathscr{F}_{W^c})(\omega)\right| \le C|V| \exp\{-\alpha d(V, W^c)\}, \tag{1.1}$$

where for $V_1, V_2 \subset Z^2$, $d(V_1, V_2)$ denotes the distance between V_1 and V_2; that is,

$$d(V_1, V_2) = \inf\{|x - y|;\ x \in V_1,\ \ y \in V_2\}.$$

This property is often called the "weak mixing property" compared with Dobrushin–Shlosman's strong mixing property (see [13] and [14]). In this paper, we need not be so serious as to distinguish these two mixing properties, and we call the above property simply "the mixing property." It is proved [14] that when $\beta < \beta_c$ or $h \neq 0$, $\mu_{\beta, h}$ has mixing property. Furthermore, let X be a $\mathscr{F}_V$-measurable random variable for a finite $V \subset Z^2$. If

$$|X| \le M \text{ for some number } M,$$

then it follows from (1.1) that

$$\left|E_{\beta, h}X - E^w_{W, \beta, h}X\right| \le CM|V| \exp(-\alpha d(V, W^c)), \tag{1.2}$$

since every $\mathscr{F}_V$-measurable function is a simple function if V is a finite subset of Z^2. Sometimes we use the mixing property (1.1) in the following form: if $V_1, V_2 \subset Z^2$ are finite sets, $V_1 \cap V_2 = \varnothing$ and A, B are cylinder sets such that $A \in \mathscr{F}_{V_1}$ and $B \in \mathscr{F}_{V_2}$, then

$$\left|\mu_{\beta, h}(A \cap B) - \mu_{\beta, h}(A)\mu_{\beta, h}(B)\right| \le C|V_1| \exp(-\alpha d(V_1, V_2))\mu_{\beta, h}(B). \tag{1.3}$$

Furthermore, as in (1.2), if X is $\mathscr{F}_{V_1}$-measurable and if $|X| \le M$, then

$$\left|E_{\beta, h}(X \mid \mathscr{F}_{V_2})(w) - E_{\beta, h}(X)\right| \le CM|V_1| \exp(-\alpha d(V_1, V_2)). \tag{1.4}$$

We prove (1.4) first. Let $V = V_1$ and $W = \{x \in Z^2;\ d(x, V_1) < d(V_1, V_2)\}$. Then $W^C \supset V_2$ and $d(V_1, W^C) = d(V_1, V_2)$. By (1.2) we have

$$\begin{aligned} -CM|V_1| \exp(-\alpha d(V_1, V_2)) &\le E^w_{W, \beta, h}(X) - E_{\beta, h}(X) \\ &\le CM|V_1| \exp(-\alpha d(V_1, V_2)). \end{aligned}$$

Note that $E^w_{W,\beta,h}(X)$ is equal to $E_{\beta,h}(X \mid \mathscr{F}_{W^c})(w)$ by the DLR equation. Taking expectation with respect to $\mu_{\beta,h}(\cdot \mid \mathscr{F}_{V_2})(w)$, we obtain

$$\left|E_{\beta,h}(X \mid \mathscr{F}_{V_2})(w) - E_{\beta,h}(X)\right| \le CM|V_1|\exp(-\alpha d(V_1, V_2)),$$

proving (1.4). Now, take 1_A as X in (1.4). Then we obtain that $M = 1$ and

$$\begin{aligned} -C|V_1|\exp(-\alpha d(V_1, V_2)) &\le \mu_{\beta,h}(A \mid \mathscr{F}_{V_2})(w) - \mu_{\beta,h}(A) \\ &\le C|V_1|\exp(-\alpha d(V_1, V_2)). \end{aligned} \tag{1.5}$$

Integrating every side of (1.5) on the set B with respect to $\mu_{\beta,h}$, we obtain

$$\begin{aligned} -C|V_1|\exp(-\alpha d(V_1, V_2))\mu_{\beta,h}(B) &\le \mu_{\beta,h}(A \cap B) - \mu_{\beta,h}(A)\mu_{\beta,h}(B) \\ &\le C|V_1|\exp(-\alpha d(V_1, V_2))\mu_{\beta,h}(B), \end{aligned}$$

which proves (1.3).

Let $\mathscr{C}_0^+$(resp., $\mathscr{C}_0^-$) be the $+$ cluster (resp., $-$ cluster) in Z^2 containing the origin and

$$\theta(\beta, h) = \mu_{\beta,h}(|\mathscr{C}_0^+| = \infty).$$

Define critical value for each fixed β as

$$h_c(\beta) = \sup\{h\colon \theta(\beta, h) = 0\}.$$

This $h_c(\beta)$ is equal to zero when $\beta \ge \beta_c$ and is positive when $\beta < \beta_c$ (see [7]). It is proved in [8] that if $\beta < \beta_c$ and $|h| < h_c(\beta)$, then

$$\mu_{\beta,h}(|\mathscr{C}_0^+| \ge n \text{ or } |\mathscr{C}_0^-| \ge n) \le C_1 \exp(-\alpha_1 n) \tag{1.6}$$

for some positive constants C_1 and α_1.

Let us consider first passage percolation on Z^2 (see [4] and [3]). Define $X(e)$ to be a random variable such that

$$X(e) = \begin{cases} 0, & \text{if } \sigma(u) = \sigma(v), \\ 1, & \text{if } \sigma(u) \ne \sigma(v), \end{cases} \tag{1.7}$$

where u, v are two vertices of the bond e in Z^2. In this paper, we always use e to represent bonds and u, v or x to represent vertices. A path $r = \{x_0, e_1, x_1, \ldots, e_n, x_n\}$ is an alternating sequence of vertices and bonds such that e_i is the bond connecting x_{i-1} and x_i and $\{x_i\}$ are vertices with $d(x_{i-1}, x_i) = 1$ for $1 \le i \le n$. For each path r define the passage time of r as

$$t(r) = \sum_{e \in r} X(e).$$

For any two sets A and B, define the first passage time from A to B by

$$T(A, B) = \inf\{t(r)\colon r \text{ a path from } A \text{ to } B\}.$$

If we focus on a special configuration w, we denote by $T(A, B)(w)$. In this paper, we would like to study the process

$$a_{0,n} = T((0, 0), (n, 0)).$$

It follows from a subadditive argument that

$$\lim_{n\to\infty} \frac{a_{0,n}}{n} = \inf_n \frac{E_{\beta,h}a_{0,n}}{n} = \nu \quad \text{a.s. and in } L_1. \tag{1.8}$$

It is proved in [4] that

$$\nu > 0 \text{ if } \beta < \beta_c \quad \text{and} \quad |h| < h_c(\beta). \tag{1.9}$$

For $|h| > h_c(\beta)$, it is easy to show (see [18]) that $\nu = 0$ since there exists an infinite $+$ cluster when h is positive, and an infinite $-$ cluster when h is negative. For $h = 0$ and $\beta > \beta_c$ with a positive or negative boundary condition, it is also easy to show that $\nu = 0$ by the same reason. The challenging question is whether $\nu = 0$ when $\beta = \beta_c$ and $h = 0$. We are unable to show it. If we consider the standard i.i.d. first passage percolation (see [10]), i.e., $\{X(e)\}$ is i.i.d., similar results as (1.8) and (1.9) have been known since 1965 and 1986, respectively (see [6] and [10]). It is of historical interest to find the convergence speed of

$$E_{\beta,h}a_{0,n} - n\nu \tag{1.10}$$

and the fluctuation of

$$a_{0,n} - E_{\beta,h}a_{0,n}. \tag{1.11}$$

For these problems, Kesten developed a remarkable martingale technique (see [11]) which gave nontrivial rates for (1.10) and (1.11). Later, Talagrand investigated (1.11) by a different way [15]. Some other studies for (1.10) can also be found in [1]. However, the methods depend heavily on the independence of $\{X(e)\}$. They do not work on our Ising passage time. Here we use another approach developed by Kesten and Zhang [12] to get the following theorems.

THEOREM 1. *If $\beta < \beta_c$ and $|h| < h_c(\beta)$, then there exists a constant $C_2 > 0$ such that*

$$n\nu \le E_{\beta,h}a_{0,n} \le n\nu + C_2(n\log^5 n)^{1/2}.$$

THEOREM 2. *If $\beta < \beta_c$ and $|h| < h_c(\beta)$, then there exist positive constants C_3 and α_2 such that for all sufficiently large n and x with $1 \le x \le \sqrt{n}$,*

$$\mu_{\beta,h}\left(\frac{|a_{0,n} - E_{\beta,h}a_{0,n}|}{n^{1/2}\log^2 n} \ge x\right) \le C_3 \exp(-\alpha_2 x^2).$$

Let $Q(n)$ denote the square $[-n,n]^2 \cap Z^2$, and let $\partial Q(n)$ be its inner boundary: set of points in $Q(n)$ such that there is a point y outside $Q(n)$ with $\|x-y\| = 1$. Another passage time,

$$c_{0,n} = T((0,0), \partial Q(n)),$$

has been considered in the literature (see [12] and [18]) since it is easy to show that there is a path (0,0) to $\partial Q(n)$ contained in $Q(n)$ which possesses the passage time $c_{0,n}$. Here we give the following theorems to deal with $c_{0,n}$.

THEOREM 3. *If $\beta < \beta_c$ and $|h| < h_c(\beta)$, then there exist $C_4, C_5 > 0$,*

$$n\nu - C_4(n \log^5 n)^{1/2} \leq E_{\beta,h} c_{0,n} \leq n\nu + C_5(n \log^5 n)^{1/2}.$$

THEOREM 4. *If $\beta < \beta_c$ and $|h| < h_c(\beta)$, then there exist positive constants C_6 and α_3 such that for every $x > 0$ and for sufficiently large n,*

$$\mu_{\beta,h}\left(\frac{|c_{0,n} - E_{\beta,h} c_{0,n}|}{n^{1/2} \log^2 n} \geq x\right) \leq C_6 \exp(-\alpha_3 x^2).$$

COROLLARY 5. *If $\beta < \beta_c$ and $|h| < h_c(\beta)$,*

$$\lim_{n\to\infty} \frac{c_{0,n}}{n} = \nu \quad a.s \text{ and in } L_1.$$

REMARK 1. The method of proof also works for i.i.d. standard first passage percolation with 0–1 valued bond on the Z^d lattice. In fact, we only need to change the closed circuits in the following proofs to the closed surfaces (see [9]). Then the same argument of the following proofs can be adapted to show Theorems 1–4 for i.i.d. first passage percolation on Z^d for $d \geq 2$. In fact, for the i.i.d. case, we could show that Theorem 2 holds without the term $\log^2 n$.

REMARK 2. Corollary 5 was proved in [6] for i.i.d. first passage percolation. Here we give a different proof.

REMARK 3. The passage time

$$b_{0,n} = T((0,0), \text{ the right boundary of } Q(n))$$

is also considered in the literature (see [10]). Since

$$c_{0,n} \leq b_{0,n} \leq a_{0,n},$$

Theorem 3 holds for $b_{0,n}$. On the other hand, we may adapt the same arguments in Section 3 to show that Theorem 4 also holds for $b_{0,n}$.

REMARK 4. Since the knowledge of Ising models for $d > 2$ is very limited, we do not know whether Theorems 1–4 hold for $d > 2$.

REMARK 5. We believe that

$$E_{\beta,h}|a_{0,n} - E_{\beta,h} a_{0,n}| = O(n^{1/3})$$

as is conjectured for i.i.d. first passage percolation for $d = 2$.

REMARK 6. It might be possible to get a better estimate such as

$$\mu_{\beta,h}\left(\frac{|\rho_{0,n} - E_{\beta,h} \rho_{0,n}|}{n^{1/2} \log^{1+\delta} n} \geq x\right) \leq C' \exp(-\alpha' x^2)$$

for some positive constants δ, C' and α' (or even better) where $\rho = a$ or c. However, it is more important to improve the power estimate for n.

REMARK 7. As a generalization of $a_{0,n}$, one also considers the vertex-to-vertex passage time $T((0,0), nu)$ which is the passage time $(0,0)$ to the nearest vertex on Z^2 to nu, for any unit vector u. If several vertices of Z^2 minimize the distance to nu, then we take $T((0,0), nu) = T((0,0), A)$ with A equal to the set of vertices of Z^2 with minimal distance to nu. The proof of Theorem 2 can be adapted to show

$$\mu_{\beta,h}\big(\big|T((0,0), nu) - E_{\beta,h}T((0,0), nu)\big| \geq x\sqrt{n}\log^2 n\big) \leq \exp(-\alpha'' x^2)$$

for some positive constant α''.

2. Concentrations at means. The bond set $\{e; X(e) = 0\}$ is divided into connected components. We call a connected component of the above set a 0-cluster. The size $|\mathscr{C}|$ of a 0-cluster $\mathscr{C}$ is the number of bonds belonging to $\mathscr{C}$. Let n be a positive integer, and we fix it. We say that a bond e in Z^2 is open if:

1. $X(e) = 0$;
2. e does not belong to a 0-cluster with size larger than $\log^2 n$, otherwise we say that e is closed.

Strictly speaking, we should use the word "n-open" for this notion of open edges. But we are fixing n, and therefore when we simply say that e is "open," it always means that the above two conditions are satisfied for e. Let

$$Z(e) = \begin{cases} 0, & \text{if } e \text{ is open,} \\ 1, & \text{if } e \text{ is closed.} \end{cases}$$

Let

$$\widehat{T}(A, B) = \inf\{\hat{t}(r) : r \text{ a path from } A \text{ to } B\}$$

and

$$\hat{a}_{0,n} = \widehat{T}((0,0), (n,0)),$$

where

$$\hat{t}(r) = \sum_{e \in r} Z(e).$$

Clearly,

$$T(A, B) \leq \widehat{T}(A, B) \quad \text{for any } A \text{ and } B.$$

By the subadditive argument, we have

$$\lim_{n\to\infty} \frac{\hat{a}_{0,n}}{n} = \hat{\nu} \quad \text{a.s. and in } L_1.$$

A path with each bond open or closed is called an open path or a closed path. Let $\mathscr{C}_n(x)$ be the open cluster containing x on $[-n, n]^2$ with free boundary condition. Namely, we delete all closed edges from $[-n, n]^2$ and we write $\mathscr{C}_n(x)$

for the connected component of the ensuing graph in $[-n, n]^2$, which contains the vertex x. Clearly,

$$|\mathscr{C}_n(x)| \le \log^2 n \quad \text{for each } x \in Q(n). \tag{2.1}$$

Next we introduce the duality of planar graphs. Define Z^* as the dual graph of Z^2 with vertices $\{v + (1/2, 1/2)\}$ for $v \in Z^2$ and bonds joining all pairs of vertices which are unit distance apart. For any bond set $A \subset Z^2$, we write $A^* \subset Z^*$ for the corresponding bonds of the dual graph of A. We declare each bond $e^* \subset Z^*$ open or closed if e is open or closed. In other words, if e^* crosses an open (closed) bond in Z^2, then e^* is open (closed). With this definition, we can obtain (see [5] for details) that if there exists a closed dual circuit D^* in $Q(n)^*$ surrounding some set $A \subset Q(n-1)$, then any path on Z^2 from A to $\partial Q(n)$ has to use at least one closed bond in D. If $|h| < h_c(\beta)$, then there is no infinite open cluster so that there are infinitely many closed dual circuits surrounding the origin. Let

$$\Lambda_1^*, \ldots, \Lambda_m^*, \ldots = \{\Lambda_m^*\}.$$

be a sequence of closed dual circuits with

$$\Lambda_i^* \cap \Lambda_j^* = \varnothing,$$

such that Λ_1^* is the innermost dual closed circuit surrounding the origin, ..., Λ_m^* is the innermost one surrounding the $m-1$th innermost one, where the innermost circuit is in the sense of the area surrounded by the circuit. Each Λ_i^* divides R^2 into two connected parts $A(\Lambda_i^*)$ and $B(\Lambda_i^*)$, where $A(\Lambda_i^*)$ contains the origin and Λ_i^* itself, and $B(\Lambda_i^*) = R^2 \setminus A(\Lambda_i^*)$ contains the infinite part. Then Λ_i^* has two vertex boundaries $\partial A(\Lambda_i^*)$ and $\partial B(\Lambda_i^*)$: the inside boundary and outside boundary such that for each $x \in \partial A(\Lambda_i^*)$ there is a path connecting x to the origin without using any bond of Λ_i, the dual set of which is Λ_i^* , and for each $y \in \partial B(\Lambda_i^*)$ there is a path connecting x to ∞ also without using any bond of Λ_i. It follows from a standard topological discussion (see [2]) that we have the following lemma.

LEMMA 1. *For each* $x \in \partial A(\Lambda_i^*)$, $i \ge 1$,

$$\widehat{T}((0,0), x) = i - 1,$$

and for each $x \in \partial B(\Lambda_i^*)$,

$$\widehat{T}((0,0), x) = i.$$

Furthermore, we shall give the following more detailed lemma.

LEMMA 2. *In the event that* $\Lambda_i^* = \Gamma^*$ *for some bond set* $\Gamma^* \subset Q(n)^*$,

$$\widehat{T}((0,0), \partial Q(n)) = i + \widehat{T}(\partial B(\Lambda_i^*), \partial Q(n)).$$

PROOF. Let path r realize $\widehat{T}((0,0), \partial Q(n))$ (such a path must exist since the bond times are 0–1 valued). Then there exists a vertex x on r with $x \in \partial B(\Gamma^*)$, and

$$\widehat{T}((0,0), \partial Q(n)) = \widehat{T}((0,0), x) + \widehat{T}(x, \partial Q(n)) \geq i + \widehat{T}(\partial B(\Gamma^*), \partial Q(n)).$$

On the other hand, let path r' realize $\widehat{T}(\partial B(\Gamma^*), \partial Q(n))$ and suppose that r' starts at $x \in \partial B(\Gamma^*)$. Then

$$\widehat{T}((0,0), \partial Q(n)) \leq \widehat{T}((0,0), x) + \widehat{T}(x, \partial Q(n)) = i + \widehat{T}(\partial B(\Gamma^*), \partial Q(n)). \quad \square$$

It follows from Proposition 2.3 in [9] and the definition of $Z(e)$ that we have the following lemma.

LEMMA 3. *The event* $\{\Lambda_i^* = \Gamma^*\}$ *for some fixed* $\Gamma^* \subset Q(n)^*$ *only depends on* $w(x)$ *for such* x*'s with* $\operatorname{dist}(x, A(\Gamma^*)) \leq \log^2 n$, *and the random variable* $\widehat{T}(\Gamma^*, \partial Q(n))$ *only depends on* $w(y)$ *for such* y's *with* $\operatorname{dist}(y, Q(n) \setminus A(\Gamma^*)) \leq \log^2 n$.

It follows from Lemma 2 and the definition of $Z(e)$ again that we have the following.

LEMMA 4. *For each* $x \in \partial A(\Lambda_i^*)$ *there exists* $y \in \partial B(\Lambda_{i-1}^*)$ *such that*

$$\|x - y\| \leq \log^2 n.$$

For $p = 1, 2, \ldots$, let

$$\mathscr{F}_p = \sigma\text{-field generated by } Z(e) \text{ for } e \in A(\Lambda_p^*),$$

where $\mathscr{F}_p$ consists of unions of sets of the form

$$\{\Lambda_p^* = \Gamma^*, (Z(e_1), \ldots, Z(e_k)) \in B\}$$

for Γ^* a dual circuit surrounding $(0,0)$, and $e_1, \ldots, e_k \subset A(\Gamma^*)$, B a k-dimensional Borel set. Here $\mathscr{F}_0$ is trivial. Clearly,

$$\mathscr{F}_i \subset \mathscr{F}_{i+1}.$$

Note that $Q(n) \subset A(\Lambda_{n+1}^*)$ and that $\widehat{T}((0,0), \partial Q(n))$ is $\mathscr{F}_n$-measurable, so that with the definition of $\mathscr{F}_n$,

$$\begin{aligned} E_{\beta,h}\big[\widehat{T}((0,0), \partial Q(n)) - E_{\beta,h}(\widehat{T}((0,0), \partial Q(n)))\big|\mathscr{F}_n\big] \\ = \widehat{T}((0,0), \partial Q(n)) - E_{\beta,h}(\widehat{T}((0,0), \partial Q(n))). \end{aligned} \tag{2.2}$$

We first show that there exists a constant $\tilde{\alpha} > 0$ such that

$$\begin{aligned} \mu_{\beta,h}\big(\big|\widehat{T}((0,0), \partial Q(n)) - E_{\beta,h}(\widehat{T}((0,0), \partial Q(n)))\big| \geq x\sqrt{n}\log^2 n\big) \\ \leq 2\exp(-\tilde{\alpha}x^2). \end{aligned} \tag{2.3}$$

To show this, we apply the Azuma–Hoeffding inequality (see [17]).

AZUMA–HOEFFDING LEMMA. *Let $M = (M_i)_{i\geq 0}$ be a martingale defined on some probability space $\{\Omega, P\}$ with $M_0 = 0$ such that, for some positive constants c_i, $i \geq 1$,*

$$|M_i - M_{i-1}| \leq c_i.$$

Then for any $x > 0$,

$$P(\sup_{i\leq k} M_i \geq x) \leq \exp\biggl(-\frac{x^2}{2\sum_{i=1}^k c_i^2}\biggr).$$

Let

$$M_i = E_{\beta,h}\{[\widehat{T}((0,0),\partial Q(n)) - E_{\beta,h}(\widehat{T}((0,0),\partial Q(n)))] \mid \mathscr{F}_i\}.$$

Since $\mathscr{F}_0$ is trivial, obviously $\{M_i\}_{i=0}^{\infty}$ is a martingale sequence with $M_0 = 0$. We now need to bound the martingale increments. Let Δ_i be the martingale increment,

$$\begin{aligned}
\Delta_i &= M_i - M_{i-1}\\
&= E_{\beta,h}\{\widehat{T}((0,0),\partial Q(n)) - E_{\beta,h}(\widehat{T}((0,0),\partial Q(n))) \mid \mathscr{F}_i\}\\
&\qquad - E_{\beta,h}\{\widehat{T}((0,0),\partial Q(n)) - E_{\beta,h}(\widehat{T}((0,0),\partial Q(n))) \mid \mathscr{F}_{i-1}\}\\
&= E_{\beta,h}\{\widehat{T}((0,0),\partial Q(n)) \mid \mathscr{F}_i\} - E_{\beta,h}\{\widehat{T}((0,0),\partial Q(n)) \mid \mathscr{F}_{i-1}\}.
\end{aligned}$$

Let E_i be the event $\{\Lambda_i^* \subset [-n,n]^2\}$. For $w \in E_i^C = \Omega \setminus E_i$, there exists a path r inside Λ_i^* such that

$$t(r) = \widehat{T}((0,0),\partial Q(n)).$$

Hence, $\widehat{T}((0,0),\partial Q(n)) I_{E_i^C}$ is $\mathscr{F}_i$-measurable so that for $w \in E_i^C$,

$$E_{\beta,h}(\widehat{T}((0,0),\partial Q(n)) \mid \mathscr{F}_i)(w) = \widehat{T}((0,0),\partial Q(n))(w). \tag{2.4}$$

If $w \in E_{i-1}^C$, then $w \in E_i^C$ since

$$\Lambda_{i-1}^* \subset A(\Lambda_i^*).$$

Therefore, for $w \in E_{i-1}^C$ it follows from (2.4) that

$$\begin{aligned}
\Delta_i &= E_{\beta,h}(\widehat{T}((0,0),\partial Q(n)) \mid \mathscr{F}_i)(w)\\
&\quad - E_{\beta,h}([\widehat{T}((0,0),\partial Q(n))] | \mathscr{F}_{i-1})(w) = 0.
\end{aligned}\tag{2.5}$$

If $w \in E_{i-1} \cap E_i^C$, then we have that

$$A(\Lambda_i^*(w)) \not\subset [-n,n]^2 \quad \text{and} \quad A(\Lambda_{i-1}^*(w)) \subset [-n,n]^2.$$

This means that $\Lambda_{i-1}^*(w)$ is within distance $\log^2 n$ from $\Lambda_i^*(w)$ and hence $\Lambda_{i-1}^*(w)$ is within distance $\log^2 n$ from $\partial Q(n)$. By Lemma 2, it follows that if $w \in E_{i-1}$, then

$$\widehat{T}((0,0),\partial Q(n)) = i - 1 + \widehat{T}(\partial B(\Lambda_{i-1}^*),\ \partial Q(n)) \geq i - 1.$$

In general, if $\Lambda_{i-1}^*(w)$ is within distance $\log^2 n$ from $\partial Q(n)$, then we have

$$\widehat{T}(\partial B(\Lambda_{i-1}^*), \partial Q(n)) \leq \log^2 n.$$

Therefore we have for $w \in E_{i-1} \cap E_i^C$,

$$0 \leq E_{\beta,h}(\widehat{T}(\partial B(\Lambda_{i-1}^*), \partial Q(n)) \mid \mathscr{F}_{i-1})(w) \leq \log^2 n \tag{2.6}$$

and also

$$\begin{aligned} i-1 &\leq E_{\beta,h}(\widehat{T}((0,0), \partial Q(n)) \mid \mathscr{F}_i)(w) \\ &= \widehat{T}((0,0), \partial Q(n))(w) \\ &\leq i-1+\log^2 n. \end{aligned} \tag{2.7}$$

Combining (2.6) with (2.7), we obtain

$$\begin{aligned} |\Delta_i| = \big| &E_{\beta,h}(\widehat{T}((0,0), \partial Q(n)) \mid \mathscr{F}_{i-1})(w) \\ &-E_{\beta,h}(\widehat{T}((0,0), \partial Q(n)) \mid \mathscr{F}_i)(w)\big| \leq \log^2 n. \end{aligned} \tag{2.8}$$

This, together with (2.5), (2.7) implies that $|\Delta_i| \leq \log^2 n$ for $w \in E_{i-1} \cap E_i^C$.

Now we focus on the case that $w \in E_i \cap E_{i-1}$. By Lemma 2,

$$E_{\beta,h}(\widehat{T}((0,0), \partial Q(n)) | \mathscr{F}_i)(w) = i + E_{\beta,h}(\widehat{T}(\partial B(\Lambda_i^*), \partial Q(n)) | \mathscr{F}_i)(w).$$

To estimate $E_{\beta,h}(\widehat{T}(\partial B(\Lambda_i^*), \partial Q(n)) | \mathscr{F}_i)(w)$, let $E_i(\Gamma^*)$ be the event,

$$\{w\colon \Lambda_i^*(w) = \Gamma^*\}$$

for some fixed dual circuit $\Gamma^* \subset [-n, n]^2$ surrounding the origin. Clearly,

$$\bigcup_{\Gamma^*} E_i(\Gamma^*) = E_i,$$

where the union is taken over all such Γ^*'s. Note that $E_i(\Gamma^*)$ only depends on $\{Z(e); e \cap A(\Gamma^*) \neq \varnothing\}$. Fix a dual circuit $\Gamma^* \subset [-n, n]^2$. If $w \in E_i(\Gamma^*)$, then

$$E_{\beta,h}(\widehat{T}(\partial B(\Lambda_i^*), \partial Q(n)) | \mathscr{F}_i)(w) = E_{\beta,h}(\widehat{T}(\partial B(\Gamma^*), \partial Q(n)) | \mathscr{F}_i)(w).$$

Let

$$\mathscr{F}(\Gamma^*) = \sigma\{Z(e)\colon e \cap A(\Gamma^*) \neq \varnothing\}.$$

Then it is easy to see that for $w \in E_i(\Gamma^*)$,

$$E_{\beta,h}(\widehat{T}(\partial B(\Gamma^*), \partial Q(n)) | \mathscr{F}_i)(w) = E_{\beta,h}(\widehat{T}(\partial B(\Gamma^*), \partial Q(n)) | \mathscr{F}(\Gamma^*))(w).$$

On $E_i(\Gamma^*)$, we can find a dual circuit κ^* surrounding Γ^* (see Figure 1) such that

$$3\log^2 n \leq d(\partial A(\kappa^*), \partial B(\Gamma^*)) \leq 4\log^2 n.$$

To show the existence of κ^*, we can find a dual circuit λ^* surrounding Γ^* such

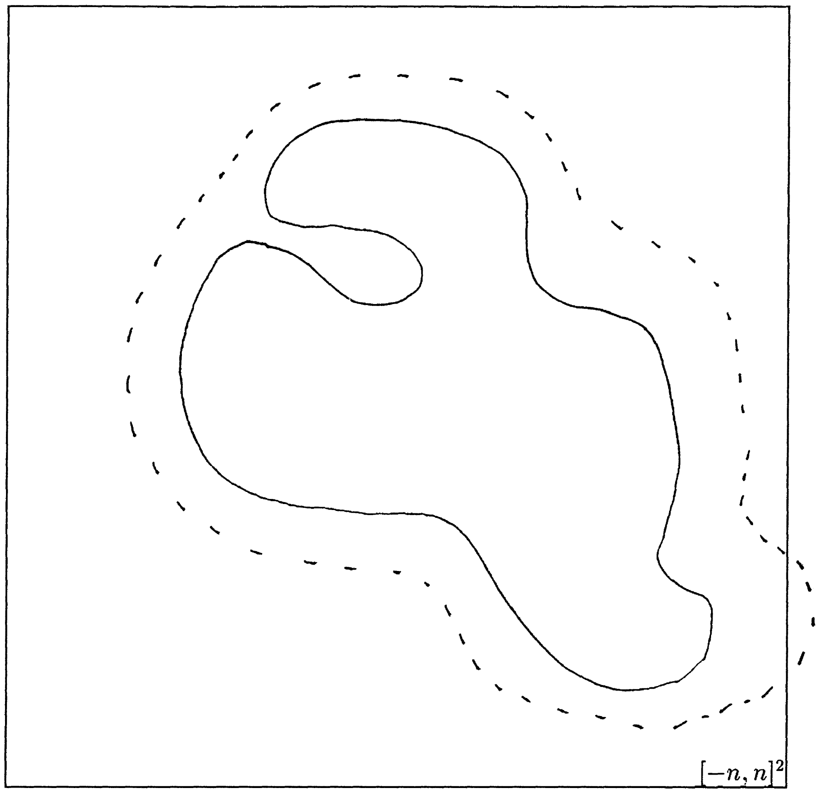

FIG. 1. *The dot curve is κ^* and the solid curve is Γ^*.*

that

$$3\log^2 n \le \|x-y\| \quad \text{for } x \in \partial B(\Gamma^*),\ \ y \in \partial A(\lambda^*).$$

Then we can shrink the area of $A(\lambda^*)$ to achieve the requirements of κ^*. In fact, for a fixed Γ^*, we may choose such κ^* by a unique way. Note that

$$\widehat{T}(\partial B(\Gamma^*), \partial Q(n)) = \widehat{T}(\partial B(\Gamma^*), Z^2\backslash Q(n-1)),$$

since $\partial Q(n)$ is a subset of $Z^2\backslash Q(n-1)$, and every path connecting $\partial B(\Gamma^*)$ with $Z^2\backslash Q(n-1)$ must pass one of the points in $\partial Q(n)$. Note also that

$$\begin{aligned}\widehat{T}(\partial B(\kappa^*), Z^2\backslash Q(n-1)) &\le \widehat{T}(\partial B(\Gamma^*), Z^2\backslash Q(n-1)) \\ &\le \widehat{T}(\partial B(\kappa^*), Z^2\backslash Q(n-1)) + 4\log^2 n.\end{aligned} \tag{2.9}$$

In fact, (2.9) is clearly true if $\kappa^* \subset [-n,n]^2$. If $\kappa^* \not\subset [-n,n]^2$, then we can find a point y in $\partial A(\kappa^*)$ which does not belong to $Q(n-1)$. Then there is a point $x \in \partial B(\Gamma^*)$ such that $\|x-y\| \le 4\log^2 n$. Let r be a shortest path connecting x

with y. Then r is also a path connecting $Z^2\backslash Q(n-1)$ with $\partial B(\Gamma^*)$. Hence,

$$\widehat{T}(\partial B(\Gamma^*), Z^2\backslash Q(n-1)) \le \hat{t}(r) \le 4\log^2 n. \tag{2.10}$$

Since $\partial A(\kappa^*) \cap Z^2\backslash Q(n-1) \ne \varnothing$, we have $\widehat{T}(\partial A(\kappa^*), Z^2\backslash Q(n-1)) = 0$. This, together with (2.10) proves (2.9) in the case that $\kappa^* \not\subset [-n, n]^2$.

Now we want to use the mixing property (1.2). The random variable in this case is $\widehat{T}(\partial B(\kappa^*), Z^2\backslash Q(n-1))$ which depends on the configuration in the set $V = \{x \in Z^2;\ d(x, B(\kappa^*) \cap Q(n)) \le \log^2 n\}$. Note that we have

$$0 \le \widehat{T}(\partial B(\kappa^*), Z^2\backslash Q(n-1)) \le \widehat{T}(\partial B(\Gamma^*), \partial Q(n)) \le n. \tag{2.11}$$

Put

$$W = \{x \in B(\Gamma^*) \cap Q(2n);\ d(x, \partial B(\Gamma^*)) \ge \log^2 n\}.$$

Then we have $W \supset V$, $d(V, W^C) \ge \log^2 n$, and $\mathscr{F}(\Gamma^*) \subset \mathscr{F}_{W^C}$. By (1.2) and (2.11) we have for $w \in E_i(\Gamma^*)$,

$$\begin{aligned} &\big|E_{\beta,h}\big[\widehat{T}(\partial B(\kappa^*), Z^2\backslash Q(n-1)) \mid \mathscr{F}_{W^C}\big](w) \\ &\qquad - E_{\beta,h}\big[\widehat{T}(\partial B(\kappa^*), Z^2\backslash Q(n-1))\big]\big| \\ &\le \text{Const.} \times n \cdot n^2 \exp(-\alpha \log^2 n) = \text{Const.} \times n^3 \exp(-\alpha\log^2 n). \end{aligned} \tag{2.12}$$

Taking conditional expectation of (2.12) with respect to $\mu_{\beta,h}(\cdot \mid \mathscr{F}(\Gamma^*))(w)$, we obtain

$$\begin{aligned} &\big|E_{\beta,h}\big[\widehat{T}(\partial B(\kappa^*), Z^2\backslash Q(n-1)) \mid \mathscr{F}(\Gamma^*)\big](w) \\ &\qquad - E_{\beta,h}\big[\widehat{T}(\partial B(\kappa^*), Z^2\backslash Q(n-1))\big]\big| \\ &\le \text{Const.} \times n^3 \exp(-\alpha\log^2 n) \end{aligned} \tag{2.13}$$

on $E_i(\Gamma^*)$. Therefore, it follows from (2.13) and (2.9) that for $w \in E_i(\Gamma^*)$,

$$\begin{aligned} &\big|E_{\beta,h}[\widehat{T}(\partial B(\Lambda_i^*), \partial Q(n))|\mathscr{F}_i](w) - E_{\beta,h}[\widehat{T}(\partial B(\Gamma^*), \partial Q(n))]\big| \\ &\le \big|E_{\beta,h}[\widehat{T}(\partial B(\kappa^*), Z^2\backslash Q(n-1))|\mathscr{F}(\Gamma^*)] \\ &\qquad -E_{\beta,h}[\widehat{T}(\partial B(\kappa^*), Z^2\backslash Q(n-1))]\big| \\ &\quad +8\log^2 n \\ &\le 8\log^2 n + \text{Const.} \times n^3 \exp(-\alpha\log^2 n). \end{aligned} \tag{2.14}$$

Since the estimate (2.14) is uniform in Γ^* such that $E_i(\Gamma^*) \ne \varnothing$, we have

$$\begin{aligned} &\big|E_{\beta,h}[\widehat{T}(\partial B(\Lambda_i^*), \partial Q(n))|\mathscr{F}_i](w) \\ &\qquad - \sum_{\Gamma^*} E_{\beta,h}[\widehat{T}(\partial B(\Gamma^*), \partial Q(n))] 1_{E_i(\Gamma^*)}(w)\big| \\ &\le 8\log^2 n + Cn^3 \exp(-\alpha\log^2 n). \end{aligned} \tag{2.15}$$

It follows from (2.15) that, for $w \in E_i \cap E_{i-1}$,

$$\begin{aligned}|\Delta_i(w)| \le{}& |i-(i-1)| \\ &+ \sum_{\Gamma_1^*, \Gamma_2^*} \left|E_{\beta,h}(\widehat{T}(\partial B(\Gamma_1^*), \partial Q(n))) - E_{\beta,h}(\widehat{T}(\partial B(\Gamma_2^*), \partial Q(n)))\right| \\ &\quad \times 1_{E_i(\Gamma_1^*)\cap E_{i-1}(\Gamma_2^*)}(w) \\ &+16\log^2 n + \text{Const.} \times n^3 \exp(-\alpha \log^2 n).\end{aligned}$$

It follows from Lemma 4 that for $\Gamma_1^*, \Gamma_2^* \subset [-n,n]^2$ with $E_i(\Gamma_1^*) \cap E_{i-1}(\Gamma_2^*) \neq \varnothing$,

$$\begin{aligned}E_{\beta,h}(\widehat{T}(\partial B(\Gamma_1^*), \partial Q(n))) &\le E_{\beta,h}(\widehat{T}(\partial B(\Gamma_2^*), \partial Q(n))) \\ &\le E_{\beta,h}(T(\partial B(\Gamma_1^*), \partial Q(n))) + \log^2 n,\end{aligned}$$

so that for $w \in E_i \cap E_{i-1}$,

$$|\Delta_i(w)| \le 17\log^2 n + 1 + \text{Const.} \times n^3 \exp(-\alpha\log^2 n). \tag{2.16}$$

It follows from (2.5), (2.8) and (2.16) that there exists C such that

$$|\Delta_i| \le C\log^2 n. \tag{2.17}$$

Finally, it follows from the Azuma–Hoeffding lemma and (2.17) that

$$\begin{aligned}\mu_{\beta,h}&(\widehat{T}((0,0), \partial Q(n)) - E_{\beta,h}\widehat{T}((0,0), \partial Q(n)) \ge x\sqrt{n}\log^2 n) \\ &= \mu_{\beta,h}(M_n \ge x\sqrt{n}\log^2 n) \\ &\le \mu_{\beta,h}\left(\sup_{k\le n} M_k \ge x\sqrt{n}\log^2 n\right) \\ &\le \exp\left(-\tfrac{1}{2}(x\sqrt{n}\log^2 n)^2 \Big/ \sum_{i=1}^n C^2\log^4 n\right) \\ &\le \exp(-x^2/2C^2).\end{aligned}$$

Similarly, we can repeat the same argument to the martingale

$$-M_i = E_{\beta,h}\big[E_{\beta,h}[\widehat{T}((0,0), \partial Q(n))] - \widehat{T}((0,0), \partial Q(n)) \mid \mathscr{F}_i\big]$$

to show

$$\mu_{\beta,h}(E_{\beta,h}[\widehat{T}((0,0), \partial Q(n))] - \widehat{T}((0,0), \partial Q(n)) \ge x\sqrt{n}\log^2 n) \le \exp(-x^2/2C^2),$$

proving (2.3). Now we show Theorem 4 from (2.3).

PROOF OF THEOREM 4. First, note that it suffices to prove Theorem 4 for $x \ge 1$. If $0 < x < 1$, then choose C_6 in Theorem 4 sufficiently large such that $C_6 e^{-1} \ge 1$; then the right-hand side of the desired inequality is always greater than 1, and Theorem 4 is trivially true.

To show Theorem 4, we need to estimate the probability

$$\mu_{\beta,h}(\widehat{T}((0,0),\partial Q(n)) - T((0,0),\partial Q(n)) \geq x\sqrt{n}).$$

Note that it follows from the definition of $\widehat{T}$ and T that

$$\widehat{T}((0,0),\partial Q(n)) \geq T((0,0),\partial Q(n)).$$

Let $E(v_1,\dots,v_m)$ be the event

$$\left\{\max\{|\mathscr{C}^-(v_i)|,|\mathscr{C}^+(v_i)|\} \geq \log^2 n \text{ for } 1 \leq i \leq m\right\}.$$

Let $\{r_n\}$ denote the paths from $(0,0)$ to $\partial Q(n)$ with passage time $T((0,0),\partial Q(n))$. If

$$\widehat{T}((0,0),\partial Q(n)) - T((0,0),\partial Q(n)) \geq x\sqrt{n},$$

then there are at least $\lfloor x\sqrt{n}\rfloor$ vertices, denoted by $\{v_1,\dots v_{\lfloor x\sqrt{n}\rfloor}\}$ in each r_n such that $|\mathscr{C}^+(v_i)| \geq \log^2 n$ or $|\mathscr{C}^-(v_i)| \geq \log^2 n$, where for a real number ξ, $\lfloor \xi \rfloor$ stands for the least integer not less than ξ. Then $E(v_1,\dots,v_m)$ occurs for some $v_1,\dots v_m$ with $\|v_i - v_j\| \geq 3\log^2 n$ if $i \neq j$, and for

$$m = \left\lfloor \frac{\lfloor x\sqrt{n}\rfloor}{9\log^4 n} \right\rfloor - 1.$$

This implies that

$$\begin{aligned} &\mu_{\beta,h}(\widehat{T}((0,0),\partial Q(n)) - T((0,0),\partial Q(n)) \geq x\sqrt{n}) \\ &\quad \leq \sum_{v_1,\dots v_m \in Q(n);\ \|v_i - v_j\| \geq 3\log^2 n\ (i\neq j)} \mu_{\beta,h}(E(v_1,\dots,v_m)). \end{aligned} \tag{2.18}$$

Let us estimate the right-hand side of (2.18). We use (1.3) for $V_1 = Q(\log^2 n) + v_1$, $V_2 = \bigcup_{2\leq j\leq m}\{Q(\log^2 n) + v_j\}$, $A = E(v_1)$ and $B = E(v_2,\dots,v_m)$. Then by (1.3) and (1.6) we obtain

$$\begin{aligned} &\mu_{\beta,h}(E(v_1,\dots,v_m)) \\ &\quad \leq \left[\mu_{\beta,h}(E(v_1)) + C(2\log^2 n + 1)^2\exp(-\alpha\log^2 n)\right]\mu_{\beta,h}(E(v_2,\dots,v_m)) \\ &\quad \leq 9\log^4 n \times \left[\exp(-\alpha_1\log^2 n) + C\exp(-\alpha\log^2 n)\right]\mu_{\beta,h}(E(v_2,\dots,v_m)). \end{aligned} \tag{2.19}$$

If n is sufficiently large, then we can make

$$9(2n+1)^2\log^4 n\left[\exp(-\alpha_1\log^2 n) + C\exp(-\alpha\log^2 n)\right] \leq 1/2,$$

where $(2n+1)^2$ is the number of points in $Q(n)$. Then, summing up both sides of (2.19) over $v_1 \in Q(n)$, we obtain for sufficiently large n,

$$\begin{aligned} &\sum_{\substack{v_1,\dots v_m \in Q(n);\\ \|v_i - v_j\| \geq 3\log^2 n\ (i\neq j)}} \mu_{\beta,h}(E(v_1,\dots,v_m)) \\ &\quad \leq \frac{1}{2} \sum_{\substack{v_2,\dots v_m \in Q(n);\\ \|v_i - v_j\| \geq 3\log^2 n\ (i\neq j)}} \mu_{\beta,h}(E(v_2,\dots,v_m)). \end{aligned} \tag{2.20}$$

Iterating (2.20), we have

$$\sum_{\substack{v_1,\dots v_m \in Q(n);\\ \|v_i - v_j\| \geq 3\log^2 n\ (i\neq j)}} \mu_{\beta,h}(E(v_1,\dots,v_m)) \leq (\tfrac{1}{2})^m \leq (\tfrac{1}{2})^{\lfloor x\sqrt{n}\rfloor/9\log^4 n - 1}. \tag{2.21}$$

By (2.21) and the fact that

$$\widehat{T}((0,0),\partial Q(n)) - T((0,0),\partial Q(n)) \leq n,$$

if n is sufficiently large, then taking $x = 1$, we have

$$\begin{aligned} E(\widehat{T}((0,0),\partial Q(n)) &\leq E(T((0,0),\partial Q(n)) + \sqrt{n} + n(\tfrac{1}{2})^{\lfloor\sqrt{n}\rfloor/9\log^4 n - 1} \\ &\leq Ec_{0,n} + 2\sqrt{n}. \end{aligned}$$

Therefore, if we also take n large such that $\log^2 n > 6$, we get by (2.3) and (2.21),

$$\begin{aligned} &\mu_{\beta,h}(|c_{0,n} - Ec_{0,n}| \geq x\sqrt{n}\log^2 n) \\ &\quad\leq \mu_{\beta,h}(|T((0,0),\partial Q(n)) - \widehat{T}((0,0),\partial Q(n))| \geq x\sqrt{n}\log^2 n/3) \\ &\quad\quad + \mu_{\beta,h}(|\widehat{T}((0,0),\partial Q(n)) - E_{\beta,h}\widehat{T}((0,0),\partial Q(n))| \geq x\sqrt{n}\log^2 n/3) \\ &\quad\leq \text{Const.} \times \exp(-\tilde{\alpha}x^2/9) + \left(\tfrac{1}{2}\right)^{2x\sqrt{n}/(27\log^2 n)} \end{aligned} \tag{2.22}$$

for some $\tilde{\alpha} > 0$. Therefore, Theorem 4 follows. □

REMARK. Equation (2.22) proves a rather stronger statement than Theorem 4. In fact, it is easy to see that the same estimate as Theorem 4 is true if $x = O(\sqrt{n}/\log^2 n)$. However, the type of estimate in Theorem 4 cannot be obtained from (2.22) if $x \gg \sqrt{n}/\log^2 n$, since in this case, the main term is the second term in the right-hand side of (2.22). This is a kind of probability estimate of moderate deviations for $c_{0,n} - E_{\beta,h}(c_{0,n})$.

PROOF OF THEOREM 2. Let

$$J = \min\{j\colon\ (n,0) \in A(\Lambda_j^*)\}.$$

Clearly we have

$$A(\Lambda_J^*) \subset A(\Lambda_{n+1}^*), \tag{2.23}$$

and by Lemma 2, we know that

$$A(\Lambda_J^*) \subset [-n\log^2 n, n\log^2 n]. \tag{2.24}$$

LEMMA 5. *There exists a path r with $t(r) = \hat{a}_{0,n}$ such that r is contained inside $A(\Lambda_J^*)$.*

PROOF. Let γ realize $\widehat{T}((0,0),(n,0))$ and, for vertices a and b on r, let $r(a,b)$ denote the portion of r connecting a and b. Since there are infinitely many closed dual circuits surrounding the origin, the existence of γ can be seen from the following reason. Suppose, contrary to the lemma, that there exists a vertex x on r outside of Λ_J^* [the first dual circuit surrounding both $(0,0)$ and $(n,0)$]. Then there exists a vertex $a \in \partial B(\Lambda_J^*)$ on $r((0,0),x)$ and vertex $b \in \partial A(\Lambda_J^*)$ on $r(x,(n,0))$ and we have

$$\begin{aligned}\widehat{T}((0,0),(n,0)) &= \hat{t}(r((0,0),x)) + \hat{t}(r(x,(n,0)))\\ &\geq \widehat{T}((0,0),a) + \widehat{T}(b,(n,0))\\ &= J + \widehat{T}(b,(n,0))\\ &> J - 1 + \widehat{T}(b,(n,0))\\ &= \widehat{T}((0,0),b) + \widehat{T}(b,(n,0)) \geq \widehat{T}((0,0),(n,0)),\end{aligned}$$

a contradiction. □

Let

$$S_i = E_{\beta,h}\{[\hat{a}_{0,n} - E_{\beta,h}(\hat{a}_{0,n})] | \mathscr{F}_i\}.$$

It follows from Lemma 5 and (2.23) that

$$\Delta_i = S_i - S_{i-1} = 0 \quad \text{if } i \geq n+1. \tag{2.25}$$

We want to use the Azuma–Hoeffding lemma again for the martingale $\{S_i\}_{i=0}^{\infty}$. To this end, we have to estimate $|\Delta_i|$ for every $i \geq 1$. [By (2.25), we only have to estimate $|\Delta_i|$ for $1 \leq i \leq n$.] The argument hereafter in this section is similar to that we made to obtain (2.14)–(2.17), but there are some necessary changes. Let us fix i with $1 \leq i \leq n$ arbitrarily, and let

$$\begin{aligned}F_1 &= \{i < J\} = \{A(\Lambda_i^*) \not\ni (n,0)\},\\ F_2 &= \{i = J\} = \{A(\Lambda_{i-1}^*) \not\ni (n,0),\ A(\Lambda_i^*) \ni (n,0)\},\\ F_3 &= \{i > J\} = \{A(\Lambda_{i-1}^*) \ni (n,0)\}.\end{aligned}$$

It is clear that $F_1, F_2, F_3 \in \mathscr{F}_i$ and $F_1 \cup F_2 \cup F_3 = \Omega$.

If $w \in F_3$, then by Lemma 5, there exists a path r in $A(\Lambda_{i-1}^*)$ which realizes $\widehat{T}((0,0),(n,0))$. Therefore if $w \in F_3$, then

$$S_{i-1}(w) = S_i(w) = \widehat{T}((0,0),(n,0))(w) - E_{\beta,h}[\widehat{T}((0,0),(n,0))]$$

and $\Delta_i = 0$.

If $w \in F_2$, we know that

$$(n,0) \in A(\Lambda_i^*) \backslash A(\Lambda_{i-1}^*),$$

and by Lemma 4, the distance between Λ_{i-1}^* and $(n,0)$ is not larger than $\log^2 n$.

Let r be a path consisting of two pieces r' and r'' such that r' connects $(n,0)$ with some point $x \in \partial B(\Lambda^*_{i-1})$ which satisfies

$$|x-(n,0)| = d((n,0), \partial B(\Lambda^*_{i-1})),$$

and r'' realizes $\widehat{T}((0,0),y)(w)$, where y is the neighboring point of x in $\partial A(\Lambda^*_{i-1})$. Then by Lemma 1, if $w \in F_2$, we have

$$i-1 \le \widehat{T}((0,0),(n,0)) \le \hat{t}(r) \le i-1+\log^2 n.$$

This means that both S_i and S_{i-1} are between $i-1$ and $i-1+\log^2 n$. Thus we have for $w \in F_2$,

$$|\Delta_i| = |S_i - S_{i-1}| \le \log^2 n.$$

Finally, let us discuss the case that $w \in F_1$. For a dual circuit Γ^* such that $(0,0) \in A(\Gamma^*)$ and $(n,0) \notin A(\Gamma^*)$, let

$$E_j(\Gamma^*) = \{\Lambda^*_j = \Gamma^*\}$$

for $j = i$ or $i-1$, as before. Note that $w \in E_j(\Gamma^*)$ implies that $j < J(w)$ and therefore Γ^* should lie inside of $[-n\log^2 n, n\log^2 n]$ by (2.24). Let κ^* be a dual circuit surrounding Γ^* such that

$$3\log^2 n \le d(\partial B(\Gamma^*), \partial A(\kappa^*)) \le 4\log^2 n. \tag{2.26}$$

If κ^* surrounds $(n,0)$, then we argue as in the case that $w \in F_2$ and obtain that $|\Delta_i| \le 4\log^2 n$, since the distance between Λ^*_j and $(n,0)$ does not exceed $4\log^2 n$ by (2.26), and the assumption that κ^* surrounds $(n,0)$. So, assume that κ^* does not surround $(n,0)$, that is, $(n,0) \in B(\kappa^*)$. In almost the same way as in (2.14), we can show that

$$\begin{aligned} &\big|E_{\beta,h}\big[\widehat{T}(\partial B(\Lambda^*_j),(n,0)) \mid \mathscr{F}_j\big] - E_{\beta,h}\big[\widehat{T}(\partial B(\Gamma^*),(n,0))\big]\big| \\ &\qquad \le 8\log^2 n + C((2n+4)\log^2 n+1)^2 n\log^2 n \exp(-\alpha\log^2 n), \end{aligned} \tag{2.27}$$

for $w \in E_j(\Gamma^*)$. Here, the factor $((2n+4)\log^2 n+1)^2 n\log^2 n$ is the only change from (2.14), and this comes from the fact that $\widehat{T}(\partial B(\kappa^*),(n,0)) \le n\log^2 n$, (2.26) and the fact that Γ^* lies inside of $[-n\log^2 n, n\log^2 n]$. By (2.27), we obtain as before,

$$\begin{aligned} |\Delta_i(w)| &= |S_i(w) - S_{i-1}(w)| \\ &\le 17\log^2 n + 1 + \text{Const.} \times n^3\log^6 n\exp(-\alpha\log^2 n) \\ &\le \text{Const.} \times \log^2 n. \end{aligned}$$

Now we are ready to use Azuma–Hoeffding's lemma to obtain

$$\mu_{\beta,h}\bigg(\frac{|a_{0,n} - E_{\beta,h}a_{0,n}|}{\sqrt{n}\log^2 n} \ge x\bigg) \le \exp(-\alpha_2' x^2)$$

for some constant $\alpha_2' > 0$. Finally, we can use the same argument as in the proof of Theorem 4 to estimate the probability

$$\mu_{\beta,h}(\hat{a}_{0,n} - a_{0,n} \geq x\sqrt{n}),$$

and obtain for sufficiently large n, for any $\sqrt{n} \geq x \geq 1$,

$$\begin{aligned} &\mu_{\beta,h}(|a_{0,n} - E_{\beta,h}a_{0,n}| \geq x\sqrt{n}\,\log^2 n) \\ &\quad \leq \text{Const.} \times \exp(-\alpha_2' x^2/9) + \left(\tfrac{1}{2}\right)^{2x\sqrt{n}/(27\log^2 n)}. \end{aligned} \tag{2.28}$$

Therefore, Theorem 2 follows. □

3. Concentrations at νn. Before we prove Theorems 1 and 3, we need to show the following lemmas. Let E_v be the event that there exists a path r with the last vertex $v \in \partial Q(n)$ starting from $(0,0)$ such that

$$t(r) = T((0,0), \partial Q(n)).$$

If there are many such v, we pick a v with the smallest x coordinate, then the smallest y coordinate. Clearly,

$$E_{\beta,h}T((0,0),\partial Q(n)) = \sum_{u\in\partial Q(n)} E_{\beta,h}(T((0,0),\partial Q(n)) \mid E_u)\mu_{\beta,h}(E_u). \tag{3.1}$$

We divide $\partial Q(n)$ into two parts:

$$\begin{aligned} \partial_1 Q(n) &= \{u \in \partial Q(n);\ \mu_{\beta,h}(E_u) \geq 1/n^2\}; \\ \partial_2 Q(n) &= \{u \in \partial Q(n);\ \mu_{\beta,h}(E_u) < 1/n^2\}. \end{aligned}$$

Since $T((0,0),\partial Q(n)) \leq n$, and the number of points in $\partial Q(n)$ is $8n$, we have

$$\sum_{u\in\partial_2 Q(n)} E_{\beta,h}\big[T((0,0),\partial Q(n)) \mid E_u\big]\mu_{\beta,h}(E_u) \leq 8n \cdot n\frac{1}{n^2} = 8. \tag{3.2}$$

So, the effect of points in $\partial_2 Q(n)$ in (3.1) is bounded. Assume first that $u \in \partial_1 Q(n)$ is on the right boundary of $\partial Q(n)$. Let E_u' be the reflected event of E_u with respect to the line

$$\{v = (v^1, v^2);\ v^1 = n + \lfloor \log^2 n \rfloor\}.$$

By symmetry, note that we can simply connect u and u' by a path with length $2\lfloor \log^2 n \rfloor$ so that

$$\begin{aligned} &E_{\beta,h}(a_{0,2n+2\lfloor \log^2 n\rfloor} \mid E_u \cap E_u') \\ &\quad \leq E_{\beta,h}(T((0,0),u) \mid E_u \cap E_u') \\ &\qquad + E_{\beta,h}(T((2n+2\lfloor \log^2 n\rfloor, 0), u') \mid E_u \cap E_u') + 2\lfloor \log^2 n \rfloor \\ &\quad = 2E_{\beta,h}(T((0,0),u) \mid E_u \cap E_u') + 2\lfloor \log^2 n \rfloor. \end{aligned} \tag{3.3}$$

Let

$$E_n^+ := \left\{ a_{0,\, 2n+2\lfloor \log^2 n \rfloor} \leq E_{\beta,\, h}(a_{0,\, 2n+2\lfloor \log^2 n \rfloor}) - \left(\frac{6}{\alpha_2} n \lfloor \log^5 n \rfloor \right)^{1/2} \right\},$$

where α_2 is the constant given in Theorem 2. Then we have

$$\begin{aligned} &E_{\beta,\, h}(a_{0,\, 2n+2\lfloor \log^2 n \rfloor};\ (E_n^+)^C \cap E_u \cap E'_u) \\ &\geq \left[E_{\beta,\, h}(a_{0,\, 2n+2\lfloor \log^2 \rfloor}) - \left(\frac{6}{\alpha_2} n \log^5 n \right)^{1/2} \right] \times \mu_{\beta,\, h}\big((E_n^+)^C \cap E_u \cap E'_u\big), \end{aligned}$$

where $E_{\beta,\, h}(X; A) = E_{\beta,\, h}(X 1_A)$. Dividing both sides of the above inequality by $\mu_{\beta,\, h}(E_u \cap E'_u)$, we obtain

$$\begin{aligned} &E_{\beta,\, h}(a_{0,\, 2n+2\lfloor \log^2 n \rfloor} \mid E_u \cap E'_u) \\ &\geq \left[E_{\beta,\, h}(a_{0,\, 2n+2\lfloor \log^2 \rfloor}) - \left(\frac{6}{\alpha_2} n \log^5 n \right)^{1/2} \right] \times \big(1 - \mu_{\beta,\, h}(E_n^+ \mid E_u \cap E'_u)\big). \end{aligned} \tag{3.4}$$

Now we estimate $\mu_{\beta,\, h}(E_n^+ \mid E_u \cap E'_u)$ for $u \in \partial_1 Q(n)$. We put $A = E_u$, $B = E'_u$, $V_1 = Q(n)$ and $V_2 = Q(n) + (2n + 2\lfloor \log^2 n \rfloor, 0)$, and use (1.3) to obtain

$$\mu_{\beta,\, h}(E_u \cap E'_u) \geq \mu_{\beta,\, h}(E_u)\mu_{\beta,\, h}(E'_u) - \text{Const.} \times n^2 \exp(-2\alpha \log^2 n). \tag{3.5}$$

By symmetry, and since $u \in \partial_1 Q(n)$, $\mu_{\beta,\, h}(E'_u) = \mu_{\beta,\, h}(E_u) \geq 1/n^2$. So if n is sufficiently large, then we have from (3.5),

$$\mu_{\beta,\, h}(E_u \cap E'_u) \geq \tfrac{1}{2} n^{-4}. \tag{3.6}$$

On the other hand, by Theorem 2 we have

$$\mu_{\beta,\, h}(E_n^+) \leq C_3 n^{-6}.$$

Therefore it follows that

$$\mu_{\beta,\, h}(E_n^+ \mid E_u \cap E'_u) \leq C_3 n^{-2}$$

for sufficiently large n. This means that in the right-hand side of (3.4), the term

$$\left[E_{\beta,\, h}(a_{0,\, 2n+2\lfloor \log^2 \rfloor}) - \left(\frac{6}{\alpha_2} n \log^5 n \right)^{1/2} \right] \times \mu_{\beta,\, h}(E_n^+ \mid E_u \cap E'_u)$$

goes to zero as n tends to infinity. Thus, there exists a positive constant C_7 such that we have

$$E_{\beta,\, h}(a_{0,\, 2n+2\lfloor \log^2 n \rfloor}) \leq E_{\beta,\, h}(a_{0,\, 2n+2\lfloor \log^2 n \rfloor} \mid E_u \cap E'_u) + C_7 (n \log^5 n)^{1/2}. \tag{3.7}$$

By mixing property we have

$$\begin{aligned} &E_{\beta,\, h}(T((0,0), u) \mid E_u \cap E'_u) \\ &\qquad \leq E_{\beta,\, h}(T((0,0), u) \mid E_u) + \text{Const.} \times n^7 \exp(-\alpha \log^2 n). \end{aligned} \tag{3.8}$$

To see this, first we use (1.4) for $V_1 = Q(n)$, $V_2 = Q(n) + (2n + 2\lfloor \log^2 n \rfloor)$, $X = T((0,0), \partial Q(n)) 1_{E_u}$, and obtain

$$E_{\beta, h}\big(T((0,0), \partial Q(n)) 1_{E_u} \mid \mathscr{F}_{V_2}\big) \le E_{\beta, h}\big(T((0,0), \partial Q(n));\ E_u\big) \\ + \text{Const.} \times n^3 \exp(-\alpha \log^2 n).$$

Taking expectation on the set E'_u with respect to $\mu_{\beta, h}$, we obtain

$$\begin{aligned} & E_{\beta, h}\big(T((0,0), \partial Q(n));\ E_u \cap E'_u\big) \\ & \le E_{\beta, h}\big(T((0,0), \partial Q(n)) 1_{E_u}\big) \mu_{\beta, h}(E'_u) + \text{Const.} \times n^3 \exp(-\alpha \log^2 n) \\ & = E_{\beta, h}\big(T((0,0), \partial Q(n)) \mid E_u\big) \mu_{\beta, h}(E_u) \mu_{\beta, h}(E'_u) \\ & \quad + \text{Const.} \times n^3 \exp(-\alpha \log^2 n). \end{aligned} \tag{3.9}$$

By (3.5) and the fact that $T((0,0), \partial Q(n)) \le n$, the right-hand side of (3.9) is not larger than

$$E_{\beta, h}\big(T((0,0), \partial Q(n)) \mid E_u\big) \mu_{\beta, h}(E_u \cap E'_u) \\ + \text{Const.} \times n^3 \exp(-\alpha \log^2 n). \tag{3.10}$$

By (3.5), (3.9) and (3.10) it follows that

$$E_{\beta, h}(T((0,0), \partial Q(n)) \mid E_u \cap E'_u) \le E_{\beta, h}(T((0,0), u) \mid E_u) \\ + \text{Const.} \times n^7 \exp(-\alpha \log^2 n),$$

which proves (3.8). Combining (3.3), (3.7) with (3.8) we have

$$\begin{aligned} & E_{\beta, h}(a_{0, 2n+2\lfloor \log^2 n \rfloor}) \\ & \le 2E_{\beta, h}\big(T((0,0), u) \mid E_u\big) + C_7 (n \log^5 n)^{1/2} + 2\lfloor \log^2 n \rfloor \\ & \quad + \text{Const.} \times n^7 \exp(-\alpha \log^2 n) \\ & \le 2E_{\beta, h}\big(T((0,0), u) \mid E_u\big) + C_8 (n \log^5 n)^{1/2}. \end{aligned} \tag{3.11}$$

for some constant $C_8 > 0$. Note that $T((0,0), u)(w) = T((0,0), \partial Q(n))(w)$ if $w \in E_u$, and hence

$$E_{\beta, h}(a_{0, 2n+2\lfloor \log^2 n \rfloor}) \le 2E_{\beta, h}\big(T((0,0), \partial Q(n)) \mid E_u\big) + C_8 (n \log^5 n)^{1/2}. \tag{3.12}$$

In the case that $u \in \partial_1 Q(n)$ is not on the right boundary of $\partial Q(n)$, we argue in a similar way as before. For example, if u is on the top side of $\partial Q(n)$, we define E'_u to be the reflected event of E_u with respect to the line

$$\{v = (v^1, v^2);\ v^2 = n + \lfloor \log^2 n \rfloor\}.$$

Then we define

$$E_n^+ = \left\{ T\big((0,0), (0, 2n + 2\lfloor \log^2 n \rfloor)\big) \le E_{\beta, h}(a_{0, 2n+2\lfloor \log^2 n \rfloor}) - \left(\frac{6}{\alpha_2} n \log^5 n \right)^{1/2} \right\}.$$

By the rotation invariance of $\mu_{\beta,h}$, $T((0,0),(0,2n+2\lfloor\log^2 n\rfloor))$ has the same distribution as $a_{0,2n+2\lfloor\log^2 n\rfloor}$, and we can argue just as before to obtain (3.12). Thus, from (3.1), (3.2) and (3.12) we obtain

$$\begin{aligned}
2E_{\beta,h}\big(T((0,0),\partial Q(n)\big) &\geq 2\sum_{u\in\partial_1 Q(n)} E_{\beta,h}\big(T((0,0),\partial Q(n))\mid E_u\big)\mu_{\beta,h}(E_u)\\
&\geq \sum_{u\in\partial_1 Q(n)} E_{\beta,h}(a_{0,2n+2\lfloor\log^2 n\rfloor})\mu_{\beta,h}(E_u) - C_8(n\log^5 n)^{1/2}\\
&\geq E_{\beta,h}(a_{0,2n+2\lfloor\log^2 n\rfloor}) - 8 - C_8(n\log^5 n)^{1/2}\\
&\geq E_{\beta,h}(a_{0,2n}) - 8 - C_8(n\log^5 n)^{1/2} - 2\log^2 n.
\end{aligned}$$

The last inequality above follows from a subadditive argument. This says that for sufficiently large n, we have

$$E_{\beta,h}a_{0,2n} \leq 2E_{\beta,h}(T((0,0),\partial Q(n))) + \text{Const.}\times(n\log^5 n)^{1/2}. \tag{3.13}$$

Note that

$$E_{\beta,h}c_{0,n} \leq E_{\beta,h}a_{0,n}. \tag{3.14}$$

Clearly, by (1.8), (3.13) and (3.14),

$$\lim_{n\to\infty}\frac{E_{\beta,h}c_{0,n}}{n} = \nu. \tag{3.15}$$

Let

$$N_n = \min\{|r|: t(r) = a_{0,n}\},$$

where $|r|$ stands for the length of the path r. We will show the following lemma.

LEMMA 6. *If $\nu > 0$, there exist $C_9 > 0$, $C_{10} > 0$ and $\alpha_4 > 0$ such that*

$$\mu_{\beta,h}(N_n \geq C_9 n) \leq C_{10}\exp(-\alpha_4 n).$$

PROOF. Lemma 6 for i.i.d. first passage percolation is proved in [10]. Here we show it for Ising first passage percolation. For number γ and a path r, let

$$t_\gamma(r) = \sum_{e\in r}[X(e)+\gamma]$$

and

$$T_\gamma(A,B) = \inf\{t_\gamma(r): r \text{ a self-avoiding path from } A \text{ to } B\}.$$

Clearly, for $\gamma > 0$,

$$a_{0,n} - \gamma N_n \geq T_{-\gamma}((0,0),(n,0)).$$

We also know that

$$a_{0,n} \leq n.$$

To show Lemma 6 we only need to show that there exists $\gamma > 0$ such that

$$\mu_{\beta,h}(T_{-\gamma}((0,0),(n,0)) < 0) \leq \text{Const.} \times \exp(-\alpha_4 n). \tag{3.16}$$

We use a standard Peierls argument to show (3.16). For a given positive integer k, partition the bonds of Z^2 into some blocks

$$\{[ik,(i+1)k) \times [jk,(j+1)k);\ i,j \in Z\},$$

where $[ik,(i+1)k) \times [jk,(j+1)k)$ is a subset of Z^2 which contains the vertices in $[ik,(i+1)k] \times [jk,(j+1)k]$ and the bonds in

$$[ik,(i+1)k] \times [jk,(j+1)k] \setminus \{x = (i+1)k\} \cup \{y = (j+1)k\}.$$

We denote $[ik,(i+1)k) \times [jk,(j+1)k)$ by $B(i,j)$. Note that two different blocks do not have an edge in common. Assume that $T_{-\gamma}((0,0),(n,0)) < 0$. Then we can find a path r from $(0,0)$ to $(n,0)$ such that $t_{-\gamma}(r) < 0$. We take such a path r and fix it. Suppose that there are m blocks which are touched by the path r. Clearly,

$$m \geq \frac{n}{k^2}. \tag{3.17}$$

We say blocks B_1 and B_2 are connected if B_1 is one of eight neighbors of B_2. Note that these m blocks are connected so that we call these m blocks a block animal. Note also that there are at most λ^m block animals which contain $B(0,0)$ and consist of m blocks for some positive constant λ. For each block $B(i,j)$ which is touched by r, we consider

$$B_1(i,j) = [(i-1)k,(i+2)k] \times [(j-1)k,(j+2)k].$$

We assume that $m > 8$. Then r has to cross $B_1(i,j) \setminus B(i,j)$. We say that the block $B_1(i,j)$ is *good* if there exist more than one disjoint dual circuits in $B_1(i,j)$ surrounding $B(i,j)$, such that every edge e which crosses one of these dual circuits satisfies $X(e) = 1$. We call such dual circuits dual 1-circuits. If $B_1(i,j)$ is not good, then we call it *bad*. It follows from (1.6) that for given $\varepsilon > 0$ there exists k such that

$$\mu_{\beta,h}(B_1(0,0) \text{ is good}) \geq 1 - \varepsilon. \tag{3.18}$$

Let us take $\gamma = 1/9k^2$. Then, since $t_{-\gamma}(r) < 0$, there are at least $m/2$ bad blocks on the block animal. For $0 \leq a,b \leq 3$, let $L_{a,b}$ denote the set of all (i,j)'s such that $(i,j) = (a,b) \bmod 4$. Therefore, if $T_{-\gamma}((0,0),(n,0)) < 0$, then we can find some a and b such that there are at least $(m/2) \times 1/16$ bad blocks on the block animal in $\{B_1(i,j); (i,j) \in L_{a,b}\}$. If (i,j) and (k,l) are from the same $L_{a,b}$, then $B_1(i,j)$ and $B_1(k,l)$ are squares of side length $3k$ which are in distance not less than k. Thus, taking k sufficiently large, by (1.3) and (3.18) we can make the conditional probability as small as we want;

$$\mu_{\beta,h}(B(i,j) \text{ is bad } \mid \mathscr{F}_{i,j}) \leq 2\varepsilon, \tag{3.19}$$

where $\mathscr{F}_{i,j}$ is the σ-field generated by

$$\{w(x): d(x,B(i,j)) \geq k\}.$$

It follows from (3.19) that

$$
\begin{aligned}
&\mu_{\beta,h}(T_{-\gamma}((0,0),(n,0) < 0)\\
&\quad\le \sum_{\substack{\Gamma:\text{block animal containing}\\ B(0,0),\,|\Gamma|>n/k^2}} \sum_{a,b=0}^{3} \mu_{\beta,h}\begin{pmatrix}\text{there are at least } |\Gamma|/32 \text{ bad blocks}\\ \text{in } L_{a,b}\cap\Gamma\end{pmatrix}\\
&\quad\le \sum_{m>n/k^2} 16\times\lambda^m\binom{m}{m/32}(2\varepsilon)^{m/32}\\
&\quad\le 16\exp(-\alpha_4 n)
\end{aligned}
$$

by taking ε small. Lemma 6 is proved. $\square$

With Lemma 6 and the method in [16], we have the following lemma.

LEMMA 7. *For each $t > 0$, if $\nu > 0$, then there exist constants $C_{11}, C_{12} > 0$ and $\alpha_5 > 0$ such that we have for sufficiently large n,*

$$\mu_{\beta,h}(a_{0,4n} \le 4t) \le C_{11}n^2(\mu_{\beta,h}(a_{0,2n} \le 2t+2\lfloor\log^2 n\rfloor))^{1/2} + C_{12}\exp(-\alpha_5\log^2 n).$$

PROOF. By Lemma 6, we have

$$\mu_{\beta,h}(a_{0,4n} \le 4t) \le \mu_{\beta,h}(a_{0,4n} \le 4t, N_{4n} < 4C_9 n) + C_{10}\exp(-4\alpha_4 n).$$

Consider a path r from $(0,0)$ to $(4n,0)$ with passage time $T((0,0),(4n,0))$ and $|r| < 4C_9 n$. For $0 \le i \le 4$, consider the first (resp., last) vertex b_i (resp., a_i) on r that has first coordinate equal to in. Clearly, there exists $0 \le i \le 3$ such that the sum of passage time on the bonds of r between a_i and b_{i+1} is at most $T((0,0),(4n,0))/4$. Given two vertices $a = (a^1, a^2)$ and $b = (b^1, b^2)$, let us denote by $H_n(a,b,t)$ the event that there exists a path ρ from a to b with passage time at most t and $|\rho| < 4C_9 n$, such that each vertex visited by ρ except a and b has a first coordinate larger than the first coordinate a^1 of a and smaller than the first coordinate b^1 of b. Then we have

$$\mu_{\beta,h}(a_{0,4n} \le 4t, N_{4n} < 4C_9 n) \le \sum_{i=0}^{3} \sum_{\substack{a=(a^1,a^2),\,a^1=in\\ b=(b^1,b^2),\,b^1=(i+1)n\\ |a^2|,\,|b^2|\le 4C_9 n}} \mu_{\beta,h}(H_n(a,b,t)).$$

Therefore, we can find some a and b with $|a^1 - b^1| = n$ such that

$$\mu_{\beta,h}(H_n(a,b,t))4(8C_9 n)^2 + C_{10}\exp(-4\alpha_4 n) \ge \mu_{\beta,h}(a_{0,4n} \le 4t). \tag{3.20}$$

Take such a and b. Let $H = H_n(a,b,t)$ and H' be its reflected event with respect to the line $\{v = (v^1, v^2);\ v^1 = b^1 + \lfloor\log^2 n\rfloor/2\}$. Then we have $H \cap H' \subset \{T(a, a + (2n + \lfloor\log^2 n\rfloor, 0)) \le 2t + \lfloor\log^2 n\rfloor\}$. Note that by definition

$H = H_n(a, b, t)$ is an event in the rectangle of vertical side length $4C_9 n$ and horizontal side length n, centered at $(a+b)/2$. By (1.3) and invariance of $\mu_{\beta, h}$ with respect to translation and reflection it follows that

$$\begin{aligned}
&\mu_{\beta, h}(a_{0, 2n} \le 2t + 2\lfloor \log^2 n \rfloor) \\
(3.21) \qquad &\qquad \ge \mu_{\beta, h}(T((0,0), (2n + \lfloor \log^2 n \rfloor, 0)) \le 2t + \lfloor \log^2 n \rfloor) \\
&\qquad \ge \mu_{\beta, h}(H_n(a, b, t))^2 - Cn4C_9 n \exp(-\alpha \log^2 n).
\end{aligned}$$

Therefore, Lemma 7 follows from (3.20) and (3.21). □

PROOF OF THEOREM 1. We follow the method in [16] to show Theorem 1. Let

$$\xi_n = E_{\beta, h} a_{0, n}.$$

Let K_0 be a sufficiently large constant, which we will specify later. We take $4t = 2\xi_{2n} - K_0(2n \log^5(2n))^{1/2}$ in Lemma 7 and obtain that

$$\begin{aligned}
&\mu_{\beta, h}(a_{0, 4n} \le 2\xi_{2n} - K_0(2n \log^5(2n))^{1/2}) \\
&\quad \le C_{11} n^2 \big[\mu_{\beta, h}(a_{0, 2n} \le \xi_{2n} - (K_0/2)(2n \log^5(2n))^{1/2} + 2\lfloor \log^2 n \rfloor)\big]^{1/2} \\
&\qquad + C_{12} \exp(-\alpha_5 \log^2 n).
\end{aligned}$$

If n is large, then $2\lfloor \log^2 n \rfloor < (K_0/4)(2n \log^5(2n))^{1/2}$, and then we apply Theorem 2 to obtain

$$\begin{aligned}
&\big[\mu_{\beta, h}(a_{0, 2n} \le \xi_{2n} - (K_0/2)(2n \log^5(2n))^{1/2} + 2\lfloor \log^2 n \rfloor)\big]^{1/2} \\
&\quad \le \text{Const.} \exp\left\{-\frac{\alpha_2}{2}(K_0/4)^2 \log(2n)\right\} \\
&\quad < \text{Const.} \times n^{-4}
\end{aligned}$$

if $K_0 \ge 8\sqrt{2/\alpha_2}$, where α_2 is the constant in Theorem 2. Therefore if n is sufficiently large, then

$$\mu_{\beta, h}(a_{0, 4n} \le 2\xi_{2n} - K_0(2n \log^5(2n))^{1/2}) \le \text{Const.} \times n^{-2}.$$

Since $0 \le a_{0, n} \le n$, it follows from this that

$$\begin{aligned}
\xi_{4n} &\ge E_{\beta, h}[a_{0, 4n};\ a_{0, 4n} \ge 2\xi_{2n} - K_0(n \log^5 n)^{1/2}] \\
&\ge [2\xi_{2n} - K_0(n \log^5 n)^{1/2}(1 - O(n^{-2})) \\
&= 2\xi_{2n} - K_0(n \log^5 n)^{1/2} + O(n^{-1}) \\
&\ge 2\xi_{2n} - (K_0 + 1)(n \log^5 n)^{1/2}
\end{aligned} \tag{3.22}$$

for sufficiently large n. Let $K = K_0 + 1$, and use (3.22) for $2^k n$ rather than n to get

$$\frac{\xi_{2^{k+2}n}}{2^{k+2}n} + (K/2)\left(\frac{\log^5 2^{k+1}n}{2^{k+1}n}\right)^{1/2} \geq \frac{\xi_{2^{k+1}n}}{2^{k+1}n}. \tag{3.23}$$

Iterating (3.23), we get

$$\frac{\xi_{2n}}{2n} \leq \widetilde{K}\left(\frac{\log^5(2n)}{2n}\right)^{1/2} + \frac{\xi_{2^{k+2}n}}{2^{k+2}n} \tag{3.24}$$

for some constant $\widetilde{K} > 0$. Theorem 1 follows from (1.8) and (3.24) by letting $k \to \infty$. $\square$

PROOF OF THEOREM 3. Theorem 3 follows from (3.13),

$$c_{0,\,n} \leq a_{0,\,n}$$

and Theorem 1. $\square$

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