The Annals of Probability
2000, Vol. 28, No. 1, 235–257

CHEEGER'S INEQUALITIES FOR GENERAL SYMMETRIC FORMS AND EXISTENCE CRITERIA FOR SPECTRAL GAP[1]

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In this paper, some new forms of Cheeger's inequalities are established for general (maybe unbounded) symmetric forms (Theorems 1.1 and 1.2): the resulting estimates improve and extend the ones obtained by Lawler and Sokal for bounded jump processes. Furthermore, some existence criteria for spectral gap of general symmetric forms or general reversible Markov processes are presented (Theorems 1.4 and 3.1), based on Cheeger's inequalities and a relationship between the spectral gap and the first Dirichlet and Neumann eigenvalues on local region.

1. Introduction. Cheeger's inequalities [2] are well known and widely used in geometric analysis; they provide a practical way to estimate the first eigenvalue of Laplacian in terms of volumes. These inequalities were established for bounded jump processes by Lawler and Sokal [8] (in which a detailed comment on the earlier study and references are included). The first aim of this paper is to establish the inequalities for general (maybe unbounded) symmetric forms.

Let $(E, \mathscr{E}, \pi)$ be a probability space satisfying $\{(x, x)\colon x \in E\} \in \mathscr{E} \times \mathscr{E}$. Consider the symmetric form D with domain $\mathscr{D}(D)$,

$$D(f, g) = \tfrac{1}{2}\int J(\mathrm{d}x, \mathrm{d}y)(f(x) - f(y))(g(x) - g(y)) + \int K(\mathrm{d}x) f(x) g(x),$$

$$f, g \in \mathscr{D}(D),$$

$$\mathscr{D}(D) = \{f \in L^2(\pi) : D(f, f) < \infty\}.$$

where J and K are nonnegative and J is symmetric: $J(dx, dy) = J(dy, dx)$. Without loss of generality, we assume that $J(\{(x, x)\colon x \in E\}) = 0$.

We are interested in the following two quantities:

$$\lambda_0 = \inf\{D(f, f)\colon \pi(f^2) = 1\}, \tag{1.1}$$

$$\lambda_1 = \inf\{D(f, f)\colon \pi(f) = 0, \pi(f^2) = 1\}. \tag{1.2}$$

We remark that in these definitions, the usual condition "$f \in \mathscr{D}(D)$" is not needed since $D(f, f) = \infty$ for all $f \in L^2(\pi) \setminus \mathscr{D}(D)$. We do not even assume in some cases the density of $\mathscr{D}(D)$ in $L^2(\pi)$. In what follows, whenever λ_1 is

Received May 1998; revised July 1999.

[1]Supported in part by NSFC Grant 19631060, Qiu Shi Science and Technology Found, DPFIHE, MCSEC and MCMCAS.

AMS 1991 *subject classifications*. 60J25, 60J75, 47A75.

Key words and phrases. Cheeger's inequality, spectral gap, Neumann and Dirichlet eigenvalue, jump process.

considered, the killing measure $K(dx)$ is set to zero. In this case, we have $\lambda_0 = 0$ and λ_1 is known as the spectral gap of the symmetric form $(D, \mathscr{D}(D))$.

Define Cheeger's constants as follows:

$$h = \inf_{\pi(A)>0} \frac{J(A \times A^c) + K(A)}{\pi(A)}, \tag{1.3}$$

$$k = \inf_{\pi(A)\in(0,1)} \frac{J(A \times A^c)}{\pi(A)\pi(A^c)}, \tag{1.4}$$

$$k' = \inf_{\pi(A)\in(0,1/2]} \frac{J(A \times A^c)}{\pi(A)} = \inf_{\pi(A)\in(0,1)} \frac{J(A \times A^c)}{\pi(A) \wedge \pi(A^c)}, \tag{1.5}$$

where $a \wedge b = \min\{a, b\}$. Clearly, $k/2 \le k' \le k$ and it is easy to see that k' can be varied over whole $(k/2, k)$. For instance, take $E = \{0, 1\}$, $K = 0$, $J(\{i\} \times \{j\}) = 1$ for $i \ne j$ and $\pi(0) = p \le 1/2$, $\pi(1) = 1 - p$. Then $k'/k = 1 - p$.

Recall that for a given reversible jump process, we have a q-pair $(q(x), q(x, dy))$: $q(x, E) \le q(x) \le \infty$ for all $x \in E$. Throughout the paper, we assume that $q(x) < \infty$ for all $x \in E$. The reversibility simply means that the measure $\pi(dx)q(x, dy)$ is symmetric, which gives us automatically a measure J. Then the killing measure is given by $K(dx) = \pi(dx)d(x)$, where $d(x) = q(x) - q(x, E)$ is called the nonconservative quantity in the context of jump processes. A jump process is called bounded if $\sup_{x\in E} q(x) < \infty$. In this case [or more generally, if $\|J(\cdot, E) + K\|_{\text{op}} < \infty$, where $\|\cdot\|_{\text{op}}$ denotes the operator norm from $L^1_+(\pi) := \{f \in L^1(\pi): f \ge 0\}$ to $\mathbb{R}_+$], for the corresponding form, we have $\mathscr{D}(D) = L^2(\pi)$. For more details, refer to [3].

THEOREM [LAWLER AND SOKAL (1988)]. *Take $J(dx, dy) = \pi(dx)q(x, dy)$ and suppose that $\|J(\cdot, E) + K/2\|_{\text{op}} \le M < \infty$. Then we have*

$$h \ge \lambda_0 \ge \frac{h^2}{2M}. \tag{1.6}$$

Next, if additionally $K = 0$, then

$$k \ge \lambda_1 \ge \max\left\{\frac{\kappa k^2}{8M}, \frac{k'^2}{2M}\right\}, \tag{1.7}$$

where

$$\kappa = \inf_{X,Y} \sup_{c\in\mathbb{R}} \frac{(\mathbb{E}|(X+c)^2 - (Y+c)^2|)^2}{1+c^2} \ge 1,$$

the infimum is taken over all i.i.d. random variables X and Y with $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = 1$.

In what follows, we consider directly the general symmetric measure J whenever possible. In other words, we do not require the existence of a kernel of a modification of $J(dx, \cdot)/\pi(dx)$, for which some extra conditions on $(E, \mathscr{E})$ are needed.

We now turn to discuss our general setup. Note that the lower bounds given in (1.6) and (1.7) decrease to zero as $M \uparrow \infty$. So the results would lose their meaning if we go directly from the bounded case to unbounded forms. More seriously, when we adopt a general approximation procedure to reduce the unbounded case to the bounded one (cf. [3], Theorem 9.12), the lower bounds given above usually vanish as we go to the limit. To overcome the difficulty, one needs some trick. Here we propose a comparison technique, that is, comparing the original form with some other forms introduced below.

Take and fix a nonnegative, symmetric function $r \in \mathscr{E}\times\mathscr{E}$ and a nonnegative function $s \in \mathscr{E}$ such that

$$\|J^{(1)}(\cdot, E) + K^{(1)}\|_{\text{op}} \le 1, \qquad L^1_+(\pi) \to \mathbb{R}_+, \tag{1.8}$$

where

$$J^{(\alpha)}(dx, \mathrm{d}y) = I_{\{r(x,y)^\alpha > 0\}} \frac{J(dx, dy)}{r(x, y)^\alpha}, \qquad K^{(\alpha)}(dx) = I_{\{s(x)^\alpha > 0\}} \frac{K(dx)}{s(x)^\alpha}, \quad \alpha \ge 0.$$

Throughout the paper, we adopt the convention that $r^0 = 1$ and $s^0 = 1$ for $r, s \ge 0$. For jump processes, one may simply choose

$$r(x, y) = q(x) \vee q(y) = \max\{q(x), q(y)\} \quad \text{and} \quad s(x) = d(x).$$

We remark that when $\alpha < 1$, the operator $J^{(\alpha)}(\cdot, E) + K^{(\alpha)}$ from $L^1_+(\pi)$ to $\mathbb{R}_+$ may no longer be bounded. Correspondingly, we have symmetric forms $D^{(\alpha)}$ defined by $(J^{(\alpha)}, K^{(\alpha)})$. Therefore, with respect to the form $D^{(\alpha)}$, according to (1.1)—(1.5), we can define $\lambda_0^{(\alpha)}$, $\lambda_1^{(\alpha)}$ and Cheeger's constants $h^{(\alpha)}$, $k^{(\alpha)}$ and $k^{(\alpha)'}$ ($\alpha \ge 0$). However, in what follows, we need only three cases, $\alpha = 0$, $1/2$ and 1. When $\alpha = 0$, we return to the original form and so the superscript "(α)" is omitted from our notations.

The next two results are our new forms of Cheeger's inequalities.

THEOREM 1.1. *Suppose that* (1.8) *holds. We have*

$$\lambda_0 \ge \frac{h^{(1/2)^2}}{2 - \lambda_0^{(1)}} \ge \frac{h^{(1/2)^2}}{1 + \sqrt{1 - h^{(1)^2}}}. \tag{1.9}$$

THEOREM 1.2. *Let* $K = 0$ *and* (1.8) *hold. Then, we have*

$$\lambda_1 \ge \left(\frac{k^{(1/2)}}{\sqrt{2} + \sqrt{2 - \lambda_1^{(1)}}} \right)^2, \tag{1.10}$$

$$\lambda_1 \ge \frac{k^{(1/2)'^2}}{1 + \sqrt{1 - k^{(1)'^2}}}. \tag{1.11}$$

When $\|J(\cdot, E)+K\|_{\text{op}} \le M < \infty$, the simplest choice of r and s are $r(x, y) \equiv M$ and $s(x) \equiv M$. Then, (1.8) holds and moreover $h^{(1/2)} = h/\sqrt{M}$, $k^{(1/2)'} =$

$k'/\sqrt{M}$, $h^{(1)} = h/M$ and $k^{(1)'} = k'/M$. Hence, by (1.9) and (1.11), we get

$$\lambda_0 \ge M\left(1 - \sqrt{1 - h^2/M^2}\right) = \frac{h^2}{M\left(1 + \sqrt{1 - h^2/M^2}\right)} \in \left[\frac{h^2}{2M}, \frac{h^2}{M}\right]$$

and

$$\lambda_1 \ge M\left(1 - \sqrt{1 - k'^2/M^2}\right) = \frac{k'^2}{M\left(1 + \sqrt{1 - k'^2/M^2}\right)} \in \left[\frac{k'^2}{2M}, \frac{k'^2}{M}\right]. \tag{1.12}$$

Therefore, for the lower bounds, (1.9) improves the second part of (1.6) and (1.11) improves the second part of (1.7). More essentially, the lower bound (1.11) is often good enough so that the approximation procedure ([3], Theorem 9.12) mentioned above becomes practical. However, we will not go in this direction. In the context of Markov chains on finite graphs, (1.12) was obtained before by Chung [6]. Applying (1.12) to $J^{(1)}$, we get $\lambda_1^{(1)} \ge 1 - \sqrt{1 - k^{(1)'2}}$. From this and (1.10), we obtain

$$\lambda_1 \ge \left(\frac{k^{(1/2)}}{\sqrt{2} + \sqrt{1 + \sqrt{1 - k^{(1)'2}}}}\right)^2$$

which is indeed controlled by (1.11) since $k^{(\alpha)} \le 2k^{(\alpha)'}$. This means that (1.11) is usually more practical than (1.10) except a good lower bound of $\lambda_1^{(1)}$ is known in advance. However, (1.10) and (1.11) are not comparable even in the case of $E = \{0, 1\}$. See also the discussion in the second paragraph below Lemma 2.2.

In view of Theorem 1.2, we have $\lambda_1 > 0$ whenever $k^{(1/2)} > 0$. We now study some more explicit conditions for the Cheeger's constants appearing in Theorem 1.2 to be positive. To state the result, we should use the operators corresponding to the forms. For a jump process, the operator corresponding to $(D^{(\alpha)}, \mathscr{D}(D^{(\alpha)}))$ can be expressed by the following simple form:

$$\Omega^{(\alpha)} f(x) = \int I_{[r(x,y)^\alpha > 0]} \frac{q(x, dy)}{r(x, y)^\alpha} [f(y) - f(x)] - I_{[s(x)^\alpha > 0]} \frac{d(x)}{s(x)^\alpha} f(x).$$

Next, we need some local quantities of λ_0 and λ_1. First, for $B \in \mathscr{E}$ with $\pi(B) \in (0, 1)$, let $\lambda_1^{(\alpha)}(B)$ and $k^{(\alpha)}(B)$ be defined by (1.2) and (1.4) with E, π and D replaced, respectively, by B, $\pi^B := \pi(\cdot \cap B)/\pi(B)$ and

$$D_B^{(\alpha)}(f, f) = \tfrac{1}{2} \int_{B \times B} J^{(\alpha)}(dx, dy)(f(y) - f(x))^2. \tag{1.13}$$

Second, define

$$\lambda_0^{(\alpha)}(B) = \inf\left\{D^{(\alpha)}(f, f) \colon \pi(f^2) = 1,\ f|_{B^c} = 0\right\}.$$

As usual, we call $\lambda_0^{(\alpha)}(B)$ and $\lambda_1^{(\alpha)}(B)$, respectively, the (generalized) first Dirichlet and Neumann eigenvalue on B. It is a simple matter to check that as in (1.7), $k^{(\alpha)}(B) \ge \lambda_1^{(\alpha)}(B)$.

For $A \in \mathscr{E}$, put $M_A^{(\alpha)} = (\text{ess sup}_\pi)_A \; J^{(\alpha)}(dx, A^c)/\pi(dx)$, where ess $\sup_\pi$ denotes the essential supremum with respect to π.

THEOREM 1.3. *Let $K = 0$. Given $\alpha \geq 0$ and $B \in \mathscr{E}$ with $\pi(B) > 1/2$, suppose that there exists a function ϕ with $\delta_1(\phi) := \text{ess sup}_{J^{(\alpha)}}|\phi(x) - \phi(y)| < \infty$ and a symmetric operator $\big(\Omega^{(\alpha)}, \mathscr{D}(\Omega^{(\alpha)})\big)$ corresponding to the form $\big(D^{(\alpha)}, \mathscr{D}(D^{(\alpha)})\big)$ such that $\mathscr{D}\big(\Omega^{(\alpha)}\big) \supset \{I_A\colon A \in \mathscr{E},\ A \subset B\}$ and $\gamma_{B^c} := -\sup_{B^c} \Omega^{(\alpha)}\phi > 0$. Then, we have*

$$k^{(\alpha)} \geq k^{(\alpha)'} \geq \frac{k^{(\alpha)}(B)\,\gamma_{B^c}\,[2\pi(B) - 1]}{k^{(\alpha)}(B)\,\delta_1(\phi)\,[2\pi(B) - 1] + 2\pi(B)^2\big[\delta_1(\phi)M_B^{(\alpha)} + \gamma_{B^c}\big]}.$$

Usually, for locally compact E, we have $k^{(\alpha)}(B) > 0$ and $M_B^{(\alpha)} < \infty$ for all compact B. Then the result means that $k^{(\alpha)'} > 0$ provided $\delta_1(\phi) < \infty$ and $\gamma_{B^c} > 0$ for large enough B.

Up to now, we have discussed the lower bound of λ_1 by using Cheeger's constants. However, Theorem 1.3 is indeed a modification of the second approach we are going to study, that is, estimating λ_1 in terms of local λ_0 and λ_1 on subsets of E. The last method has been used recently in the context of diffusions by Wang [10] and is extended here to general reversible processes. The details of the next two results for the general situation are delayed to Section 3. Here, we restrict ourselves to the symmetric forms introduced above.

This is the place to state our first criterion for $\lambda_1 > 0$.

THEOREM 1.4. *Let $K = 0$. Then for any $A \subset B$ with $0 < \pi(A)$, $\pi(B) < 1$, we have*

$$\frac{\lambda_0(A^c)}{\pi(A)} \geq \lambda_1 \geq \frac{\lambda_1(B)[\lambda_0(A^c)\pi(B) - 2M_A\pi(B^c)]}{2\lambda_1(B) + \pi(B)^2[\lambda_0(A^c) + 2M_A]}. \tag{1.14}$$

As we mentioned before, usually $\lambda_1(B) > 0$ for all compact B. Hence the result means that $\lambda_1 > 0$ iff $\lambda_0(A^c) > 0$ for some compact A, because we can first fix such an A and then make B large enough so that the right-hand side of (1.14) becomes positive.

Finally, we present an upper bound of λ_1 which provides us a necessary condition for $\lambda_1 > 0$ and can qualitatively be sharp as illustrated by Example 4.5. For some related works, refer to [1] and references therein.

THEOREM 1.5. *Let $K = 0$, $r > 0$, J-a.e. and* (1.8) *hold. If there exists $\phi \geq 0$ such that*

$$0 < \delta_2(\phi) := \text{ess sup}_J|\phi(x) - \phi(y)|^2 r(x, y) < \infty,$$

then

$$\lambda_1 \leq \frac{\delta_2(\phi)}{4} \inf\Big\{\varepsilon^2 : \varepsilon \geq 0,\ \pi\big(e^{\varepsilon\phi}\big) = \infty\Big\}.$$

Consequently, $\lambda_1 = 0$ *if there exists* $\phi \geq 0$ *with* $0 < \delta_2(\phi) < \infty$ *such that* $\pi(e^{\varepsilon\phi}) = \infty$ *for all* $\varepsilon > 0$. *In particular, when* $J(dx, dy) = \pi(dx)q(x, dy)$, $\delta_2(\phi)$ *can be replaced by* $\delta_2'(\phi) := \operatorname{ess\,sup}_\pi \int |\phi(x) - \phi(y)|^2 q(x, dy) < \infty$, *without using the function* r *and* (1.8).

To have a test for the new forms of Cheeger's constants, we introduce the following result.

COROLLARY 1.6. *Let* $J(dx, dy) = j(x, y)\pi(dx)\pi(dy)$ *for some symmetric function* $j(x, y)$ *having the properties:* $j(x, x) = 0$ *and* $j(x) := \int j(x, y)\pi(dy) < \infty$ *for all* $x \in E$. *Take* $r(x, y) = j(x) \vee j(y)$. *Then*

$$k^{(\alpha)'} \geq \frac{1}{2} \inf_{x \neq y} \frac{j(x, y)}{[j(x) \vee j(y)]^\alpha}. \tag{1.15}$$

PROOF. Denote by $C^{(\alpha)}$ the right-hand side of (1.15). Note that

$$\begin{aligned}\frac{J^{(\alpha)}(A \times A^c)}{\pi(A)} &= \frac{1}{\pi(A)} \int_{A \times A^c} \pi(dx)\pi(dy) \frac{j(x, y)}{[j(x) \vee j(y)]^\alpha} \\ &\geq \inf_{x \neq y} \frac{j(x, y)}{[j(x) \vee j(y)]^\alpha} \pi(A^c) = 2C^{(\alpha)}\pi(A^c).\end{aligned}$$

Hence $k^{(\alpha)'} = \inf_{\pi(A) \in (0, 1/2]} J^{(\alpha)}(A \times A^c)/\pi(A) \geq C^{(\alpha)}$ as required. □

The corollary shows that our results are meaningful in a very general setup. Here are two more explicit examples.

1. Let $j(x, y) = 1$ for $x \neq y$ and $j(x, x) = 0$. Then, by (1.15), we have $k^{(\alpha)'} \geq 1/2$. Hence $\lambda_1 \geq 1/2(2 + \sqrt{3})$ by (1.11). The precise value of λ_1 is equal to 1.
2. Let $E = \mathbb{Z}$ and $j(x, y) = |x^2 - y^2|$. Suppose that $c := \pi(x^2) < \infty$. Then $j(x) \leq x^2 + c$ for all x and $k^{(1/2)'} \geq \dfrac{1}{2} \inf_{x \neq y} \dfrac{|x| + |y|}{\sqrt{x^2 + y^2 + c}} \geq \dfrac{1}{2\sqrt{c + 1}}$. Hence $\lambda_1 \geq k^{(1/2)'^2}/2 \geq 1/8(c + 1)$.

Certainly, the estimate (1.15) is very rough. However, Theorems 1.1 and 1.2 can actually be sharp as illustrated by Examples 4.6 and 4.7 in Section 4.

We mention that the study on the leading eigenvalue of a bounded integral operator is indeed included in our general setup. Consider the operator P on $L^2(\pi)$: $Pf(x) = \int p(x, dy)f(y)$, generated by a nonnegative kernel $p(x, dy)$ with $M := \sup_x p(x, E) < \infty$. Let $\pi(dx)p(x, dy)$ be symmetric for a moment. Clearly, the spectrum of P on $L^2(\pi)$ is determined by that of $M - P$. Note that

$$\begin{aligned}\langle f, (M - P)f \rangle_\pi &= \tfrac{1}{2} \int \pi(dx)p(x, dy)\big[f(x) - f(y)\big]^2 \\ &\quad + \int \pi(dx)\big[M - p(x, E)\big]f(x)^2.\end{aligned}$$

Thus, the largest (nontrivial) eigenvalue of the integral operator P can be deduced from λ_0 or λ_1 treated in the paper. Finally, by using a symmetrizing procedure, all the results presented here can be extended to nonsymmetric forms. Refer to [3], Chapter 9, or [8], for instance.

The remainder of the paper is organized as follows. Section 2 is devoted to the proofs of Theorems 1.1—1.3. At the end of the section, a different approach for handling unbounded symmetric forms is presented. A general existence criterion for spectral gap is presented in Section 3, which also contains the proofs of Theorems 1.4 and 1.5. All the results concerning the spectral gap are illustrated by Markov chains in the last section.

2. Proofs of Theorems 1.1—1.3. We begin this section with the functional representation of Cheeger's constants. The proof is essentially the same as in [8] and [9], Section 3.3, for the bounded situation and hence omitted.

LEMMA 2.1. *For every $\alpha \geq 0$, we have*

$$h^{(\alpha)} = \inf\left\{\frac{1}{2}\int J^{(\alpha)}(dx, dy)|f(x) - f(y)| + K^{(\alpha)}(f)\colon\ f \geq 0,\ \pi(f) = 1\right\},$$

$$\begin{aligned} k^{(\alpha)} &= \inf\left\{\int J^{(\alpha)}(dx, dy)|f(x) - f(y)|\colon\ f \in L^1_+(\pi),\ \int\pi(dx)\pi(dy)|f(x) - f(y)| = 1\right\} \\ &= \inf\left\{\int J^{(\alpha)}(dx, dy)|f(x) - f(y)|\colon\ f \in L^1_+(\pi),\ \pi(|f - \pi(f)|) = 1\right\}, \end{aligned}$$

$$k^{(\alpha)'} = \inf\left\{\frac{1}{2}\int J^{(\alpha)}(dx, dy)|f(x) - f(y)|\colon\ f \in L^1_+(\pi),\ \min_{c\in\mathbb{R}}\pi(|f - c|) = 1\right\}.$$

PROOF OF THEOREM 1.1. The idea of the proof is based on [8].

Let $E^* = E \cup \{\infty\}$. For any $f \in \mathscr{E}$, define f^* on E^* by setting $f^* = fI_E$. Next, define $J^{*(\alpha)}$ on $E^* \times E^*$ by

$$J^{*(\alpha)}(C) = \begin{cases} J^{(\alpha)}(C), & C \in \mathscr{E} \times \mathscr{E}, \\ K^{(\alpha)}(A), & C = A \times \{\infty\} \text{ or } \{\infty\} \times A,\ A \in \mathscr{E}, \\ 0, & C = \{\infty\} \times \{\infty\}. \end{cases}$$

We have $J^{*(\alpha)}(dx, dy) = J^{*(\alpha)}(dy, dx)$ and

$$\int J^{(\alpha)}(dx, E)f(x)^2 + K^{(\alpha)}(f^2) = \int J^{*(\alpha)}(dx, E^*)f^*(x)^2, \tag{2.1}$$

$$D^{(\alpha)}(f, f) = \frac{1}{2}\int J^{*(\alpha)}(dx, dy)(f^*(y) - f^*(x))^2, \tag{2.2}$$

$$\begin{aligned}(2.3)\qquad &\tfrac{1}{2}\int J^{(\alpha)}(dx,dy)|f(y)-f(x)|+\int K^{(\alpha)}(dx)|f(x)|\\ &\quad=\tfrac{1}{2}\int J^{*(\alpha)}(dx,dy)|f^*(y)-f^*(x)|.\end{aligned}$$

Therefore, for f with $\pi(f^2) = 1$, by (2.1)–(2.3), (1.8), Lemma 2.1 and the Cauchy–Schwarz inequality,

$$\begin{aligned} h^{(1)^2} &\le \left\{\tfrac{1}{2}\int J^{*(1)}(dx,dy)\big|f^*(y)^2 - f^*(x)^2\big|\right\}^2\\ &\le \tfrac{1}{2}D^{(1)}(f,f)\int J^{*(1)}(dx,dy)\big[f^*(y)+f^*(x)\big]^2\\ &= \tfrac{1}{2}D^{(1)}(f,f)\bigg\{2\int J^{*(1)}(dx,dy)\big[f^*(y)^2+f^*(x)^2\big]\\ &\qquad\qquad - \int J^{*(1)}(dx,dy)\big[f^*(y)-f^*(x)\big]^2\bigg\}\\ &\le D^{(1)}(f,f)[2-D^{(1)}(f,f)].\end{aligned}$$

This implies that $D^{(1)}(f,f) \ge 1-\sqrt{1-h^{(1)^2}}$ and so

$$\lambda_0^{(1)} \ge 1-\sqrt{1-h^{(1)^2}}. \tag{2.4}$$

Next, by (1.8), Lemma 2.1 and another use of the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}(2.5)\qquad h^{(1/2)^2} &\le \left\{\tfrac{1}{2}\int J^{*(1/2)}(dx,dy)\big|f^*(y)^2 - f^*(x)^2\big|\right\}^2\\ &\le \tfrac{1}{2}D(f,f)\int J^{*(1)}(dx,dy)\big[f^*(y)+f^*(x)\big]^2\\ &\le D(f,f)\big[2-D^{(1)}(f,f)\big] \le D(f,f)\big[2-\lambda_0^{(1)}\big].\end{aligned}$$

From this and (2.4), the required assertion follows. □

PROOF OF THEOREM 1.2. (a) First, we prove (1.10). Let $f \in \mathscr{D}(D)$ with $\pi(f) = 0$ and $\pi(f^2) = 1$. Set $g = f + c$, $c \in \mathbb{R}$. Similarly to (2.5), we have

$$\begin{aligned}\left\{\int J^{(1/2)}(dx,dy)\big|g(y)^2 - g(x)^2\big|\right\}^2 &\le 4D(f,f)\big[2(1+c^2) - D^{(1)}(f,f)\big]\\ &\le 4D(f,f)[2(1+c^2)-\beta]\end{aligned}$$

for all β: $0 \le \beta < \lambda_1^{(1)} \le 2$. Hence by Lemma 2.1, we have

$$D(f,f) \ge \frac{1}{4[2(1+c^2)-\beta]}\left\{\int J^{(1/2)}(dx,dy)\big|g(y)^2-g(x)^2\big|\right\}^2 \\ \ge \frac{\kappa_\beta}{4}k^{(1/2)2}, \tag{2.6}$$

where κ_β is the same as κ defined below (1.7) but replacing the denominator $1+c^2$ with $2(1+c^2)-\beta$. To estimate κ_β, we adopt an optimizing procedure which will be used several times subsequently. Set $\gamma = \mathbb{E}|X| \in (0,1]$. It is known that

$$\lim_{c\to\pm\infty}\frac{\big(\mathbb{E}\big|(X+c)^2-(Y+c)^2\big|\big)^2}{2(1+c^2)-\beta} = 2(\mathbb{E}|X-Y|)^2 \ge 2(\mathbb{E}|X|)^2 = 2\gamma^2$$

and when $c = 0$, $\mathbb{E}\big|X^2-Y^2\big| \ge 2(1-\mathbb{E}|X|) = 2(1-\gamma)$ (cf. [8] or [3], Section 9.2]). Thus,

$$\kappa_\beta \ge \inf_{\gamma\in(0,1]}\max\left\{2\gamma^2, \frac{4(1-\gamma)^2}{2-\beta}\right\}. \tag{2.7}$$

We now need an elementary fact.

LEMMA 2.2. *Let f and g be continuous functions on $[0,1]$ and satisfy $f(0) < g(0)$ and $f(1) > g(1)$. Suppose that f is increasing and g is decreasing. Then*

$$\inf_{\gamma\in[0,1]}\max\{f(\gamma),\ g(\gamma)\} = f(\gamma_0),$$

where γ_0 is the unique solution to the equation $f = g$ on $[0,1]$.

Applying Lemma 2.2 to (2.7), we get

$$\kappa_\beta \ge \frac{4}{\big(\sqrt{2}+\sqrt{2-\beta}\big)^2}.$$

Combining this with (2.6) and then letting $\beta \uparrow \lambda_1^{(1)}$, we obtain (1.10).

It is worthy to mention that the estimate just proved can be sharp. To see this, simply consider $E = \{0,1\}$, $J(\{i\},\{j\}) = 1(i \ne j)$ and $\pi_0 = \pi_1 = 1/2$. Then $k^{(1/2)} = \lambda_1^{(1)} = \lambda_1 = 2$. Moreover, the same example shows that in contrast to (1.9), the analog of (1.9) "$\lambda_1 \ge k^{(1/2)^2}/\big[4\big(2-\lambda_1^{(1)}\big)\big]$" or "$\lambda_1 \ge k^{(1/2)'2}/\big[2-\lambda_1^{(1)}\big)\big]$" does not hold.

(b) For any $B \subset E$ with $\pi(B) > 0$, define a local form as follows:

$$\widetilde{D}_B^{(\alpha)}(f,f) = \tfrac{1}{2}\int_{B\times B} J^{(\alpha)}(dx,dy)[f(y)-f(x)]^2 + \int_B J^{(\alpha)}(dx,B^c)f(x)^2.$$

Obviously, $\widetilde{D}_B^{(\alpha)}(f,f) = \widetilde{D}_B^{(\alpha)}(fI_B, fI_B)$. Moreover, it is easy to see that

$$\lambda_0(B) = \inf\big\{\widetilde{D}_B(f,f):\ \pi(f^2I_B) = 1\big\}.$$

Let

$$h_B^{(\alpha)} = \inf_{A\subset B,\, \pi(A)>0} \frac{J^{(\alpha)}(A\times(B\setminus A))+J^{(\alpha)}(A\times B^c)}{\pi(A)} \tag{2.8}$$
$$= \inf_{A\subset B,\, \pi(A)>0} \frac{J^{(\alpha)}(A\times A^c)}{\pi(A)}.$$

Applying Theorem 1.1 to the local form on $L^2(B, \mathscr{E}\cap B, \pi^B)$ generated by $J^B = \pi(B)^{-1}J|_{B\times B}$ and $K^B = J(\cdot, B^c)|_B$, we obtain $\lambda_0(B) \geq h_B^{(1/2)^2}/\Big[1+\sqrt{1-h_B^{(1)^2}}\Big]$.

We now come to another key point of the proof. In [8], the proof is based on the estimate $\lambda_1 \geq \inf_B\{\lambda_0(B)\vee\lambda_0(B^c)\}$. However, we are unable to prove this in the present setup. Instead, we prove the following weaker result, which is enough for our purpose:

$$\lambda_1 \geq \inf_{\pi(B)\leq 1/2} \lambda_0(B).$$

For each $\varepsilon > 0$, choose f_ε with $\pi(f_\varepsilon) = 0$ and $\pi(f_\varepsilon^2) = 1$ such that $\lambda_1 + \varepsilon \geq D(f_\varepsilon, f_\varepsilon)$. Next, choose c_ε such that $\pi(f_\varepsilon < c_\varepsilon)$, $\pi(f_\varepsilon > c_\varepsilon) \leq 1/2$. Set $f_\varepsilon^\pm = (f_\varepsilon - c_\varepsilon)^\pm$ and $B_\varepsilon^\pm = \{f_\varepsilon^\pm > 0\}$. Then

$$\begin{aligned}
\lambda_1 + \varepsilon &\geq D(f_\varepsilon - c_\varepsilon,\, f_\varepsilon - c_\varepsilon)\\
&= \tfrac{1}{2}\int J(dx,dy)\big[\big|f_\varepsilon^+(y) - f_\varepsilon^+(x)\big| + \big|f_\varepsilon^-(y) - f_\varepsilon^-(x)\big|\big]^2\\
&\geq \tfrac{1}{2}\int J(dx,dy)\big(f_\varepsilon^+(y) - f_\varepsilon^+(x)\big)^2 + \tfrac{1}{2}\int J(dx,dy)\big(f_\varepsilon^-(y) - f_\varepsilon^-(x)\big)^2\\
&\geq \lambda_0\big(B_\varepsilon^+\big)\pi\big(\big(f_\varepsilon^+\big)^2\big) + \lambda_0\big(B_\varepsilon^-\big)\pi\big(\big(f_\varepsilon^-\big)^2\big)\\
&\geq \inf_{\pi(B)\leq 1/2} \lambda_0(B)\pi\big(\big(f_\varepsilon^+\big)^2 + \big(f_\varepsilon^-\big)^2\big)\\
&= (1 + c_\varepsilon^2) \inf_{\pi(B)\leq 1/2} \lambda_0(B) \geq \inf_{\pi(B)\leq 1/2} \lambda_0(B).
\end{aligned}$$

Because ε is arbitrary, we obtain the required conclusion.

Finally, combining the above two assertions, we obtain

$$\begin{aligned}
\lambda_1 &\geq \inf_{\pi(B)\leq 1/2} \frac{h_B^{(1/2)^2}}{1+\sqrt{1-h_B^{(1)^2}}} \geq \inf_{\pi(B)\leq 1/2} \frac{\inf_{\pi(B)\leq 1/2} h_B^{(1/2)^2}}{1+\sqrt{1-h_B^{(1)^2}}}\\
&\geq \frac{\inf_{\pi(B)\leq 1/2} h_B^{(1/2)^2}}{1+\sqrt{1-\inf_{\pi(B)\leq 1/2} h_B^{(1)^2}}} = \frac{k^{(1/2)'2}}{1+\sqrt{1-k^{(1)'2}}}.
\end{aligned}$$

□

PROOF OF THEOREM 1.3. The proof is split into two lemmas given below. Noticing that α is fixed, we may and will omit the superscript "(α)" everywhere in the next two lemmas and their proofs for simplicity. □

LEMMA 2.3. *Let $B \in \mathscr{E}$ with $2\pi(B) > 1$. Then*

$$k' \geq \frac{h_{B^c} k(B)(2\pi(B)-1)}{k(B)(2\pi(B)-1) + 2\pi(B)^2(M_B + h_{B^c})},$$

where h_B is defined by (2.8).

PROOF. We need only to consider the case that $h_{B^c} k(B) > 0$. For any $A \in \mathscr{E}$ with $\pi(A) \in (0, 1/2]$, let $\gamma = \pi(AB)/\pi(A)$. Then

$$\begin{aligned} \frac{J(A \times A^c)}{\pi(A)} &= \frac{1}{2\pi(A)} \int J(dx, dy)\big[I_A(y) - I_A(x)\big]^2 \\ &\geq \frac{1}{2\pi(A)} \int_{B \times B} J(dx, dy)\big[I_A(y) - I_A(x)\big]^2 \qquad (2.9) \\ &\geq \frac{k(B)\pi^B(A)\pi^B(A^c)}{\pi(A)} \geq \frac{\pi(B) - 1/2}{\pi(B)^2} k(B)\gamma. \end{aligned}$$

Here, in the last step, we have used $\pi(AB) \leq \pi(A) \leq 1/2$. On the other hand, we have

$$\begin{aligned} h_{B^c}\pi(AB^c) &\leq \tfrac{1}{2} \int J(dx, dy)\big[I_{AB^c}(x) - I_{AB^c}(y)\big]^2 \\ &= \tfrac{1}{2} \int J(dx, dy)\big|I_{A^c \cup B}(x) - I_{A^c \cup B}(y)\big|. \end{aligned}$$

Noticing that J is symmetric and

$$\big|I_{A^c \cup B}(x) - I_{A^c \cup B}(y)\big| \leq \big|I_{A^c}(x) - I_{A^c}(y)\big| + I_{B \times B^c + B^c \times B}\big|I_{AB}(x) - I_{AB}(y)\big|,$$

we obtain

$$h_{B^c}(1 - \gamma) = \frac{h_{B^c}\pi(AB^c)}{\pi(A)} \leq \frac{J(A \times A^c)}{\pi(A)} + M_B\gamma.$$

Combining this with (2.9) and applying Lemma 2.2, we get

$$\begin{aligned} \frac{J(A \times A^c)}{\pi(A)} &\geq \inf_{\gamma \in [0,1]} \max\Big\{(\pi(B) - 1/2)\pi(B)^{-2}k(B)\gamma,\, h_{B^c} - \big(M_B + h_{B^c}\big)\gamma\Big\} \\ &= \frac{h_{B^c} k(B)(2\pi(B)-1)}{k(B)(2\pi(B)-1) + 2\pi(B)^2(M_B + h_{B^c})}. \end{aligned}$$ □

LEMMA 2.4. *Let ϕ satisfy $\delta_1(\phi) < \infty$. If $\gamma_B = -\sup_B \Omega\phi > 0$, then $h_B \geq \gamma_B/\delta_1(\phi) > 0$.*

PROOF. For any $A \subset B$, we have

$$\begin{aligned} \gamma_B \pi(A) &\leq \int_A [-\Omega\phi] d\pi = \frac{1}{2} \int J(dx, dy)(I_A(x) - I_A(y))(\phi(x) - \phi(y)) \\ &\leq \frac{\delta_1(\phi)}{2} \int J(dx, dy)|I_A(x) - I_A(y)| = \delta_1(\phi) J(A \times A^c). \end{aligned}$$

Hence, $h_B \geq \gamma_B/\delta_1(\phi)$. □

To conclude this section, we discuss a different way to deal with general symmetric forms. In contrast to the previous approach, we now keep (J, K) to be the same but change the L^2-space. To do so, let p be a measurable function and satisfy $\alpha_p := \operatorname{ess\,inf}_\pi p > 0$, $\beta_p := \pi(p) < \infty$ and $\|J(\cdot, E) + K\|_{\text{op}} \le \beta_p (L^1_+(\pi_p) \to \mathbb{R}_+)$, where $\pi_p = p\pi/\beta_p$. For jump processes, one may take $p(x) = q(x) \vee r$ for some $r \ge 0$. From this, one sees the main restriction of the present approach: $\int \pi(dx) q(x) < \infty$, since we require that $\pi(p) < \infty$. Except this point, the approach is not comparable with the previous one (see Examples 4.5 and 4.7 below).

Next, define h_p, k_p and k'_p by (1.3)–(1.5), respectively, with π replaced by π_p and then divided by β_p. For instance, $k'_p = \inf_{\pi_p(A) \le 1/2} J(A \times A^c)/\pi(pI_A)$.

THEOREM 2.5. *Let p, α_p, β_p and π_p be given above. Define $\lambda_{p,i}$ $(i = 0, 1)$ by* (1.1) *and* (1.2) *with π replaced by π_p. Then, we have*

$$\lambda_i \ge \frac{\alpha_p}{\beta_p} \lambda_{p,i}, \qquad i = 0, 1. \tag{2.10}$$

In particular,

$$\lambda_0 \ge \alpha_p \Big(1 - \sqrt{1 - h_p^2}\,\Big) \tag{2.11}$$

and when $K = 0$,

$$\lambda_1 \ge \max\left\{\frac{\kappa}{8} \alpha_p k_p^2, \alpha_p \Big(1 - \sqrt{1 - k_p'^2}\,\Big)\right\}. \tag{2.12}$$

PROOF. (a) We prove that $L^\infty(\pi)$ is dense in $\mathscr{D}(D)$ in the D-norm: $\|f\|_D^2 = D(f, f) + \pi(f^2)$. The proof is similar to [3], Lemma 9.7. First, we show that $L^\infty(\pi) \subset \mathscr{D}(D)$. Because $1 \in L^1(\pi_p)$ and $\|J(\cdot, E) + K\|_{\text{op}} \le \beta_p$, we have $J(E, E) + K(E) \le \beta_p < \infty$. Thus,

$$\begin{aligned} D(f, f) &\le \int J(dx, dy)\big[f(y)^2 + f(x)^2\big] + \int K(dx) f(x)^2 \\ &\le 2\|f\|_\infty^2 \big(J(E, E) + K(E)\big) < \infty, \end{aligned}$$

and hence $f \in \mathscr{D}(D)$. Next, let $f \in \mathscr{D}(D)$ and set $f_n = (-n) \vee (f \wedge n)$. Then $f_n \in \mathscr{D}(D)$,

$$|f_n(y) - f_n(x)| \le |f(y) - f(x)| \quad \text{and} \quad |f_n(x)| \le |f(x)| \tag{2.13}$$

for all x, y and n. Clearly, $\pi\big((f_n - f)^2\big) \to 0$. Moreover, since $D(f_n - f, f_n - f) \le 4D(f, f) < \infty$ by (2.13), we have $D(f_n - f, f_n - f) \to 0$ by (2.13) and the dominated convergence theorem. Therefore, $\|f_n - f\|_D \to 0$.

(b) Here, we prove (2.10) for $i = 1$ only since the proof for $i = 0$ is similar and even simpler. Then, (2.11) and (2.12) follows from (1.7) and the comment right after Theorem 1.2 with $M = \beta_p$.

Because $L^\infty(\pi) \subset L^2(\pi_p)$ and $L^2(\pi_p)$ is just the domain of the form $D(f, f)$ on $L^2(\pi_p)$, by definition of λ_1 and $\lambda_{p,1}$, it suffices to show that $\pi_p(f^2) - \pi_p(f)^2 \ge [\pi(f^2) - \pi(f)^2]\alpha_p/\beta_p$ for every $f \in L^\infty(\pi)$. The proof goes as follows:

$$\begin{aligned}\pi_p(f^2) - \pi_p(f)^2 &= \inf_{c\in\mathbb{R}} \int \big(f(x) - c\big)^2 \pi_p(dx)\\ &= \beta_p^{-1} \inf_{c\in\mathbb{R}} \int \big(f(x) - c\big)^2 p(x)\pi(dx)\\ &\ge \frac{\alpha_p}{\beta_p} \inf_{c\in\mathbb{R}} \int \big(f(x) - c\big)^2 \pi(dx)\\ &= \frac{\alpha_p}{\beta_p}\big[\pi(f^2) - \pi(f)^2\big]. \qquad \square\end{aligned}$$

3. A criterion for the existence of spectral gap. Proofs of Theorems 1.4 and 1.5. To state our main criterion, we need some preparation.

Let E be a locally compact separable metric space with Borel field $\mathscr{E}$ and π be a probability measure with $\mathrm{supp}(\pi) = E$. Denote by $C_b(E)$ [resp. $C_0(E)$] the set of all bounded continuous functions (resp. with compact support) on E.

Next, let $(D, \mathscr{D}(D))$ be a regular conservative Dirichlet form on $L^2(\pi)$. By Beurling-Deny's formula, the form can be expressed as follows:

$$\begin{aligned}D(f, f) &= D^{(c)}(f, f)\\ &\quad + \tfrac{1}{2} \int J(dx, dy)(f(x) - f(y))^2, \qquad f \in \mathscr{D}(D) \cap C_0(E),\end{aligned} \tag{3.1}$$

where $\mathscr{D}(D^{(c)}) = \mathscr{D}(D) \cap C_0(E)$ and satisfies a strong local property; J is a symmetric Radon measure on the product space $E \times E$ off diagonal. Moreover, there exists a finite, nonnegative Radon measure $\mu^c_{\langle f\rangle}$ such that

$$D^{(c)}(f, f) = \tfrac{1}{2} \int_E d\mu^c_{\langle f\rangle}, \qquad f \in \mathscr{D}(D) \cap C_b(E).$$

THEOREM 3.1. *Let $\mathscr{C} \subset \mathscr{D}(D) \cap C_0(E)$ be dense in $\mathscr{D}(D)$ in the D-norm: $\|f\|_D^2 = D(f, f) + \pi(f^2)$. Set $\mathscr{C}_L = \{f + c\colon f \in \mathscr{C}, c \in \mathbb{R}\}$. Given $A, B \in \mathscr{E}$, $A \subset B$ with $0 < \pi(A), \pi(B) < 1$. Suppose that the following conditions hold:*

(i) *There exists a conservative Dirichlet form $(D_B, \mathscr{D}(D_B))$ on the square-integrable functions on B with respect to π^B such that $\mathscr{C}_L|_B \subset \mathscr{D}(D_B)$ and*

$$D(f, f) \ge D_B(fI_B, fI_B), \qquad f \in \mathscr{C}_L.$$

(ii) *There exists a function $h \in \mathscr{C}_L$: $0 \le h \le 1$, $h|_A = 0$ and $h|_{B^c} = 1$ such that*

$$c(h) := \sup_{f\in\mathscr{C}_L} \frac{1}{\pi(f^2 I_B)} \left[\frac{1}{2}\int f^2 d\mu^c_{\langle h\rangle} + \int_{B\times A^c} J(dx, dy)\big[f(1-h)(y) - f(1-h)(x)\big]^2\right] < \infty.$$

Then we have

$$\frac{\lambda_0(A^c)}{\pi(A)} \ge \lambda_1 \ge \frac{\lambda_1(B)[\lambda_0(A^c)\pi(B) - 2c(h)\pi(B^c)]}{2\lambda_1(B) + \pi(B)^2[\lambda_0(A^c) + 2c(h)]}.$$

PROOF. The upper bound is easy. Simply take $f \in \mathscr{D}(D)$ with $f|_A = 0$ and $\pi(f^2) = 1$. Then

$$\pi(f^2) - \pi(f)^2 = 1 - \pi\big(fI_{A^c}\big)^2 \ge 1 - \pi\big(f^2\big)\pi\big(A^c\big) = 1 - \pi\big(A^c\big) = \pi(A).$$

Hence $\lambda_1 \le D(f,f)/\pi(A)$ which gives us $\lambda_1 \le \lambda_0\big(A^c\big)/\pi(A)$.

For the lower bound, let $f \in \mathscr{C}_L$ with $\pi(f) = 0$ and $\pi\big(f^2\big) = 1$. Set $\gamma = \pi\big(f^2 I_B\big)$.

(a) By condition (i), we have

$$\begin{aligned} D(f,f) &\ge D_B(fI_B, fI_B) \ge \lambda_1(B)\pi(B)^{-1}\big[\pi\big(f^2 I_B\big) - \pi(B)^{-1}\pi\big(fI_B\big)^2\big] \\ &= \lambda_1(B)\pi(B)^{-1}\big[\pi\big(f^2 I_B\big) - \pi(B)^{-1}\pi\big(fI_{B^c}\big)^2\big] \\ &\ge \lambda_1(B)\pi(B)^{-1}\big[\gamma - \pi(B)^{-1}\pi\big(f^2 I_{B^c}\big)\pi\big(B^c\big)\big] \\ &= \lambda_1(B)\pi(B)^{-2}\big[\gamma - \pi\big(B^c\big)\big]. \end{aligned} \tag{3.2}$$

(b) Let ρ be the metric in E. By the construction of $\mu^c_{\langle f\rangle}$ (cf. [7], Section 3.2), there exist a sequence of relatively compact open sets G_ℓ increasing to E, a sequence of symmetric, nonnegative Radon measures σ_{β_n} and a sequence δ_ℓ such that

$$\int_E g\,d\mu^c_{\langle f\rangle} = \lim_{\ell\to\infty}\lim_{\beta_n\to\infty}\beta_n\int_{G_\ell\times G_\ell,\,\rho(x,y)<\delta_\ell}[f(x)-f(y)]^2 g(x)\sigma_{\beta_n}(dx,dy),$$
$$f, g \in \mathscr{D}(D)\cap C_0(E).$$

From this and

$$[(fh)(x) - (fh)(y)]^2 \le 2h(y)^2[f(x)-f(y)]^2 + 2f(x)^2[h(x)-h(y)]^2,$$

it follows that

$$\int d\mu^c_{\langle fh\rangle} \le 2\int h^2 d\mu^c_{\langle f\rangle} + 2\int f^2 d\mu^c_{\langle h\rangle},$$

first for $f,\ h \in \mathscr{D}(D)\cap C_0(E)$ and then for $f,\ h \in \mathscr{D}(D)\cap C_b(E)$ (cf. [7], Section 3.2). Hence

$$D^{(c)}(fh, fh) = \tfrac{1}{2}\int d\mu^c_{\langle fh\rangle} \le 2D^{(c)}(f,f) + \int f^2 d\mu^c_{\langle h\rangle}. \tag{3.3}$$

On the other hand, since

$$|(fh)(x)-(fh)(y)| \le |f(x)-f(y)| + I_{B\times A^c\cup A^c\times B}(x,y)|f(1-h)(x) - f(1-h)(y)|,$$

we have

$$
\begin{aligned}
&\int J(dx,dy)[(fh)(x)-(fh)(y)]^2\\
(3.4)\qquad &\quad\le 2\int J(dx,dy)[f(x)-f(y)]^2\\
&\qquad +4\int_{B\times A^c} J(dx,dy)[f(1-h)(x)-f(1-h)(y)]^2.
\end{aligned}
$$

Thus, combining (3.1), (3.3), (3.4) with condition (ii), we get

$$
\begin{aligned}
(3.5)\qquad D(fh,fh)&\le 2D(f,f)+\int f^2 d\mu^c_{\langle h\rangle}\\
&\quad +2\int_{B\times A^c} J(dx,dy)\big[f(1-h)(x)-f(1-h)(y)\big]^2\\
&\le 2D(f,f)+2c(h)\pi\big(f^2 I_B\big)=2D(f,f)+2\gamma c(h).
\end{aligned}
$$

That is,

$$
\begin{aligned}
(3.6)\qquad D(f,f)&\ge \tfrac12 D(fh,fh)-\gamma c(h)\ge \tfrac12\lambda_0\big(A^c\big)\pi\big(f^2h^2\big)-\gamma c(h)\\
&\ge \tfrac12\lambda_0\big(A^c\big)\pi\big(f^2 I_{B^c}\big)-\gamma c(h)=\tfrac12\lambda_0\big(A^c\big)(1-\gamma)-\gamma c(h).
\end{aligned}
$$

Combining (3.2) with (3.5), we obtain

$$
\begin{aligned}
(3.7)\qquad D(f,f)&\ge \inf_{\gamma\in[0,1]}\max\left\{\frac{\lambda_1(B)}{\pi(B)^2}\big(\gamma-\pi(B^c)\big),\ \frac12\lambda_0\big(A^c\big)(1-\gamma)-\gamma c(h)\right\}\\
&=\lambda_1(B)\pi(B)^{-2}\big(\gamma_0-\pi\big(B^c\big)\big).
\end{aligned}
$$

The assertion of the theorem now follows from (3.6) and Lemma 2.2. □

Theorem 3.1 is effective for diffusions as was shown in [10] with a more direct proof (in this case the Dirichlet form is explicit). We now apply the theorem to jump processes.

PROOF OF THEOREM 1.4. First, the topological assumptions of Theorem 3.1 are unnecessary in the present context. To see that condition (i) is fulfilled, simply take D_B to be the one defined by (1.13). For condition (ii), take $h=I_{A^c}$. Then

$$
\int_{B\times A^c} J(dx,dy)[(fI_A)(x)-(fI_A)(y)]^2\le\int_{B\times A^c} J(dx,dy)f(x)^2\le M_A\pi\big(f^2 I_B\big).
$$

This means that condition (ii) holds with $c(h)=M_A$. We have thus proved Theorem 1.4. □

The application of Theorem 3.1 (or Theorem 1.4) requires some estimates of $\lambda_0(A^c)$ and $\lambda_1(B)$, which may be obtained from Theorems 1.1 and 1.2. These estimates are usually in the qualitative sense good enough for $\lambda_1(B)$, for which there are also quite a lot of publications, including the authors' study, in the past years. However, for $\lambda_0(A^c)$, the bound presented above may not be sharp enough, especially in the unbounded situation. For this reason, we now introduce a different result.

THEOREM 3.2. *Let E be a metric space with Borel field $\mathscr{E}$ and let (x_t) be a reversible right-continuous Markov process valued in E with weak generator Ω. Suppose that the corresponding Dirichlet form is regular. Next, fix a closed set B. Suppose additionally that the following conditions hold:*

(i) *There exists a function ϕ satisfying $\phi|_B = 0$, $\phi|_{B^c} > 0$ and $\sup_{B^c} \Omega\phi/\phi =: -\delta < 0$.*

(ii) *There exists a sequence of open sets (E_n): $E_0 \supset B$, $E_n \uparrow E$ such that ϕ is bounded below on each $E_n \setminus B$ by a positive constant.*

(iii) *The first Dirichlet eigenfunction of Ω on each $E_n \setminus B$ is bounded above.*

Then we have $\lambda_0(B^c) \geq \delta$. In particular, for jump processes, the condition "$\phi|_B = 0$" given in (i) *can be removed.*

Clearly, conditions (ii) and (iii) with compact B are fulfilled for diffusions or Markov chains. Thus, the key condition here is the first one.

PROOF OF THEOREM 3.2. The last assertion follows by replacing ϕ with ϕI_{B^c}. Indeed,

$$\begin{aligned}\Omega(\phi I_{B^c})(x) &= \int q(x, dy)\big[(\phi I_{B^c})(y) - (\phi I_{B^c})(x)\big]\\ &\leq \int q(x, dy)[\phi(y) - (\phi I_{B^c})(x)] = \Omega\phi(x)\\ &\leq -\delta(\phi I_{B^c})(x) \quad \text{on } B^c.\end{aligned}$$

We are now going to prove the main assertion of the theorem. Set $\tau_B = \inf\{t \geq 0\colon\ x_t \in B\}$. Then, by condition (i) plus a truncating argument if necessary, we get

$$\mathbb{E}^x \phi(x_{t\wedge\tau_B}) \leq \phi(x)e^{-\delta t}, \qquad t \geq 0, x \notin B.$$

Next, let $u_n\,(\geq 0)$ be the first Dirichlet eigenfunction of Ω on $E_n \setminus B$. Set $\tau = \inf\{t \geq 0\colon\ x_t \notin E_n \setminus B\}$. Then, by conditions (ii) and (iii), there exists $c_1 > 0$ such that $u(x_{t\wedge\tau}) \leq c_1\phi(x_{t\wedge\tau_B})$ and so

$$\begin{aligned}u_n(x)\exp(-\lambda_0(E_n \setminus B)t) = \mathbb{E}^x u_n(x_{t\wedge\tau}) &\leq c_1\mathbb{E}^x\phi(x_{t\wedge\tau_B})\\ &\leq c_1\phi(x)e^{-\delta t}, \qquad x \in E_n \setminus B.\end{aligned}$$

This implies that $\lambda_0(E_n \setminus B) \geq \delta$. Finally, because the Dirichlet form is regular, it is easy to show that $\lambda_0(B^c) = \lim_{n\to\infty} \lambda_0(E_n \setminus B)$ and so the required assertion follows. □

For the remainder of this section, we turn to study the upper bound of λ_1.

Let $(D, \mathscr{D}(D))$ be a general conservative Dirichlet form and let $P(t, x, dy)$ be the corresponding transition probability. Fix $\phi \geq 0$. Suppose that $\phi \wedge n \in$

$\mathscr{D}(D)$ for every $n \geq 1$. Set $f_n = \exp[\varepsilon(\phi \wedge n)/2]$. Since the function $e^{\alpha x}$ is locally Lipschitz continuous and $\phi \wedge n$ is bounded, by the elementary spectral representation theory, we have

$$\begin{aligned} D(f_n, f_n) &= \lim_{t \to 0} \frac{1}{2t} \int \pi(dx)P(t, x, dy)[f_n(x) - f_n(y)]^2 \\ &\leq \frac{\varepsilon^2}{4} C(\phi, n) \lim_{t \to 0} \frac{1}{2t} \int \pi(dx)P(t, x, dy)[(\phi \wedge n)(x) - (\phi \wedge n)(y)]^2 \\ &\leq \frac{\varepsilon^2}{4} C(\phi, n) D(\phi \wedge n, \phi \wedge n) < \infty, \end{aligned}$$

where $C(\phi, n)$ is the Lipschitz norm of $e^{\varepsilon x/2}$ on the range of $\phi \wedge n$. This leads us to introduce the following constant:

$$\delta(\varepsilon, \phi) = \varepsilon^{-2} \sup_{n \geq 1} D(f_n, f_n)/\pi(f_n^2).$$

THEOREM 3.3. *Let $(D, \mathscr{D}(D))$, ϕ, f_n and $\delta(\varepsilon, \phi)$ be as above. Then, we have*

$$\lambda_1 \leq \inf\{\varepsilon^2 \delta(\varepsilon, \phi): \ \pi(e^{\varepsilon\phi}) = \infty\}.$$

PROOF. We need to show that if $\pi(e^{\varepsilon\phi}) = \infty$, then $\lambda_1 \leq \varepsilon^2 \delta(\varepsilon, \phi)$. For $n \geq 1$, we have

$$\lambda_1 \leq \frac{D(f_n, f_n)}{\pi(f_n^2) - \pi(f_n)^2}. \tag{3.8}$$

For every $m \geq 1$, choose $r_m > 0$ such that $\pi(\phi \geq r_m) \leq 1/m$. Then

$$\pi(I_{[\phi \geq r_m]} f_n^2)^{1/2} \geq \sqrt{m}\, \pi(I_{[\phi \geq r_m]} f_n) \geq \sqrt{m}\, \pi(f_n) - \sqrt{m} \exp(\varepsilon r_m/2).$$

Hence

$$\pi(f_n)^2 \leq \Big[\sqrt{\pi(f_n^2)}/\sqrt{m} + \exp(\varepsilon r_m/2)\Big]^2. \tag{3.9}$$

On the other hand, by assumption, we have

$$D(f_n, f_n) \leq \varepsilon^2 \delta(\varepsilon, \phi)\pi(f_n^2). \tag{3.10}$$

Noticing that $\pi(f_n^2) \uparrow \infty$, combining (3.9) with (3.7) and (3.8) and then letting $n \uparrow \infty$, we obtain

$$\lambda_1 \leq \varepsilon^2 \delta(\varepsilon, \phi)/[1 - m^{-1}].$$

The proof is completed by setting $m \uparrow \infty$. □

PROOF OF THEOREM 1.5. It suffices to prove the first assertion because the remainder of the proof is similar. Let f_n be given as in Theorem 3.3. Note that by the mean value theorem, $|e^A - e^B| \le |A-B|e^{A\vee B} = |A-B|(e^A \vee e^B)$ for all $A,\ B \ge 0$. Hence,

$$\begin{aligned} D(f_n, f_n) &= \frac{1}{2}\int J(dx, dy)[f_n(x) - f_n(y)]^2 \\ &\le \frac{\varepsilon^2}{8}\int J^{(1)}(dx, dy)[\phi(x) - \phi(y)]^2 r(x, y)\big[f_n(x)\vee f_n(y)\big]^2 \\ &\le \frac{\varepsilon^2}{4}\delta_2(\phi)\pi\big(f_n^2\big). \end{aligned}$$

The conclusion now follows from Theorem 3.3 with $\delta(\varepsilon, \phi) = \frac{1}{4}\delta_2(\phi)$. □

4. Spectral gap for Markov chains. Usually, the power of a result for general jump processes should be justified by Markov chains.

Let E be countable and (q_{ij}) be a regular and irreducible Q-matrix, reversible with respect to $\pi = (\pi_i)$. As usual, let $q_i = \sum_{j\ne i} q_{ij}$. Then $K = 0$ and $\Omega f(i) = \sum_{j\ne i} q_{ij}[f_j - f_i]$. The density of the symmetric measure with respect to the counting measure becomes $J(i, j) = \pi_i q_{ij}\,(i \ne j)$. For simplicity, we consider only two typical situations: $E = \mathbb{Z}_+$ or $E = \mathbb{Z}^d$ and take $r(i, j) = 1/(q_i \vee q_j)$. Denote by $|i|$ the L^1-norm, that is, $|i| = \sum_{k=1}^d |i_k|$ for $i = (i_1, \dots, i_d) \in \mathbb{Z}^d$.

A combination of Theorem 1.2 and the next result provides us with a simple condition for the existence of spectral gap for birth–death processes and the result seems to be new, to our knowledge, even for such a simple situation (cf. [4]).

THEOREM 4.1. *Consider the birth–death process on $\mathbb{Z}_+$ with birth rates (b_i) and death rates (a_i):*

(i) *Take $r_{ij} = (a_i + b_i) \vee (a_j + b_j)(i \ne j)$. Then $k^{(\alpha)'} > 0$ (equivalently, $k^{(\alpha)} > 0$) iff there exists a constant $c > 0$ such that*

$$\frac{\pi_i a_i}{[(a_i + b_i)\vee(a_{i-1} + b_{i-1})]^\alpha} \ge c\sum_{j\ge i}\pi_j, \qquad i \ge 1. \tag{4.1}$$

Then, we indeed have $k^{(\alpha)'} \ge c$. Furthermore,

$$k^{(\alpha)} \ge \inf_{i\ge 1}\frac{\pi_i a_i}{[(a_i + b_i)\vee(a_{i-1} + b_{i-1})]^\alpha(1 - \pi_i)\sum_{j\ge i}\pi_j}.$$

(ii) *Let $\sum_i \pi_i(a_i + b_i) < \infty$. Take $p_i = a_i + b_i$. Then we have $k'_p > 0$ (equivalently, $k_p > 0$) iff $\inf_{i\ge 1}\frac{\pi_i a_i}{\sum_{j\ge i}\pi_j p_j} > 0$ and moreover,*

$$k'_p \ge \inf_{i\ge 1}\frac{\pi_i a_i}{\sum_{j\ge i}\pi_j p_j}, \qquad k_p \ge \inf_{i\ge 1}\frac{\pi_i a_i}{(1 - \pi_i p_i/\beta_p)\sum_{j\ge i}\pi_j p_j}.$$

Roughly speaking, (4.1) holds if π_j has exponential decay. For polynomial decay, (4.1) can still be true when $\alpha = 1/2$. See Example 4.5.

PROOF OF THEOREM 4.1. Here we prove part (i) only since the proof of part (ii) is similar.

(a) Let $k^{(\alpha)} > 0$. Take $A = I_i = \{i, i+1, \dots\}$ for a fixed $i > 0$ and

$$J^{(\alpha)}(i,j) = \frac{\pi_i q_{ij}}{[q_i \vee q_j]^\alpha} = \begin{cases} \dfrac{\pi_i a_i}{[(a_i+b_i)\vee(a_{i-1}+b_{i-1})]^\alpha} =: \pi_i\tilde{a}_i, & \text{if } j = i-1, \\ \dfrac{\pi_i b_i}{[(a_i+b_i)\vee(a_{i+1}+b_{i+1})]^\alpha} =: \pi_i\tilde{b}_i, & \text{if } j = i+1. \end{cases}$$

Then

$$k^{(\alpha)'} \le k^{(\alpha)} \le \frac{J^{(\alpha)}(A\times A^c)}{\pi(A)\pi(A^c)} = \frac{\pi_i\tilde{a}_i}{\left(\sum_{j\ge i}\pi_j\right)\left(\sum_{j<i}\pi_j\right)} \le \frac{\pi_i\tilde{a}_i}{\pi_0\sum_{j\ge i}\pi_j}.$$

This proves the necessity of the condition.

(b) Next, assume that the condition holds. Then for each A with $\pi(A) \in (0,1)$, since the symmetry of A and A^c, we may assume that $0 \notin A$. Set $i_0 = \min A \ge 1$. Then, $A \subset I_{i_0}$, $A^c \subset E\setminus\{i_0\}$ and so

$$\frac{J^{(\alpha)}(A\times A^c)}{\pi(A)\wedge\pi(A^c)} \ge \frac{\pi_{i_0}\tilde{a}_{i_0}}{\sum_{j\ge i_0}\pi_j} \ge c, \qquad \frac{J^{(\alpha)}(A\times A^c)}{\pi(A)\pi(A^c)} \ge \frac{\pi_{i_0}\tilde{a}_{i_0}}{(1-\pi_{i_0})\sum_{j\ge i_0}\pi_j}.$$

Because A is arbitrary, we obtain the required assertions. □

THEOREM 4.2. *Let $E = \mathbb{Z}_+$. Suppose that (q_{ij}) has finite range, that is, there exists $R > 0$ such that $q_{ij} = 0$ whenever $|i-j| > R$. Then, we have $\lambda_1 > 0$ provided*

$$\limsup_{i\to\infty}\sum_j \frac{q_{ij}}{\sqrt{q_i \vee q_j}}(j-i) < 0.$$

PROOF. Simply take $\phi_i = i+1$ and $B = \{0,1,\dots,n\}$ for large n in Theorem 1.3 and then apply Theorem 1.2. □

Similarly, we have the following result.

THEOREM 4.3. *Let $E = \mathbb{Z}^d$. Suppose that (q_{ij}) has finite range. Then, we have $\lambda_1 > 0$ provided*

$$\limsup_{|i|\to\infty}\sum_j \frac{q_{ij}}{\sqrt{q_i\vee q_j}}\big[|j|-|i|\big] < 0.$$

PROOF. Take $\phi_i = |i|+1$ in Theorem 1.3 and then apply Theorem 1.2. □

THEOREM 4.4. *Let $E = \mathbb{Z}^d$. If there exists a positive function ϕ such that*

$$\limsup_{|i|\to\infty}\Omega\phi/\phi < 0,$$

then $\lambda_1 > 0$.

PROOF. Apply Theorem 1.2, Theorem 3.2 and then Theorem 1.4 to the finite sets $\{i: |i| \le n\}$. □

The following example, taken from [4], is especially rare and interesting since it exhibits the critical phenomena for the existence of spectral gap. It is now used to justify the power of our results and we should see soon what will happen. Similar examples for diffusion were given in [5] and [10].

EXAMPLE 4.5. *Let $E = \mathbb{Z}_+$ and $a_i = b_i = i^\gamma\,(i \ge 1)$ for some $\gamma > 0$, $a_0 = 0$ and $b_0 = 1$. Then $\lambda_1 > 0$ iff $\gamma \ge 2$.*

PROOF. (a) By part (i) of Theorem 4.1, we have $k^{(1/2)} > 0$ iff $\gamma \ge 2$. Thus, by Theorem 1.2, we have $\lambda_1 > 0$ for all $\gamma \ge 2$.

(b) Applying Theorem 1.5 to $\phi_i = 1 + i^{1-\gamma/2}$, it follows that $\lambda_1 = 0$ for all $\gamma \in (1, 2)$.

(c) The conditions of Theorem 4.2 hold whenever $\gamma \ge 2$. Hence $\lambda_1 > 0$ for all $\gamma \ge 2$.

(d) Next, taking $\phi_i = \sqrt{i}(i \ge 1)$, we see that $\Omega\phi(i)/\phi_i = -\frac{1}{4}i^{\gamma-2} + O\big(i^{\gamma-3}\big)$. Then

$$\lim_{i\to\infty} \frac{1}{\phi_i}\Omega\phi(i) = \begin{cases} -\infty, & \text{if } \gamma > 2,\\ -\frac{1}{4}, & \text{if } \gamma = 2.\end{cases}$$

By Theorem 4.4, we have $\lambda_1 > 0$ for all $\gamma \ge 2$.

On the other hand, take $f_n(i) = i^{(\gamma-1)}/2 \wedge n^{(\gamma-1)}/2$ and $A = \{0\}$. Then

$$\begin{aligned}\lambda_0(A^c) &\le \liminf_{n\to\infty} \frac{\sum_{i,j\ge 0}\pi_i q_{ij}[f_n(j) - f_n(i)]^2}{2\sum_{i\ge 0}\pi_i f_n(i)^2}\\ &= \liminf_{n\to\infty} \frac{\sum_{i\ge 0}\pi_i q_{i,i+1}[f_n(i+1) - f_n(i)]^2}{2\sum_{i\ge 0}\pi_i f_n(i)^2}\\ &\le \liminf_{n\to\infty} \frac{1 + (\gamma-1)^2\sum_{i=1}^n i^{\gamma-3}}{\sum_{i=1}^n i^{-1}} = 0, \qquad 1 < \gamma < 2.\end{aligned}$$

By Theorem 1.4, we get $\lambda_1 \le \lambda_0(A^c)/\pi(A) = 0$. The case that $\gamma \le 1$ can be ignored since then the chain is not positive recurrent. □

Thus, we have seen that all the results presented in this paper, except Theorem 2.5 which does not work for this example, are qualitatively sharp for this example since every one covers the required region and there is no gap left. Finally, taking $\alpha = 0$ in part (i) of Theorem 4.1, we obtain $k \ge \big(\sum_{i=1}^\infty i^{-\gamma}\big)^{-1} > 0$ for all $\gamma > 1$. In other words, we have $k > 0$ but $\lambda_1 = 0$ for all $\gamma \in (1, 2)$. Therefore, the condition "$k > 0$" is not enough but "$k^{(1/2)} > 0$" is sufficient for $\lambda_1 > 0$.

The next two examples show that the two approaches used in the paper for Cheeger's inequalities may all attain sharp estimates but they are not comparable (remember that Theorem 2.5 is not suitable for Example 4.5). We mention that as far as we know, no optimal estimate provided by Cheeger's technique has appeared before.

EXAMPLE 4.6. *Let* $E = \mathbb{Z}_+$ *and take* $a_i \equiv a$ *and* $b_i \equiv b$ *with* $a > b > 0$. *Then, both Theorem* 1.2 *and Theorem* 2.5 *are sharp.*

PROOF. This is a standard example which is often used to justify the power of a method. It is well known that $\lambda_1 = \left(\sqrt{a} - \sqrt{b}\right)^2$ (cf. [3], Example 9.22 and [4]).

(a) By part (i) of Theorem 4.1, we have

$$k^{(\alpha)'} \geq \inf_{i \geq 1} \frac{\pi_i a_i}{(a+b)^\alpha \sum_{j \geq i} \pi_j} = \frac{a-b}{(a+b)^\alpha}.$$

Then, by Theorem 1.2, we get $\lambda_1 \geq \left(\sqrt{a} - \sqrt{b}\right)^2$.

(b) Take $p_i \equiv a + b$. Then by part (ii) of Theorem 4.1,

$$k_p' \geq \inf_{i \geq 1} \frac{\pi_i a_i}{\sum_{j \geq i} \pi_j p_j} = \frac{a-b}{a+b}.$$

The same estimate as in (a) now follows from Theorem 2.5. □

EXAMPLE 4.7. *Let* $E = \mathbb{Z}_+$ *and take* $q_{0k} = \beta_k > 0$ (*be careful to distinguish the sequence* (β_k) *and the constant* β_p), $q_{k0} = 1/2$ *for* $k \geq 1$ *and* $q_{ij} = 0$ *for all other* $i \neq j$. *Assume that* $q_0 = \sum_{k \geq 1} \beta_k < \infty$. *Then, Theorem* 2.5 *is sharp for all* q_0 *but Theorem* 1.2 *is sharp only for* $q_0 \leq 1/2$.

PROOF. From $\pi_0 q_{0k} = \pi_k q_{k0}$, it follows that $\pi_k = 2\pi_0 \beta_k$, $k \geq 1$ and $\pi_0 = (1+2q_0)^{-1}$. An interesting point of the example is that the decay of $\sum_{j \geq i} \pi_j$ as $i \to \infty$ can be arbitrarily slow, not necessarily exponential. The last condition is necessary for $\lambda_1 > 0$ for the birth–death processes with rates bounded below (by a positive constant) and above [cf. [3], Corollary 9.19 (4)].

(a) Take $p_i = q_i \vee (1/2)$, then $\alpha_p = 1/2$. Without loss of generality, assume that $0 \notin A$. Then

$$\begin{aligned}
\frac{1}{\beta_p} \frac{J(A \times A^c)}{\pi_p(A) \wedge \pi_p(A^c)} &= \frac{\sum_{i \in A} \pi_i q_{i0}}{\left(\sum_{i \in A} 2\pi_0 \beta_i p_i\right) \wedge \left(\pi_0 p_0 + \sum_{i \notin A,\, i \neq 0} 2\pi_0 \beta_i p_i\right)} \\
&= \frac{\sum_{i \in A} \beta_i}{\left(\sum_{i \in A} 2\beta_i p_i\right) \wedge \left(p_0 + \sum_{i \notin A,\, i \neq 0} 2\beta_i p_i\right)} \\
&= \frac{\sum_{i \in A} \beta_i}{\left(\sum_{i \in A} \beta_i\right) \wedge \left(p_0 + \sum_{i \notin A,\, i \neq 0} \beta_i\right)} \geq 1.
\end{aligned}$$

This gives us $k_p' \geq 1$ and hence by Theorem 2.5,

$$\lambda_1 \geq \alpha_p \left(1 - \sqrt{1 - {k_p'}^2}\right) \geq 1/2.$$

Actually, every equality in the last line must hold.

(b) Again, assume that $0 \notin A$. Then

$$\frac{J^{(\alpha)}(A \times A^c)}{\pi(A) \wedge \pi(A^c)} = \frac{\sum_{i \in A} \pi_i q_{i0} (q_i \vee q_0)^{-\alpha}}{\left(\sum_{i \in A} 2\pi_0 \beta_i\right) \wedge \left(\pi_0 + \sum_{i \notin A,\, i \neq 0} 2\pi_0 \beta_i\right)}$$

$$= \frac{1}{2} \frac{\sum_{i \in A} 2\beta_i}{\left(\frac{1}{2} \vee q_0\right)^{\alpha} \left[\left(\sum_{i \in A} 2\beta_i\right) \wedge \left(1 + \sum_{i \notin A,\, i \neq 0} 2\beta_i\right)\right]}$$

$$= \frac{1}{2} \frac{1}{\left(\frac{1}{2} \vee q_0\right)^{\alpha}} \frac{\sum_{i \in A} \beta_i}{\left(\sum_{i \in A} \beta_i\right) \wedge \left(1/2 \sum_{i \notin A,\, i \neq 0} \beta_i\right)}$$

$$= \frac{1}{2} \frac{1}{\left(\frac{1}{2} \vee q_0\right)^{\alpha}} \frac{1}{1 \wedge \left[\left(1/2 + \sum_{i \notin A,\, i \neq 0} \beta_i\right) \Big/ \sum_{i \in A} \beta_i\right]}.$$

Because $\left(1/2 + \sum_{i \notin A,\, i \neq 0} \beta_i\right) \Big/ \sum_{i \in A} \beta_i$ decreases when A increases, by setting $A = \{i\}$ for a large enough $i \neq 0$, it follows that

$$k^{(\alpha)'} = \inf_{A:\, 0 \notin A} \frac{J^{(\alpha)}(A \times A^c)}{\pi(A) \wedge \pi(A^c)} = \frac{1}{2}\left(\frac{1}{2} \vee q_0\right)^{-\alpha}.$$

By Theorem 1.2, we get

$$\lambda_1 \geq \tfrac{1}{2}\left\{1 \vee (2q_0) + \sqrt{\left(1 \vee (2q_0)\right)^2 - 1}\right\}^{-1}.$$

Thus, the lower bound is equal to $1/2 = \lambda_1$ iff $q_0 \leq 1/2$. □

The following counterexample shows the limitation of Cheeger's inequalities. Of course, the example can be easily handled with the help of some comparison technique. However, this suggests to us that sometimes it is necessary to examine a model carefully before applying the inequalities.

EXAMPLE 4.8. *Consider the birth–death process with* $a_{2i-1} = (2i-1)^2$, $a_{2i} = (2i)^4$ *and* $b_i = a_i$ *for all* $i \geq 1$. *Then, we have* $k^{(1/2)'} = 0$ *and so Theorem* 1.2 *is not applicable.*

PROOF. First, applying Theorem 4.4 to $\phi_i = \sqrt{i}$ or comparing the chain with the one with rates $a_i = b_i = (2i)^2$, one sees that $\lambda_1 > 0$. Next, because $\mu_i = 1/a_i$ (and hence $\pi_i = \mu_i/Z$, where Z is the normalizing constant), we have $\sum_{j \geq i} \mu_j = O(i^{-1})$. However, $\sqrt{a_i \vee a_{i-1}} = O(i^2)$. Hence $\sup_{i \geq 1} \sqrt{a_i \vee a_{i-1}} \sum_{j \geq i} \mu_j = \infty$. This gives us $k^{(1/2)'} = 0$ by part (i) of Theorem 4.1.

Note that the choice $r_{ij} = q_i \vee q_j\, (i \neq j)$ is usually not optimal in the sense for which (1.8) often becomes inequality rather than equality. However, the improvement provided by an optimal r_{ij} is still not enough to cover this example and so the problem is really due to the limitation of the technique. □

Acknowledgment. We are grateful to a referee for very careful comments on the first version of the paper, which was distributed as MSRI Preprint No. 1998-024. The results were also announced in *Chinese Sci. Bull.* (1998) **43** 1475–1477.

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