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LARGE DEVIATION OF DIFFUSION PROCESSES WITH DISCONTINUOUS DRIFT AND THEIR OCCUPATION TIMES[1]

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For the system of d-dim stochastic differential equations,

$$dX^\varepsilon(t) = b(X^\varepsilon(t))\,dt + \varepsilon dW(t), \qquad t \in [0, 1],$$
$$X^\varepsilon(0) = x^0 \in R^d,$$

where b is smooth except possibly along the hyperplane $x_1 = 0$, we shall consider the large deviation principle for the law of the solution diffusion process and its occupation time as $\varepsilon \to 0$. In other words, we consider $P(\|X^\varepsilon - \varphi\| < \delta, \|u^\varepsilon - \psi\| < \delta)$ where $u^\varepsilon(t)$ and $\psi(t)$ are the occupation times of X^ε and φ in the positive half space $\{x \in R^d\colon x_1 > 0\}$, respectively. As a consequence, an unified approach of the lower level large deviation principle for the law of $X^\varepsilon(\cdot)$ can be obtained.

1. Introduction In this paper, we are concerned with the large deviation principle (abbreviated as l.d.p. in the sequel) of the d-dimensional stochastic differential equations

$$\begin{aligned} dX^\varepsilon(t) &= b(X^\varepsilon(t))\,dt + \varepsilon dW(t), \qquad t \in [0, 1],\\ X^\varepsilon(0) &= x^0 \in R^d, \end{aligned} \tag{1.1}$$

where b is a bounded smooth vector field except possibly along the hyperplane $\{x = (x_1, x_2, \ldots, x_d) = (x_1, \bar{x}) \in R^d, x_1 = 0\}$ with left and right-hand side limits. Instead of the usual l.d.p. concerning only the law of trajectories of the diffusion,

$$P(\|X^\varepsilon - \varphi\| < \delta) \sim \exp\left(-\frac{I(\phi)}{\varepsilon^2}\right), \tag{1.2}$$

we shall consider the l.d.p. for the law of the diffusion and its occupation time in the positive halfspace,

$$P(\|X^\varepsilon - \varphi\| < \delta,\ \|u^\varepsilon - \psi\| < \delta) \sim \exp\left(-\frac{I(\phi, \psi)}{\varepsilon^2}\right) \tag{1.3}$$

for some suitable rate function $I(\cdot, \cdot)$ where

$$u^\varepsilon(t) = \int_0^t \chi_{(0,\infty)}(X_1^\varepsilon(s))\,ds. \tag{1.4}$$

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Obviously, u^ε is an occupation time of X^ε in the positive half space $H^+ := \{(x_1, \bar{x}) \in R^d;\ x_1 > 0\}$. Here in general, for a continuous function $f\colon [0,1] \to R^d$, an occupation time of f in H^+ is defined to be an absolutely continuous function g satisfying

$$(1.5)\qquad \dot{g}(t) \in [0,1] \text{ if } f_1(t) = 0 \quad \text{and} \quad \dot{g}(t) = \begin{cases} 1, & \text{if } f_1(t) > 0,\\ 0, & \text{if } f_1(t) < 0,\end{cases}$$

(f_1 is the first component of f). Such a g is unique if and only if $m\{t\colon f_1(t) = 0\} = 0$ where $m(\cdot)$ is the Lebesque measure. In particular, u^ε defined above is unique for X^ε because the time $X^\varepsilon(\cdot)$ spends in $\{x \in R^d;\ x_1 = 0\}$ has 0 Lebesque measure with probability 1. In any case, we write $g \in H^+(f)$ if (1.5) is satisfied.

Throughout the paper $\|\cdot\|$ shall mean the supnorm. For any vector $x \in R^d$ or any R^d-valued function φ, we shall write $x = (x_1, \bar{x})$ or $\phi = (\phi_1, \bar{\phi})$ to emphasize their first components.

If φ_1 is never 0, then $H^+(\varphi)$ is a singleton and it is easy to see that for any $\gamma > 0$, there exists an δ such that $\|X^\varepsilon - \varphi\| < \delta$ implies $\|u^\varepsilon - \psi\| < \gamma$ for $\psi \in H^+(\varphi)$ (see Lemma 6.5); hence the occupation times u^ε and ψ in (1.3) are redundant. This is also true for φ with $m\{t\colon \varphi_1(t) = 0\} = 0$ because the (bounded) derivatives of u^ε and ψ then differ on a set whose Lebesgue measure is small if $\|X^\varepsilon - \varphi\|$ is small (see Lemma 6.5). However, when $\varphi \equiv 0$ or $\varphi = 0$ in a subinterval of $[0,1]$, $H^+(\varphi)$ contains more than one element and various choices of $\psi \in H^+(\varphi)$ in (1.3) sometimes can yield more detailed information than (1.2).

The large deviation principle for the small perturbed diffusion processes is well understood if the drift and the diffusion coefficient are smooth (see [3], [13]). However, Markov processes with discontinuous transition arise naturally in a broad range of applications. The dynamics of a physical system in a discontinuous media and the queueing networks are some interesting examples. There are also many interesting results concerning the l.d.p. for the discrete time Markov processes of this type which were obtained in [1, 2, 5, 9, 10, 12, 17]. See [1] for a nice review.

The problem considered here has been previously studied in [15] and [16]. In those works, $b(\cdot)$ is assumed stable, that is, $\inf_{x_2}(b(0^-, x_2) - b(0^+, x_2)) > 0$, and the l.d.p. of (1.2) was obtained. The stable case is simpler because, as in the case that $b(\cdot)$ is smooth, the solution process can be expressed as a continuous mapping of $\varepsilon W(\cdot)$ and therefore the contraction principle is in force to yield the l.d.p. from that of Wiener measures. In [7], the l.d.p. in the one-dimensional case without assuming the stability of the drift was obtained. The basic tool used there was the Cameron–Martin–Girsanov change of measure formula. A totally different approach for (1.2) using a weak convergence argument was adopted in [6] and the rate function $I(\varphi)$ was represented in a variational form [see (2.6) in the next section]. Their arguments also work for the cases of nonconstant diffusion coefficient, but the existence of strong solution for the dynamics was required.

One advantage of considering (1.3) by adding one more component is that the complicated-looking variational form in [6] becomes transparent through the contraction of ψ in $I(\varphi, \psi)$. [See (2.5) in the next section as well as some explanation in Remark 2.3.] Especially, the proof of our main result for the case of tangential drifts given in Section 3 illustrates the usefulness of this consideration. It is shown that the coupled process (diffusion, occupation time of the diffusion) is a continuous mapping of the coupled process (Wiener process, occupation time of the Wiener process). Then the contraction principle can be applied to obtain the l.d.p.

Our approach is motivated by [7] and [12] and is based on the Cameron–Martin–Girsanov theorem with techniques such as the local time, the Skorohod equation and ergodic properties of diffusion processes. In a forthcoming paper, we shall study the l.d.p. for the stochastic differential equations with nonsmooth diffusion coefficients by extending our present techniques. However, it seems necessary to include a third component, the local time, in (1.3) for such processes.

The organization of the paper is as follows. In Section 2, we shall give the necessary definitions and exact statement of the main theorem. Section 3 will be devoted to the l.d.p. in the case of (1.1) with tangential drift. In Section 4, we prove some estimates for the local time of the 1-dim Wiener process which is essential to the proof of our main theorem. The proof of our main theorem for the general cases will be given in Section 5. In Section 6, we collect lemmas and proofs concerning functions on [0,1] used in this paper and, in particular, we prove the lower semicontinuity of the rate function for l.d.p.

2. Definitions and statement of the main theorem. Let b^+ and b^- be bounded vector fields in R^d with bounded derivatives up to the second order. For the system of stochastic differential equations in R^d,

$$\begin{aligned} dX^\varepsilon(t) &= b(X^\varepsilon(t))\,dt + \varepsilon dW(t), \qquad t \in [0,1], \\ X^\varepsilon(0) &= x^0 \in R^d, \end{aligned} \tag{2.1}$$

where

$$b(x) = \begin{cases} b^+(x), & \text{if } x_1 > 0, \\ b^-(x), & \text{if } x_1 \le 0, \end{cases} \tag{2.2}$$

$x = (x_1, \bar{x})$. Let $u^\varepsilon(t) = \int_0^t \chi_{(0,\infty)}(X_1^\varepsilon(s))\,ds$ be the occupation time of the solution $X^\varepsilon(\cdot)$ in the positive half space H^+, where

$$H^+ = \{x = (x_1, \bar{x}), x_1 > 0\}.$$

Such a process is unique because the time $X_1^\varepsilon(\cdot)$ spends at 0 has Lebesgue measure 0 with probability 1. The coupled process $(X^\varepsilon(\cdot), u^\varepsilon(\cdot))$ has sample paths in $C_{x_0}([0,1]\colon R^d) \times AC_0^+[0,1]$. [Here $C_{x_0}([0,1]\colon R^d)$ is the set of all continuous functions from $[0,1]$ to R^d starting from x^0, and $AC_0^+[0,1]$ is the set of all absolutely continuous functions on $[0,1]$ starting from 0 with derivatives between 0 and 1.] Obviously, $C_{x^0}([0,1]\colon R^d)$ and $AC_0^+[0,1]$ are complete metric spaces when equipped with the supnorm. In the sequel, we shall use the notation

$$\mathfrak{C} = C_{x^0}([0,1]\colon R^d) \times AC_0^+[0,1].$$

For any pair $(\varphi, \psi) \in \mathfrak{C}$ with $\psi \in H^+(\varphi)$ and a function f on R^d such that

$$f(x) = \begin{cases} f^+(x), & \text{if } x_1 > 0, \\ f^-(x), & \text{if } x_1 \le 0, \end{cases}$$

$x = (x_1, \bar{x})$, we define

$$f_{\varphi,\psi}(t) = f^+(\varphi(t))\dot{\psi}(t) + f^-(\varphi(t))(1 - \dot{\psi}(t)). \tag{2.3}$$

The main theorem of the paper can be stated as follows.

THEOREM 2.1. *Let $X^\varepsilon(t)$ be the solution of* (2.1) *and $X^\varepsilon = \{X^\varepsilon(t), 0 \le t \le 1\}$. Define $u^\varepsilon = \{u^\varepsilon(t), 0 \le t \le 1\}$ with $u^\varepsilon(t) = \int_0^t \chi_{(0,\infty)}(X_1^\varepsilon(s))\,ds$. Then the family of probability distributions on $\mathfrak{C}$ induced by the processes $(X^\varepsilon, u^\varepsilon)$, $\varepsilon > 0$, satisfy the large deviation principle with the following rate function $I(\cdot,\cdot)$: for $(\varphi, \psi) \in \mathfrak{C}$, φ absolutely continuous and $\psi \in H^+(\varphi)$,*

$$\begin{aligned} I(\varphi, \psi) = {} & \tfrac{1}{2} \int_{\varphi_1(t) \ne 0} |\dot{\varphi}(t) - b(\varphi(t))|^2\,dt \\ & + \tfrac{1}{2} \int_{\varphi_1(t)=0,\, b_1^-(\varphi(t)) \ge b_1^+(\varphi(t))} |\dot{\varphi}(t) - b_{\varphi,\psi}(t)|^2\,dt \\ & + \tfrac{1}{2} \int_{\varphi_1(t)=0,\, b_1^+(\varphi(t)) > b_1^-(\varphi(t))} (|\dot{\bar{\varphi}}(t) - \bar{b}_{\varphi,\psi}(t)|^2 + (b_1^2)_{\varphi,\psi}(t))\,dt. \end{aligned} \tag{2.4}$$

For all other pairs $(\varphi, \psi) \in \mathfrak{C}$, we set $I(\varphi, \psi) = \infty$.

Note that from (2.3), we have

$$\begin{aligned} (b_1^2)_{\varphi,\psi} &= b_1^{+2}(\varphi(t))\dot{\psi}(t) + b_1^{-2}(\varphi(t))(1 - \dot{\psi}(t)) \\ &= (b_{\varphi,\psi}(t))_1^2 + (b_1^+(\varphi(t)) - b_1^-(\varphi(t)))^2 \dot{\psi}(t)(1 - \dot{\psi}(t)). \end{aligned}$$

Therefore,

$$\begin{aligned} I(\varphi, \psi) = {} & \tfrac{1}{2} \int_0^1 |\dot{\varphi}(t) - b_{\varphi,\psi}(t)|^2\,dt \\ & + \tfrac{1}{2} \int_{\varphi_1(t)=0,\, b_1^-(\varphi(t)) < b_1^+(\varphi(t))} (b_1^+(\varphi(t)) - b_1^-(\varphi(t)))^2 \dot{\psi}(t)(1 - \dot{\psi}(t))\,dt. \end{aligned}$$

The second term is the main difference between stable and unstable regions and it reflects the difficulty for $X_1^\varepsilon(\cdot)$ to stay close to 0 in the unstable case.

Readers are referred to [8], [13], [21] for the definition and motivation of l.d.p. We shall, however, adopt the following slightly different formulation in the present set-up because it is more intuitive and fits our methodology.

In general, a family of probability measure $\{P^\varepsilon\}_{\varepsilon>0}$ on a metric space (X, ρ) is said to satisfy the l.d.p. with the rate function $I(\cdot)$ if the following conditions are satisfied:

(i) I: $X \to [0, \infty]$ is lower semicontinuous.
(ii) For each $r > 0$, $\{x \in X; I(x) \le r\}$ is precompact.
(iii) For any $R > 0$, there exists a compact set K such that for any $\delta > 0$, $P^\varepsilon(B_\delta(K)^c) \le \exp(-R/\varepsilon^2)$ if ε is small,
(iv) $\lim_{\delta\to 0} \liminf_{\varepsilon\to 0} \varepsilon^2 \log P^\varepsilon(B_\delta(x)) = \lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \varepsilon^2 \log P^\varepsilon(B_\delta(x)) = -I(x)$.

Here, $B_\delta(x)$ and $B_\delta(K)$ denote the δ-neighborhoods of x and K, respectively, and $B_\delta(x)^c$ and $B_\delta(K)^c$ are their complements. In terms of [8], (i)–(iv) imply that $\{P^\varepsilon\}$ satisfy the l.d.p. with a good rate function (see [7]). Here (i), (ii) and (iii) are easy to verify and most of the effort will be devoted to (iv).

Since the projection $(\varphi, \psi) \to \varphi$ from $\mathfrak{C} = C_{x^0}([0, 1]; R^d) \times AC_0^+[0, 1]$ into $C_{x^0}([0, 1]; R^d)$ is continuous, by the contraction principle [21], we have the following corollary.

COROLLARY 2.2. *Let X^ε be the solution process of (2.1) as defined in Theorem 2.1. Then the laws of X^ε on $C_{x^0}([0, 1]; R^d)$, $\varepsilon > 0$, satisfy the large deviation principle with the rate function $I(\cdot)$ as follows: for an absolutely continuous function $\varphi \in C_{x^0}([0, 1]; R^d)$,*

$$I(\varphi) = \int_0^1 L(\varphi(t), \dot{\varphi}(t))\, dt,$$

where

$$L(x, p) = \begin{cases} \frac{1}{2}|p - b(x)|^2, & \text{if } x_1 \neq 0, \\ \inf\limits_{0<\beta<1} \left\{\frac{1}{2}|p - (b^+(x)\beta + b^-(x)(1-\beta))|^2\right\}, & \\ & \text{if } x_1 = 0 \text{ and } b_1^-(x) \ge b_1^+(x), \\ \inf\limits_{0<\beta<1} \left\{\frac{1}{2}(|\bar{p} - (\bar{b}^+(x)\beta + \bar{b}^-(x)(1-\beta))|^2 \right. & \\ \qquad \left. + b_1^+(x)^2\beta + b_1^-(x)^2(1-\beta))\right\}, & \\ & \text{if } x_1 = 0 \text{ and } b_1^-(x) < b_1^+(x). \end{cases} \tag{2.5}$$

For all other $\varphi \in C_{x^0}([0, 1]; R^d)$, $I(\varphi) = \infty$.

Remark 2.3. Corollary 2.2 was also obtained in [6] where the rate function was expressed as follows. For absolutely continuous functions $\varphi \in C_{x^0}([0, 1]; R^d)$,

$$I(\varphi) = \int_0^1 \bar{L}(\varphi(t), \dot{\varphi}(t))\, dt,$$

where

$$(2.6)\qquad \bar{L}(x,p)=\begin{cases} L^+(x,p)=\frac{1}{2}|p-b^+(x)|^2, & \text{if } x_1>0,\\ L^-(x,p)=\frac{1}{2}|p-b^-(x)|^2, & \text{if } x_1<0,\\ L^0(x,p), & \text{if } x_1=0 \end{cases}$$

and

$$L^0(x,p)=\inf\{\beta L^+(x,p^+)+(1-\beta)L^-(x,p^-)\}$$

with inf taken over all possible β, p^+ and p^- satisfying $\beta p^+ + (1-\beta)p^- = p$, $p_1^+ < 0$ and $p_1^- > 0$. For all other $\varphi \in C_{x^0}([0,1];R^d)$, $I(\varphi)=\infty$. It is not difficult to see that (2.5) and (2.6) coincide.

Similarly, by (2.4),

$$I(\varphi,\psi)=\int_0^T L(\varphi(t),\dot\varphi(t),\dot\psi(t))dt,$$

where

$$L(x,p,\beta)=\begin{cases} \frac{1}{2}|p-b(x)|^2, & \text{if } x_1>0, \beta=1 (x_1<0, \beta=0),\\ \frac{1}{2}|p-(b^+(x)\beta+b^-(x)(1-\beta))|^2, & \text{if } x_1=0 \text{ and } b_1^-(x)\ge b_1^+(x),\\ \frac{1}{2}(|\bar p-(\bar b^+(x)\beta+\bar b^-(x)(1-\beta))|^2, +b_1^+(x)^2\beta+b_1^-(x)^2(1-\beta)), & \\ & \text{if } x_1=0 \text{ and } b_1^-(x)<b_1^+(x) \end{cases}$$

and is defined to be ∞ for all other (x,p,β). We can also show that

$$L(x,p,\beta)=\begin{cases} L^+(x,p)=\frac{1}{2}|p-b^+(x)|^2, & \text{if } x_1>0, \beta=1,\\ L^-(x,p)=\frac{1}{2}|p-b^-(x)|^2, & \text{if } x_1<0, \beta=0,\\ L^0(x,p,\beta), & \text{if } x_1=0 \end{cases}$$

and is equal to ∞ for all other (p,β), where

$$L^0(x,p,\beta)=\inf\{\beta L^+(x,p^+)+(1-\beta)L^-(x,p^-)\}$$

with inf taken over all possible p^+ and p^- satisfying $\beta p^+ + (1-\beta)p^- = p$, $p_1^+ < 0$ and $p_1^- > 0$.

3. The l.d.p. for the case with tangential drift. In this section we shall discuss the l.d.p. of $(X^\varepsilon, u^\varepsilon)$ defined in Section 2 for the case with tangential drift, that is,

$$b(x)=(0,\bar b(x)),\qquad x\in R^d,$$

with $\bar b(x)\in R^{d-1}$ and

$$(3.1)\qquad \bar b(x)=\begin{cases} \bar b^+(x), & \text{if } x_1>0,\\ \bar b^-(x), & \text{if } x_1\le 0, \end{cases}$$

$x = (x_1, \bar{x})$, where $\bar{b}^+(\cdot), \bar{b}^-(\cdot)$ are smooth functions defined on R^d. We shall first establish the l.d.p. for the Wiener process and then for the general cases of tangential drift by the contraction principle.

LEMMA 3.1. *Let $w(t)$ be the standard one-dimensional Wiener process. Define*

$$u_0^\varepsilon(t) = \int_0^T \chi_{(0,\infty)}(\varepsilon w(s))\, ds$$

be the occupation time of εw in $(0, \infty)$. Then the laws of $(\varepsilon w, u_0^\varepsilon)$ $(= \{(\varepsilon w(t), u_0^\varepsilon(t)); t \in [0,1]\})$, $\varepsilon > 0$, on $\mathfrak{C}$(with $d = 1$) satisfy the large deviation principle with the rate function $I_0(\cdot,\cdot)$ as follows. For an absolutely continuous function φ and $\psi \in H^+(\varphi)$,

$$I_0(\varphi, \psi) = \tfrac{1}{2} \int_0^1 |\dot{\varphi}(t)|^2\, dt$$

For all other pairs (φ, ψ), $I_0(\varphi, \psi) = \infty$.

PROOF. Let φ be an absolutely continuous function in $[0, 1]$ with finite $\int_0^1 |\dot{\varphi}(t)|^2\, dt$ and $\psi \in H^+(\varphi)$. We shall show that

$$\begin{aligned}
&\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \varepsilon^2 \log P\{\|\varepsilon w - \varphi\| \le \delta, \|u_0^\varepsilon - \psi\| \le \delta\} \\
&\quad = \lim_{\delta\to 0} \liminf_{\varepsilon\to 0} \varepsilon^2 \log P\{\|\varepsilon w - \varphi\| \le \delta, \|u_0^\varepsilon - \psi\| \le \delta\} \\
&\quad = -\tfrac{1}{2} \int_0^1 |\dot{\varphi}(t)|^2\, dt.
\end{aligned} \tag{3.2}$$

If φ is never 0 or φ equals 0 with Lebesgue measure 0, then the result follows from Schilder's l.d.p. [19].

By Lemma 6.4 we may consider without loss of generality the case where $x^0 = 0$, $\varphi \equiv 0$ and $\dot{\psi} \equiv \beta \in [0, 1]$. Since

$$P\{\|\varepsilon w - \varphi\| \le \delta, \|u_0^\varepsilon - \psi\| \le \delta\} \le P\{\|\varepsilon w - \varphi\| \le \delta\}$$

for any ψ, hence

$$\begin{aligned}
&\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \varepsilon^2 \log P\{\|\varepsilon w\| \le \delta, \|u_0^\varepsilon - \psi\| \le \delta\} \\
&\quad \le \lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \varepsilon^2 \log P\{\|\varepsilon w\| \le \delta\} = 0.
\end{aligned}$$

On the other hand, if $\beta = 1$, let the function f be linear between $[0, \delta/2)$ with slope 1 and constant in $[\delta/2, 1]$. Then

$$P\{\|\varepsilon w\| \le \delta, \|u_0^\varepsilon - \psi\| \le \delta\} \ge P\left\{\|\varepsilon w - f\| \le \frac{\delta}{2}\right\}$$

and

$$\begin{aligned}
&\lim_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon^2\log P\{\|\varepsilon w\|\le\delta,\ \|u_0^\varepsilon-\psi\|\le\delta\}\\
&\quad\ge\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\varepsilon^2\log P\left\{\|\varepsilon w-f\|\le\frac{\delta}{2}\right\}\\
&\quad=\lim_{\delta\to 0}\left(-\frac{1}{2}\int_0^1\dot f^2(t)\,dt\right)=\lim_{\delta\to 0}\left(-\frac{\delta}{4}\right)=0.
\end{aligned}$$

Therefore, we have proved (3.2) for this case. Similarly for $\beta=0$.

For $\beta\in(0,1)$ and any δ, we shall construct a continuous piecewise linear function f such that $\|f\|<\delta/2, m(t\colon f(t)=0)=0, \int_0^1\dot f^2(t)\,dt<\delta$ and $\|g-\psi\|<\delta/2$ for the occupation time g of f in $(0,\infty)$ [occupation time for f is unique since $m(t\colon f(t)=0)=0$]. It then follows from Lemma 6.5 that there exists an $\gamma<\delta/2$ such that $\|v-g\|<\delta/2$ if $\|h-f\|<\gamma$ and v is an occupation time of h in $(0,\infty)$. Thus

$$P\{\|\varepsilon w\|\le\delta,\ \|u_0^\varepsilon-\psi\|\le\delta\}\ge P\{\|\varepsilon w-f\|\le\gamma\}$$

and

$$\begin{aligned}
&\lim_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon^2\log P\{\|\varepsilon w\|\le\delta,\ \|u_0^\varepsilon-\psi\|\le\delta\}\\
&\quad\ge\lim_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon^2\log P\left\{\|\varepsilon w-f\|\le\frac{\delta}{2}\right\}\\
&\quad=\lim_{\delta\to 0}\left(-\frac{1}{2}\int_0^1\dot f^2(t)\,dt\right)=\lim_{\delta\to 0}\left(-\frac{\delta}{2}\right)=0.
\end{aligned}$$

Therefore, we have (3.2) in this case. We shall construct f as follows. Here g denotes an occupation time of f in $(0,\infty)$.

Let $a_0=0, a_1, a_2,\dots,$ be the increasing sequence in $[0,1]$ such that $a_1=\delta/2(1-\beta), a_2=a_1+\delta/\beta, a_3=a_2+\delta/(1-\beta), a_4=a_3+\delta/\beta$ and so on. In $[0,a_1]$, f is defined to be linear in $[0,\gamma]$ and $[a_1-\gamma,a_1]$ with slope $+1$ and -1, respectively. In $[\gamma,a_1-\gamma], f$ is defined to be the constant γ. Hence in $[0,a_1], 0\le f(t)\le\gamma$ and $0\le g(t)-\psi(t)\le\delta/2$. In $[a_1,a_2]$, f is defined to be linear in $[a_1,a_1+\gamma]$ and $[a_2-\gamma,a_2]$ with slope -1 and $+1$, respectively. In $[a_1+\gamma,a_2-\gamma]$, f is defined to be the constant $-\gamma$. Obviously in $[a_1,a_2], -\gamma\le f\le 0$ and $|g(t)-\psi(t)|\le\delta/2$. The construction now repeats and it is easy to see that $\|f\|<\gamma$ and $\|g-\psi\|\le\delta/2$ for any g. Since $\int_{a_{2k}}^{a_{2k+2}}\dot f^2(t)dt=4\gamma$ in each $[a_{2k},a_{2k+2}]$, thus

$$\int_0^1\dot f^2(t)\,dt\le 4\gamma\frac{1}{\delta/2(1-\beta)+\delta/2\beta}=\frac{8\gamma\beta(\beta-1)}{\delta}.$$

If we take $\gamma<\delta^2/8\beta(\beta-1)$, then $\int_0^1\dot f^2(t)\,dt<\delta$. Such an f thus satisfies all the conditions necessary. □

It is then trivial to have the following corollary.

COROLLARY 3.2. *Let $W(t)$ be the standard d-dimensional Wiener process. Then the laws of εW and the occupation time of εW in H^+, $\varepsilon > 0$, satisfy the large deviation principle with the rate function $I_0(\cdot,\cdot)$. For an absolutely continuous R^d-valued function φ and $\psi \in H^+(\varphi)$,*

$$I_0(\varphi,\psi) = \tfrac{1}{2}\int_0^1 |\dot\varphi(t)|^2\,dt.$$

For all other (φ,ψ), $I_0(\varphi,\psi) = \infty$.

Let b^+ and b^- be bounded smooth vector fields with $b_1^+ = b_1^- = 0$ and let $b(\cdot)$ be defined as in (3.1). We next consider an integral equation which is similar to that in [13] page 103. Let F be a function from $\mathfrak{C}$ to $\mathfrak{C}$ defined as the following:

$$\begin{aligned} &F(z,u) = (x,u) \\ &\text{if } x(t) = \int_0^t (b^+(x(s))\dot u(s) + b^-(x(s))(1-\dot u(s)))\,ds + z(t). \end{aligned} \tag{3.3}$$

LEMMA 3.3. *F is well defined and continuous.*

PROOF. The proof is exactly the same as that of a similar equation in [13] page 104.

Let $X^\varepsilon(t)$ be the diffusion process satisfying

$$\begin{aligned} dX^\varepsilon(t) &= b(X^\varepsilon(t))\,dt + \varepsilon dW(t), \\ X^\varepsilon(0) &= x^0, \end{aligned} \tag{3.4}$$

where $b(x)$ is as in (3.1). We denote $u^\varepsilon(t)$ the occupation time of $X^\varepsilon(t)$ in H^+; $(X^\varepsilon, u^\varepsilon) = \{(X^\varepsilon(t), u^\varepsilon(t)), 0 \le t \le 1\}$. The next is our main result in this section which asserts that the l.d.p. holds for the laws of $(X^\varepsilon, u^\varepsilon)$, $\varepsilon > 0$. Its proof follows from Corollary 3.2, Lemma 3.3 and the contraction principle. We recall (2.3) for the definition of $b_{\varphi,\psi}(t)$. $\square$

THEOREM 3.4. *Let $(X^\varepsilon, u^\varepsilon)$ be the process defined above. Then the laws of $(X^\varepsilon, u^\varepsilon)$ on $\mathfrak{C}$, $\varepsilon > 0$, satisfy the large deviation principle with the rate function $I(\cdot,\cdot)$. For the absolutely continuous R^d-valued function φ and $\psi \in H^+(\varphi)$,*

$$\begin{aligned} I(\varphi,\psi) &= \tfrac{1}{2}\int_0^1 |\dot\varphi(t) - b_{\varphi,\psi}(t)|^2\,dt \\ &= \tfrac{1}{2}\int_0^1 \dot\varphi_1^2(t)dt + \tfrac{1}{2}\int_0^1 |\dot{\bar\varphi}(t) - \bar b_{\varphi,\psi}(t)|^2\,dt. \end{aligned}$$

For all other (φ,ψ), $I(\varphi,\psi) = \infty$.

PROOF. Let F be the function (3.3). Let $W(t)$ be the d-dimensional Wiener process and u_0^ε be the occupation time of εW in H^+. Then we have

$$(X^\varepsilon(t), u^\varepsilon(t)) = F(\varepsilon W, u_0^\varepsilon)(t).$$

By the contraction principle and Corollary 3.2, the laws of $(X^{\varepsilon}, u^{\varepsilon})$ on $\mathfrak{C}$ satisfy the l.d.p. with the rate function defined by

$$I(\varphi, \psi) = I(F^{-1}(\varphi, \psi)) = \tfrac{1}{2}\int_0^1 |\dot{\varphi}(t) - b_{\varphi,\psi}(t)|^2\, dt.$$

This completes the proof. □

4. Some estimates for the local time of the one-dimensional Wiener process. We shall give the proof of Theorem 2.1 for the general case in the next section. To prepare for this purpose, we shall establish some estimates for the local time of the 1-dim Wiener process in this section.

In the following, the local time $\ell(t)$ for a continuous semimartingale $m(t)$ at 0 is defined as the increasing process such that

$$|m(t)| = |m(0)| + \int_0^t \operatorname{sgn}(m(s))\, dm(s) + \ell(t).$$

See [22].

Let w be the standard one-dimensional Wiener process starting from 0 and u^{ε} be the process of occupation time of εw in $(0, \infty)$. Denote by ℓ^{ε} the local time of $\varepsilon w(\cdot)$ at 0 up to time t. Then

$$\ell^{\varepsilon}(t) = \varepsilon^2 \lim_{\delta\to 0} \frac{1}{\delta}\int_0^t \chi_{\{|\varepsilon w(s)|\le\delta\}}\, ds.$$

See [14]. See also [12] for some calculation concerning local time of a Markov diffusion process. We have $\ell^{\varepsilon}(t) = \varepsilon\ell_0(t)$, $\ell_0(t)$ is the local time of $w(\cdot)$ at 0.

Lemma 4.1. *Let w and ℓ^{ε} be defined as above. Then for any $r > 0$,*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log E\left\{\exp\left(r\frac{\ell^{\varepsilon}(1)}{\varepsilon^2}\right)\right\} = \frac{r^2}{2}.$$

In particular,

$$\lim_{r\to 0}\lim_{\varepsilon\to 0} \varepsilon^2 \log E\left\{\exp\left(r\frac{\ell^{\varepsilon}(1)}{\varepsilon^2}\right)\right\} = 0.$$

Proof. Since

$$|\varepsilon w(t)| = \varepsilon\int_0^t \operatorname{sgn} w(s)\, dw(s) + \ell^{\varepsilon}(1) = \varepsilon\widetilde{w}(t) + \ell^{\varepsilon}(1),$$

where $\widetilde{w}(t)$ is a standard Brownian motion, we have $\ell^\varepsilon(1) = -\inf_{t\le 1}\varepsilon\widetilde{w}(t)$ by Skorohod representation [14]. Thus

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon^2\log E\left\{\exp\left(r\frac{\ell^\varepsilon(1)}{\varepsilon^2}\right)\right\}\\
&\quad=\lim_{\varepsilon\to 0}\varepsilon^2\log E\left\{\exp\left(-\frac{r}{\varepsilon^2}\cdot\inf_{t\le 1}\varepsilon\widetilde{w}(t)\right)\right\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Laplace–Varadhan's method)}\\
&\quad=\sup_{\varphi}\left(-\inf_{t\le 1} r\varphi(t)-\frac{1}{2}\int_0^1\dot\varphi^2(t)\,dt\right)\\
&\quad=\sup_{\varphi}\left(r\int_0^1\dot\varphi(t)\,dt-\frac{1}{2}\int_0^1\dot\varphi^2(t)\,dt\right)\\
&\quad=\sup_{\varphi}\left(r\left(\int_0^1\dot\varphi^2(t)dt\right)^{\frac{1}{2}}-\frac{1}{2}\int_0^1\dot\varphi^2(t)dt\right)=\frac{r^2}{2}.
\end{aligned}$$

□

LEMMA 4.2. *Let w, u^ε and ℓ^ε be deined as above, ψ be a nonegative, absolutely continuous function such that $\dot\psi(t)\in[0,1]$. Then for any $r>0$,*

$$\lim_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon^2\log E\left\{\exp\left(-r\frac{\ell^\varepsilon(1)}{\varepsilon^2}\right);\|\varepsilon w\|\le\delta,\ \|u^\varepsilon-\psi\|\le\delta\right\}=0.$$

PROOF. By Lemma 6.4, we can without loss of generality assume that $\dot\psi(t)=\beta$ throughout $[0,1]$ and $0<\beta<1$. Let c be a fixed positive constant and let $x^\varepsilon(t)$ be the solution of the following stochastic differential equation:

$$\begin{aligned}
dx_0^\varepsilon(t)&=v(x^\varepsilon(t))\,dt+\varepsilon dw_0(t),\\
x^\varepsilon(0)&=0,
\end{aligned}$$

where

$$v(x)=\begin{cases}-\dfrac{c}{\beta}, & x>0,\\[2mm] \dfrac{c}{1-\beta}, & x\le 0.\end{cases}$$

Let $V(x)=\int_0^x v(y)\,dy$. By Lemma 6.7 for any $\theta>0$ there is δ so small that

$$|V(x(1))|+\left|\int_0^1 v_{x,u}^2(s)\,ds-\int_0^1 v_{0,\psi}^2(s)\,ds\right|<\theta,$$

when $x(\cdot)\in B_\delta(0)$ and $u(\cdot)\in H^+(x(\cdot))$ satisfying $u(\cdot)\in B_\delta(\psi)$. Let $\tilde u^\varepsilon$ be the occupation time of x^ε in $(0,\infty)$ and $\tilde\ell^\varepsilon(t)$ be the local time of x^ε at 0. By the

Cameron–Martin–Girsanov theorem,

$$\begin{aligned}
&E\biggl\{\exp\biggl(-r\frac{\ell^\varepsilon(1)}{\varepsilon^2}\biggr); \|\varepsilon w\|\le\delta,\ \|u^\varepsilon-\psi\|\le\delta\biggr\}\\
&\quad= E\biggl\{\exp\biggl(-r\frac{\tilde\ell^\varepsilon(1)}{\varepsilon^2}\biggr)\\
&\qquad\qquad\times\exp\biggl(\int_0^1\frac{-v(x^\varepsilon(t))}{\varepsilon^2}\,dx^\varepsilon(t)+\frac{1}{2\varepsilon^2}\int_0^1 v^2(x^\varepsilon(t))\,dt\biggr);\\
&\qquad\qquad\qquad\qquad\qquad\|x^\varepsilon\|\le\delta,\ \|\tilde u^\varepsilon-\psi\|\le\delta\biggr\}
\end{aligned}$$

by Tanaka's formula

$$\begin{aligned}
&= E\biggl\{\exp\biggl(-r\frac{\tilde\ell^\varepsilon(1)}{\varepsilon^2}\biggr)\exp\biggl(-\frac{1}{\varepsilon^2}V(x^\varepsilon(1))-\frac12\biggl(\frac{c}{\beta}+\frac{c}{1-\beta}\biggr)\frac{\tilde\ell^\varepsilon(1)}{\varepsilon^2}\\
&\qquad\qquad+\frac{1}{2\varepsilon^2}\int_0^1 v^2(x^\varepsilon(t))\,dt\biggr); \|x^\varepsilon\|\le\delta,\ \|\tilde u^\varepsilon-\psi\|\le\delta\biggr\}
\end{aligned}$$

(4.1)

$$\begin{aligned}
&= E\biggl\{\exp\biggl(-\biggl(r+\frac{c}{2\beta(1-\beta)}\biggr)\frac{\tilde\ell^\varepsilon(1)}{\varepsilon^2}\biggr)+\frac{1}{2\varepsilon^2}\int_0^1 v^2(x^\varepsilon(t))\,dt;\\
&\qquad\qquad\qquad\qquad\|x^\varepsilon\|\le\delta,\ \|\tilde u^\varepsilon-\psi\|\le\delta\biggr\}\exp\biggl(\frac{-\theta}{\varepsilon^2}\biggr)\\
&= E\biggl\{\exp\biggl(-\biggl(r+\frac{c}{2\beta(1-\beta)}\biggr)\frac{\tilde\ell^\varepsilon(1)}{\varepsilon^2}\biggr); \|x^\varepsilon\|\le\delta,\ \|\tilde u^\varepsilon-\psi\|\le\delta\biggr\}\\
&\quad\times\exp\biggl(\frac{-\theta}{\varepsilon^2}+\frac{c^2}{2\varepsilon^2\beta(1-\beta)}\biggr)\\
&= E\biggl\{\exp\biggl(-r'\frac{\tilde\ell^\varepsilon(1)}{\varepsilon^2}\biggr); \|x^\varepsilon\|\le\delta,\ \|\tilde u^\varepsilon-\psi\|\le\delta\biggr\}\\
&\quad\times\exp\biggl(\frac{-\theta}{\varepsilon^2}+\frac{c^2}{2\varepsilon^2\beta(1-\beta)}\biggr),
\end{aligned}$$

where $r'=r+(c/2\beta(1-\beta))$.

We now rescale $x^\varepsilon(t)$ as follows: let $y(t)=(1/\varepsilon^2)x^\varepsilon(\varepsilon^2t)$. Then

$$\begin{aligned}
y(t)&=\frac{1}{\varepsilon^2}\int_0^{\varepsilon^2t}v(x^\varepsilon(s))\,ds+\frac{\varepsilon w(\varepsilon^2t)}{\varepsilon^2}\\
&=\int_0^t v(x^\varepsilon(\varepsilon^2s))\,ds+\widetilde w(t)\biggl(\widetilde w(t)=\frac{1}{\varepsilon}w(\varepsilon^2t)\text{ is a Brownian motion}\biggr)\\
&=\int_0^t v(y(s))\,ds+\widetilde w(t).
\end{aligned}$$

Thus $y(t)$ satisfies

$$\begin{aligned}
dy(t)&=v(y(t))\,dt+d\widetilde w(t),\\
y(0)&=0
\end{aligned}$$

and is ergodic (see [18], page 219, Corollary 1.11) with the stationary distribution $\mu(dx) = f(x)dx$, where

$$f(x) = \begin{cases} 2c\exp\left(\dfrac{-2c}{\beta}x\right), & x > 0, \\ 2c\exp\left(\dfrac{2c}{1-\beta}x\right), & x \le 0. \end{cases}$$

Since

$$\tilde{u}^\varepsilon(t) = \int_0^t \chi_{(0,\infty)}(x^\varepsilon(s))\,ds = \varepsilon^2 \int_0^{\varepsilon^{-2}t} \chi_{(0,\infty)}(y(s))\,ds$$

and

$$\lim_{\varepsilon\to 0} \frac{\tilde{u}^\varepsilon(t)}{t} = \lim_{\varepsilon\to 0} \frac{\varepsilon^2}{t} \int_0^{t/\varepsilon^2} \chi_{(0,\infty)}(y(s))\,ds = \mu((0,\infty)) = \beta$$

uniformly over any compact subinterval of $(0, 1]$ by the ergodic theorem, we conclude that $\|\tilde{u}^\varepsilon - \psi\| < \delta$ when ε is small. Hence

$$\lim_{\varepsilon\to 0} P\{\|x^\varepsilon\| \le \delta, \|\tilde{u}^\varepsilon - \psi\| \le \delta\} = 1. \tag{4.2}$$

Since

$$\begin{aligned} |x^\varepsilon(t)| &= \int_0^t \operatorname{sgn} x^\varepsilon(s)\,dx^\varepsilon(s) + \tilde{\ell}^\varepsilon(t) \\ &= \int_0^t \operatorname{sgn} x^\varepsilon(s) v(x^\varepsilon(s))\,ds + \int_0^t \varepsilon \operatorname{sgn} x^\varepsilon(s)\,d\tilde{w}(s) + \tilde{\ell}^\varepsilon(t), \end{aligned}$$

by the Skorohod representation,

$$\tilde{\ell}^\varepsilon(1) = -\inf_{t\le 1}\left(\int_0^t \operatorname{sgn} x^\varepsilon(s) v(x^\varepsilon(s))\,ds + \varepsilon\hat{w}(t)\right).$$

Here

$$\hat{w}(t) = \int_0^t \operatorname{sgn} x^\varepsilon(s)\,d\tilde{w}(s)$$

is a one-dimensional Brownian motion. Since

$$\begin{aligned} -\tilde{\ell}^\varepsilon(1) &= \inf_{t\le 1}\left(\int_0^t \operatorname{sgn} x^\varepsilon(s) v(x^\varepsilon(s))\,ds + \varepsilon\hat{w}(t)\right) \\ &\ge \int_0^1 \operatorname{sgn} x^\varepsilon(s) v(x^\varepsilon(s))\,ds + \inf_{t\le 1} \varepsilon\hat{w}(t) \\ &\ge -\left(2c + \frac{c\delta}{\beta(1-\beta)}\right) + \inf_{t\le 1} \varepsilon\hat{w}(t) \end{aligned}$$

if $\|\tilde{u}^\varepsilon - \psi\| \le \delta$, then

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon^2 \log E\left\{\exp\left(-r'\frac{\tilde{\ell}^\varepsilon(1)}{\varepsilon^2}\right); \|x^\varepsilon\| \le \delta, \|\tilde{u}^\varepsilon - \psi\| \le \delta\right\} \\
&\quad \ge \lim_{\varepsilon\to 0} \varepsilon^2 \log E\left\{\exp\left(\frac{1}{\varepsilon^2}\left(-r'\left(2c + \frac{c\delta}{\beta(1-\beta)}\right) + r' \inf_{t\le 1} \varepsilon\hat{w}(t)\right)\right);\right. \\
&\qquad\qquad \left. \|x^\varepsilon\| \le \delta, \|\tilde{u}^\varepsilon - \psi\| \le \delta\right\} \\
&\quad \ge -r'\left(2c + \frac{c\delta}{\beta(1-\beta)}\right) + \lim_{\varepsilon\to 0} \varepsilon^2 \log E\left\{\exp\left(\frac{r'}{\varepsilon^2} \inf_{t\le 1} \varepsilon\hat{w}(t)\right); \|\varepsilon\hat{w}\| \le \delta,\right. \\
&\qquad\qquad \left. \|x^\varepsilon\| \le \delta, \|\tilde{u}^\varepsilon - \psi\| \le \delta\right\} \\
&\quad \ge -r'\left(2c + \frac{c\delta}{\beta(1-\beta)}\right) \\
&\qquad + \lim_{\varepsilon\to 0} \varepsilon^2 \log\left(\exp\left(\frac{-r'\delta}{\varepsilon^2}\right) P\left\{\|\varepsilon\hat{w}\| \le \delta, \|x^\varepsilon\| \le \delta, \|\tilde{u}^\varepsilon - \psi\| \le \delta\right\}\right) \\
&\quad = -r'\left(2c + \frac{c\delta}{\beta(1-\beta)}\right) - r'\delta.
\end{aligned}$$

The proof is now complete by letting c and δ [hence θ in (4.1)] approach 0. $\square$

With similar techniques, we have the following refinement of Lemma 4.2.

LEMMA 4.3. *Let $w(t)$ be the one-dimensional Wiener process, ℓ^ε be the local time of εw at 0 and $\psi(t) = \beta t, 0 \le \beta \le 1$. Then for any $r > 0$,*

$$\lim_{\delta\to 0} \liminf_{\varepsilon\to 0} \varepsilon^2 \log E\left\{\exp\left(r\frac{\ell^\varepsilon(1)}{\varepsilon^2}\right); \|\varepsilon w\| \le \delta, \|u^\varepsilon - \psi\| \le \delta\right\} = 2\beta(1-\beta)r^2$$

and

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \varepsilon^2 \log E\left\{\exp\left(r\frac{\ell^\varepsilon(1)}{\varepsilon^2}\right); \|\varepsilon w\| \le \delta, \|u^\varepsilon - \psi\| \le \delta\right\} = 2\beta(1-\beta)r^2.$$

PROOF. We may assume that $0 < \beta < 1$. Following the proof of Lemma 4.2, we have from (4.1) that for any $\theta > 0$,

$$\begin{aligned}
&E\left\{\exp\left(r\frac{\ell^\varepsilon(1)}{\varepsilon^2}\right); \|\varepsilon w\| \le \delta, \|u^\varepsilon - \psi\| \le \delta\right\} \\
&\qquad \ge E\left\{\exp\left(\left(r - \frac{c}{2\beta(1-\beta)}\right)\frac{\tilde{\ell}^\varepsilon(1)}{\varepsilon^2}\right); \|x^\varepsilon\| \le \delta, \|\tilde{u}^\varepsilon - \psi\| \le \delta\right\} \\
&\qquad\quad \times \exp\left(\frac{-\theta}{\varepsilon^2} + \frac{c^2}{2\varepsilon^2\beta(1-\beta)}\right)
\end{aligned}$$

and

$$
\begin{aligned}
&E\left\{\exp\left(r\frac{\ell^\varepsilon(1)}{\varepsilon^2}\right); \|\varepsilon w\| \le \delta,\, \|u^\varepsilon - \psi\| \le \delta\right\} \\
&\quad \le E\left\{\exp\left(\left(r - \frac{c}{2\beta(1-\beta)}\right)\frac{\tilde{\ell}^\varepsilon(1)}{\varepsilon^2}\right); \|x^\varepsilon\| \le \delta,\, \|\tilde{u}^\varepsilon - \psi\| \le \delta\right\} \\
&\qquad \times \exp\left(\frac{\theta}{\varepsilon^2} + \frac{c^2}{2\varepsilon^2\beta(1-\beta)}\right)
\end{aligned}
$$

when δ is small. If we choose $c > 0$ so that $r - (c/2\beta(1-\beta)) = 0$, then

$$
\begin{aligned}
&\exp\left(\frac{\theta}{\varepsilon^2} + \frac{2\beta(1-\beta)r^2}{\varepsilon^2}\right) P\{\|x^\varepsilon\| \le \delta,\, \|\tilde{u}^\varepsilon - \psi\| \le \delta\} \\
&\quad \ge E\left\{\exp\left(r\frac{\ell^\varepsilon(1)}{\varepsilon^2}\right); \|\varepsilon w\| \le \delta,\, \|u^\varepsilon - \psi\| \le \delta\right\} \\
&\quad \ge \exp\left(-\frac{\theta}{\varepsilon^2} + \frac{2\beta(1-\beta)r^2}{\varepsilon^2}\right) P\{\|x^\varepsilon\| \le \delta,\, \|\tilde{u}^\varepsilon - \psi\| \le \delta\}.
\end{aligned}
$$

Using (4.2), the lemma follows by letting $\delta \to 0$ (hence $\theta \to 0$). □

REMARK 4.4. If ψ is an absolutely continuous function satisfying $\dot{\psi}(t) \in [0, 1]$ and $\psi(0) = 0$, we can prove, using Lemma 4.3, approximating ψ by a piecewise linear function and using an argument involving conditioning, that

$$
\begin{aligned}
&\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} \varepsilon^2 \log E\left\{\exp\left(r\frac{\ell^\varepsilon(1)}{\varepsilon^2}\right); \|\varepsilon w\| \le \delta,\, \|u^\varepsilon - \psi\| \le \delta\right\} \\
&\quad = 2r^2 \int_0^1 \dot{\psi}(t)(1 - \dot{\psi}(t))\, dt
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{\delta\to 0}\liminf_{\varepsilon\to 0} \varepsilon^2 \log E\left\{\exp\left(r\frac{\ell^\varepsilon(1)}{\varepsilon^2}\right); \|\varepsilon w\| \le \delta,\, \|u^\varepsilon - \psi\| \le \delta\right\} \\
&\quad = 2r^2 \int_0^1 \dot{\psi}(t)(1 - \dot{\psi}(t))\, dt.
\end{aligned}
$$

More generally, we have the following result. The detail of its proof is omitted.

LEMMA 4.5. *Let $c(t)$, $t \in [0, 1]$ be a continuous real-valued function. Then*

$$
\begin{aligned}
&\lim_{\delta\to 0}\limsup_{\varepsilon\to 0} \varepsilon^2 \log E\left\{\exp\left(\frac{1}{\varepsilon^2}\int_0^1 c(t) d\ell^\varepsilon(t)\right); \|\varepsilon w\| \le \delta,\, \|u^\varepsilon - \psi\| \le \delta\right\} \\
&\quad = 2\int_0^1 c^+(t)^2 \dot{\psi}(t)(1 - \dot{\psi}(t))\, dt
\end{aligned}
$$

and

$$\lim_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon^2\log E\left\{\exp\left(\frac{1}{\varepsilon^2}\int_0^1 c(t)\,d\ell^\varepsilon(t)\right);\|\varepsilon w\|\le\delta,\|u^\varepsilon-\psi\|\le\delta\right\}$$
$$=2\int_0^1 c^+(t)\dot\psi(t)(1-\dot\psi(t))\,dt,$$

where $c^+(t)=c(t)$ *if* $c(t)\ge 0$ *and* $c^+(t)=0$ *otherwise.*

5. The l.d.p. for the general case. In this section, we shall prove the l.d.p. for the system (2.1) for the general case. Throughout, (φ,ψ) will be a pair in $\mathfrak{C}$ with $\psi\in H^+(\varphi)$(see Section 1 for the definition). Let $X^\varepsilon(\cdot)$ be the solution of (2.1) and $u^\varepsilon(\cdot)$ be the occupation time of X^ε in H^+ defined in (1.4). We shall consider

$$P(\|X^\varepsilon-\varphi\|<\delta,\|u^\varepsilon-\psi\|<\delta).$$

By the Cameron–Martin–Girsanov theorem,

$$\begin{aligned}&P(\|X^\varepsilon-\varphi\|<\delta,\|u^\varepsilon-\psi\|<\delta)\\&=E\left\{\exp\left(\int_0^1 c(\widetilde X^\varepsilon(t))d\widetilde X^\varepsilon(t)-\int_0^1\widetilde b(\widetilde X^\varepsilon(t))c(\widetilde X^\varepsilon(t))\,dt\right.\right.\\&\qquad\left.\left.-\tfrac12\int_0^1\varepsilon^2|c(\widetilde X^\varepsilon(t))|^2\,dt\right);(\widetilde X^\varepsilon,\tilde u^\varepsilon)\in B_\delta(\varphi,\psi)\right\},\end{aligned}$$

where

$$\begin{aligned}d\widetilde X^\varepsilon(t)&=\widetilde b(\widetilde X^\varepsilon(t))\,dt+\varepsilon d\widetilde W(t),\\ \widetilde X^\varepsilon(0)&=x^0\end{aligned}\tag{5.1}$$

and $\tilde u^\varepsilon$ is the occupation time of $\tilde X^\varepsilon$. Here $\widetilde W$ is a d-dimensional Wiener process, $\widetilde b$ is any bounded vector field and $c(x)$ satisfies $b(x)=\widetilde b(x)+\varepsilon^2c(x)$. To make Ito's formula applicable, we choose

$$\widetilde b(x)=\left(0,b_2(x)-\frac{\partial}{\partial x_2}F(x),\dots,b_d(x)-\frac{\partial}{\partial x_d}F(x)\right),\tag{5.2}$$

where $F(x)=\int_0^{x_1}b_1(t,\bar x)\,dt$ for $x=(x_1,\bar x)$, that is,

$$F(x)=\int_0^{x_1}b_1^+(t,\bar x)\,dt\text{ if }x_1>0\quad\text{and}\quad F(x)=\int_0^{x_1}b_1^-(t,\bar x)\,dt\qquad\text{if }x_1\le 0.$$

Then

$$c(x)=\frac{1}{\varepsilon^2}(b(x)-\widetilde b(x))=\frac{1}{\varepsilon^2}\left(b_1(x),\frac{\partial}{\partial x_2}F(x),\dots,\frac{\partial}{\partial x_d}F(x)\right)=\frac{1}{\varepsilon^2}\nabla F(x)$$

and

$$\begin{aligned}
&P\{(X^\varepsilon, u^\varepsilon) \in B_\delta(\varphi, \psi)\}\\
&= E\biggl\{\exp\biggl(\int_0^1 \frac{1}{\varepsilon^2}\nabla F(\widetilde{X}^\varepsilon(t))d\widetilde{X}^\varepsilon(t) - \frac{1}{\varepsilon^2}\int_0^1 (\tilde{b}\nabla F)(\widetilde{X}^\varepsilon(t))\,dt\\
&\qquad\qquad - \frac{1}{2\varepsilon^2}\int_0^1 |\nabla F(\widetilde{X}^\varepsilon(t))|^2\,dt\biggr);\\
&\qquad\qquad (\widetilde{X}^\varepsilon, \tilde{u}^\varepsilon) \in B_\delta(\varphi, \psi)\biggr\}
\end{aligned}$$

Since by Tanaka's formula,

$$\begin{aligned}
F(\widetilde{X}^\varepsilon(t)) &= F(\widetilde{X}^\varepsilon(0)) + \int_0^t (\nabla F)(\widetilde{X}^\varepsilon(s))d\widetilde{X}^\varepsilon(s)\\
&\quad + \frac{\varepsilon^2}{2}\int_0^t \sum_{i,j}\frac{\partial^2 F}{\partial x_i \partial x_j}(\widetilde{X}^\varepsilon(s))\,ds\\
&\quad + \frac{1}{2}\int_0^t (b_1^+(\widetilde{X}^\varepsilon(s)) - b_1^-(\widetilde{X}^\varepsilon(s)))\,d\tilde{\ell}^\varepsilon(s),
\end{aligned}$$

where $\tilde{\ell}^\varepsilon(t)$ is the local time of $\widetilde{X}_1^\varepsilon(\cdot)$ at 0 up to time t, we have

$$\begin{aligned}
&P\{(X^\varepsilon, u^\varepsilon) \in B_\delta(\varphi, \psi)\}\\
&= E\biggl\{\exp\biggl(\frac{F(\widetilde{X}^\varepsilon(1)) - F(\widetilde{X}^\varepsilon(0))}{\varepsilon^2}\\
&\qquad - \frac{1}{2\varepsilon^2}\int_0^1 (b_1^+(\widetilde{X}^\varepsilon(t)) - b_1^-(\widetilde{X}^\varepsilon(t))\,d\tilde{\ell}^\varepsilon(t)\\
&\qquad - \frac{1}{2}\int_0^1 \sum_{i,j}\frac{\partial^2 F}{\partial x_i \partial x_j}(\widetilde{X}^\varepsilon(t))\,dt - \frac{1}{\varepsilon^2}\int_0^1 (\tilde{b}\nabla F)(\widetilde{X}^\varepsilon(t))\,dt\\
&\qquad + \frac{1}{2\varepsilon^2}\int_0^1 |\nabla F|^2(\widetilde{X}^\varepsilon(t))\,dt\biggr);\\
&\qquad (\widetilde{X}^\varepsilon, \tilde{u}^\varepsilon) \in B_\delta(\varphi, \psi)\biggr\}
\end{aligned} \tag{5.3}$$

In $B_\delta(\varphi, \psi)$, we have the following estimates (5.4)–(5.6): since F is continuous, for any $\gamma > 0$ there exists a $\delta > 0$ such that

$$|(F(X(1)) - F(X(0))) - (F(\varphi(1)) - F(\varphi(0)))| < \gamma \quad \text{if } X(\cdot) \in B_\delta(\varphi). \tag{5.4}$$

There is an M such that

$$\left|\sum_{i,j}\frac{\partial^2 F}{\partial x_i \partial x_j}(x)\right| \le M \tag{5.5}$$

by the boundedness of $\partial^2 b_1^+/(\partial x_i \partial x_j)$, $\partial^2 b_1^-/(\partial x_i \partial x_j)$.

Since

$$\int_0^1 (\tilde{b}\nabla F)(X(t))\,dt = \int_0^1 \left(\left(b_2 - \frac{\partial F}{\partial x_2}\right)\frac{\partial F}{\partial x_2} + \cdots + \left(b_d - \frac{\partial F}{\partial x_d}\right)\frac{\partial F}{\partial x_d}\right)(X(t))\,dt,$$

thus

$$\begin{aligned}
&\int_0^1 (\tilde{b}\nabla F)(X(t))\,dt + \frac{1}{2}\int_0^1 |\nabla F|^2(X(t))\,dt \\
&\quad = \int_0^1 \sum_{i=2}^d \left(b_i \frac{\partial F}{\partial x_i}\right)(X(t))\,dt - \frac{1}{2}\int_0^1 \sum_{i=2}^d \left|\frac{\partial F}{\partial x_i}(X(t))\right|^2 dt + \frac{1}{2}\int_0^1 b_1^2(X(t))\,dt \\
&\quad = \int_0^1 (b\nabla F)(X(t))\,dt - \frac{1}{2}\int_0^1 |(\nabla F)|^2(X(t))\,dt
\end{aligned}$$

and by Lemma 6.7, for any $\gamma > 0$, there exists a $\delta > 0$ such that if $(X, u) \in B_\delta(\varphi, \psi)$ with

$$u(t) = \int_0^t \chi_{(0,\infty)}(X_1(s))\,ds,$$

we have

$$\begin{aligned}
&\left|\left(\int_0^1 (\tilde{b}\nabla F)(X(t))\,dt + \tfrac{1}{2}\int_0^1 |\nabla F|^2(X(t))\,dt\right)\right. \\
&\qquad \left. - \left(\int_0^1 (b\nabla F)_{\varphi,\psi}(t)\,dt - \tfrac{1}{2}\int_0^1 |\nabla F|^2_{\varphi,\psi}(t)\,dt\right)\right| < \gamma.
\end{aligned} \tag{5.6}$$

From these estimates, the only term in (5.3) that remains to be estimated now is

$$\begin{aligned}
I = E\Bigg\{&\exp\left(\frac{-1}{2\varepsilon^2}\int_0^1 (b_1^+(\tilde{X}^\varepsilon(t)) - b_1^-(\tilde{X}^\varepsilon(t)))\,d\tilde{\ell}^\varepsilon(t)\right); \\
&\qquad\qquad (\tilde{X}^\varepsilon, \tilde{u}^\varepsilon) \in B_\delta(\varphi, \psi)\Bigg\},
\end{aligned} \tag{5.7}$$

where $\tilde{X}^\varepsilon(t)$ satisfies (5.1).

For any $\gamma > 0$, let δ be so small that

$$|b_1^+(X(t)) - b_1^+(\varphi(t))| + |b_1^-(X(t)) - b_1^-(\varphi(t))| < \gamma, \qquad t \in [0,1] \tag{5.8}$$

if $\|X - \varphi\| < \delta$. In (5.7), using (5.8) and

$$\begin{aligned}
&b_1^+(\tilde{X}^\varepsilon(t)) - b_1^-(\tilde{X}^\varepsilon(t)) \\
&\quad = ((b_1^+(\tilde{X}^\varepsilon(t)) - b_1^-(\tilde{X}^\varepsilon(t))) \\
&\qquad -(b_1^+(\varphi(t)) - b_1^-(\varphi(t)))) + (b_1^+(\varphi(t)) - b_1^-(\varphi(t))),
\end{aligned}$$

we have

$$(5.9)\qquad \begin{aligned} I \le\ & E\biggl\{\exp\biggl(-\frac{1}{2\varepsilon^2}\int_0^1 \bigl(b_1^+(\varphi(t)) - b_1^-(\varphi(t)) - \gamma\bigr)\,d\tilde{\ell}^\varepsilon(t)\biggr);\\ &\qquad\qquad (\widetilde{X}^\varepsilon, \tilde{u}^\varepsilon) \in B_\delta(\varphi, \psi)\biggr\},\\ I \ge\ & E\biggl\{\exp\biggl(-\frac{1}{2\varepsilon^2}\int_0^1 (b_1^+(\varphi(t)) - b_1^-(\varphi(t)) + \gamma)d\tilde{\ell}^\varepsilon(t)\biggr);\\ &\qquad\qquad (\widetilde{X}^\varepsilon, \tilde{u}^\varepsilon) \in B_\delta(\varphi, \psi)\biggr\}. \end{aligned}$$

Therefore, it remains to estimate the following expectation:

$$(5.10)\qquad \mathrm{II} = E\biggl\{\exp\biggl(\frac{1}{\varepsilon^2}\int_0^1 c(t)\,d\tilde{\ell}^\varepsilon(t)\biggr); (\widetilde{X}^\varepsilon, \tilde{u}^\varepsilon) \in B_\delta(\varphi, \psi)\biggr\}.$$

Here $c(t)$ is a continuous function. In our case,

$$(5.11)\qquad c(t) = -\tfrac{1}{2}(b_1^+(\varphi(t)) - b_1^-(\varphi(t)) \pm \gamma).$$

where $\gamma \to 0$ as $\delta \to 0$.

To treat (5.10), since $\widetilde{b}_1 = 0$, by Lemma 6.7, the function

$$(\varphi, \psi) \to (G(\varphi, \psi), \psi) \quad \text{from } \mathfrak{C} \text{ to } \mathfrak{C}$$

is continuous, where

$$G(\varphi, \psi)(t) = \varphi(t) - \varphi(0) - \int_0^t \widetilde{b}_{\varphi,\psi}(s)\,ds.$$

Thus for any $\gamma > 0$ and $(X, u) \in B_\delta(\varphi, \psi)$ with $u(t) = \int_0^t \chi_{(0,\infty)}(X_1(s))\,ds$,

$$\left|\biggl(X(t) - \int_0^t \widetilde{b}(X(s))\,ds\biggr) - \biggl(\varphi(t) - \int_0^t \widetilde{b}_{\varphi,\psi}(s)\,ds\biggr)\right| < \gamma, \qquad t \in [0, 1]$$

for δ small. By (5.1), $\varepsilon W(t) = G(\widetilde{X}^\varepsilon, \tilde{u}^\varepsilon)(t)$. Therefore, for any $\gamma > 0$ and δ small,

$$\mathrm{II} \le E\biggl\{\exp\biggl(\frac{1}{\varepsilon^2}\int_0^1 c(t)\,d\tilde{\ell}^\varepsilon(t)\biggr); \|\varepsilon W - h\| \le \gamma, \|\tilde{u}^\varepsilon - \psi\| \le \delta\biggr\},$$

where

$$h(t) = \varphi(t) - \varphi(0) - \int_0^t \widetilde{b}_{\varphi,\psi}(s)\,ds.$$

Since $\widetilde{X}_1^\varepsilon = \varepsilon W_1$ and $h_1(t) = G(\varphi, \psi)_1(t) = \varphi_1(t)$ from (5.1), we have

$$\mathrm{II} \le E\biggl\{\exp\biggl(\frac{1}{\varepsilon^2}\int_0^1 c(t)\,d\tilde{\ell}^\varepsilon(t)\biggr); \|\varepsilon W_1 - \varphi_1\| \le \gamma, \|\tilde{u}^\varepsilon - \psi\| \le \delta\biggr\} P\{\|\varepsilon \bar{W} - \bar{h}\| \le \gamma\}.$$

Then

$$
\begin{aligned}
&\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\varepsilon^2\log \mathrm{II}\\
&\qquad \le \lim_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon^2\log E\bigg\{\exp\bigg(\frac{1}{\varepsilon^2}\int_0^1 c(t)\,d\tilde{\ell}^{\varepsilon}(t)\bigg);\\
&\qquad\qquad\qquad\qquad\qquad\qquad \|\varepsilon W_1-\varphi_1\|\le\delta,\ \|\tilde{u}^{\varepsilon}-\psi\|\le\delta\bigg\}\\
&\qquad\quad +\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\varepsilon^2\log P\{\|\varepsilon\bar{W}-\bar{h}\|\le\delta\}\\
&\qquad = \lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\varepsilon^2\log E\bigg\{\exp\bigg(\frac{1}{\varepsilon^2}\int_0^1 c(t)\,d\tilde{\ell}^{\varepsilon}(t)\bigg);\\
&\qquad\qquad\qquad\qquad\qquad\qquad \|\varepsilon W_1-\varphi_1\|\le\delta,\ \|\tilde{u}^{\varepsilon}-\psi\|\le\delta\bigg\}\\
&\qquad\quad -\frac{1}{2}\int_0^1|\dot{\bar{\varphi}}(t)-\tilde{\bar{b}}_{\varphi,\psi}(t)|^2\,dt\\
&\qquad = -\frac{1}{2}\int_0^1|\dot{\varphi}_1(t)|^2dt+2\int_0^1 c^+(t)^2\dot{\psi}(t)(1-\dot{\psi}(t))\,dt\\
&\qquad\quad -\frac{1}{2}\int_0^1|\dot{\bar{\varphi}}(t)-\tilde{\bar{b}}_{\varphi,\psi}(t)|^2\,dt.
\end{aligned}
\tag{5.12}
$$

Here in the last step we use Lemma 4.5, and $c^+(t)=c(t)$ if $c(t)>0$, $c^+(t)=0$ otherwise.

Similarly,

$$
\begin{aligned}
\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\varepsilon^2\log \mathrm{II} \ge{}& -\tfrac{1}{2}\int_0^1|\dot{\varphi}_1(t)|^2\,dt+2\int_0^1 c^+(t)^2\dot{\psi}(t)(1-\dot{\psi}(t))\,dt\\
&-\tfrac{1}{2}\int_0^1|\dot{\bar{\varphi}}(t)-\tilde{\bar{b}}_{\varphi,\psi}(t)|^2\,dt.
\end{aligned}
\tag{5.13}
$$

We now combine (5.3)–(5.7), (5.9)–(5.12) to conclude that

$$
\begin{aligned}
&\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\varepsilon^2\log P\{(X^{\varepsilon},u^{\varepsilon})\in B_{\delta}(\varphi,\psi)\}\\
&\quad = F(\varphi(1))-F(\varphi(0))-\int_0^1(b\nabla F)_{\varphi,\psi}(t)\,dt\\
&\qquad +\tfrac{1}{2}\int_0^1|\nabla F|^2_{\varphi,\,\psi}(t)\,dt-\tfrac{1}{2}\int_0^1|\dot{\varphi}(t)-\tilde{b}_{\varphi,\psi}(t)|^2\,dt\\
&\qquad +\tfrac{1}{2}\int_0^1((b_1^-(\varphi(t))-b_1^+(\varphi(t)))^+)^2\dot{\psi}(t)(1-\dot{\psi}(t))\,dt.
\end{aligned}
\tag{5.14}
$$

However,

$$
\begin{aligned}
&\tfrac{1}{2}\int_0^1 \Big|\dot\varphi(t) - \widetilde{b}_{\varphi,\psi}(t)|^2\,dt \\
&\quad = \tfrac{1}{2}\int_0^1 |\dot\varphi(t) - b_{\varphi,\psi}(t) + (\nabla F)_{\varphi,\psi}(t)|^2\,dt \\
&\quad = \tfrac{1}{2}\int_0^1 \big(|\dot\varphi(t) - b_{\varphi,\psi}(t)|^2 \\
&\qquad\qquad + |(\nabla F)_{\varphi,\psi}|^2(t) + 2(\dot\varphi(t) - b_{\varphi,\psi}(t))(\nabla F)_{\varphi,\psi}(t)\big)\,dt \\
&\quad = \tfrac{1}{2}\int_0^1 |\dot\varphi(t) - b_{\varphi,\psi}(t)|^2\,dt + \tfrac{1}{2}\int_0^1 |(\nabla F)_{\varphi,\psi}|^2(t)\,dt \\
&\qquad - \int_0^1 b_{\varphi,\psi}(t)\nabla F_{\varphi,\psi}(t)\,dt + \int_0^1 \dot\varphi(t)(\nabla F)_{\varphi,\psi}(t)\,dt.
\end{aligned}
\tag{5.15}
$$

We know

$$
\begin{aligned}
\int_0^1 \dot\varphi(t)(\nabla F)_{\varphi,\psi}(t)\,dt &= \int_0^1 \dot\varphi(t)\nabla F(\varphi(t))\,dt \\
&= F(\varphi(1)) - F(\varphi(0)).
\end{aligned}
\tag{5.16}
$$

The first relation in (5.16) can be proved by considering the integration on the sets

$$A = \{t \in [0,1]; \varphi_1(t) = 0\}, \qquad B = \{t \in [0,1]; \varphi_1(t) \neq 0\}$$

separately. For almost all $t \in A$, $\dot\varphi_1(t) = 0$ and $\overline{\nabla F}(\varphi(t)) = 0$; therefore,

$$\dot\varphi(t)(\nabla F)_{\varphi,\psi}(t) = \dot\varphi(t)\nabla F(\varphi(t)) = 0.$$

On the other hand, for almost all $t \in B$,

$$\dot\varphi(t)(\nabla F)_{\varphi,\psi}(t) = \dot\varphi(t)\nabla F(\varphi(t)).$$

This proves (5.16).

Similarly, we have

$$
\begin{aligned}
&\tfrac{1}{2}\int_0^1 |\nabla F|^2_{\varphi,\psi}(t)\,dt - \tfrac{1}{2}\int_0^1 |(\nabla F)_{\varphi,\psi}|^2(t)\,dt \\
&\quad = \tfrac{1}{2}\int_0^1 |b_1|^2_{\varphi,\psi}(t)\,dt - \tfrac{1}{2}\int_0^1 |(b_1)_{\varphi,\psi}|^2\,dt \\
&\quad = \tfrac{1}{2}\int_0^1 (b_1^+(\varphi(t)) - b_1^-(\varphi(t)))^2\dot\psi(t)(1-\dot\psi(t))\,dt
\end{aligned}
\tag{5.17}
$$

and

$$
\begin{aligned}
&\int_0^1 b_{\varphi,\psi}(t)\cdot\nabla F_{\varphi,\psi}(t)\,dt - \int_0^1 (b\cdot\nabla F)_{\varphi,\psi}(t)\,dt \\
&\quad = -\int_0^1 (b_1^+(\varphi(t)) - b_1^-(\varphi(t)))^2\dot\psi(t)(1-\dot\psi(t))\,dt.
\end{aligned}
\tag{5.18}
$$

Combining (5.14)–(5.18), we have

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\varepsilon^2\log P((X^\varepsilon,u^\varepsilon)\in B_\delta(\varphi,\psi))\le -I(\varphi,\psi).$$

The proof that $\lim_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon^2\log P((X^\varepsilon,u^\varepsilon)\in B_\delta(\varphi,\psi))\ge -I(\varphi,\psi)$ is similar. The proof of Theorem 2.1. is now complete. □

6. Some properties of $I(\cdot,\cdot)$. In this section we shall establish the lower semicontinuity of $I(\cdot,\cdot)$ defined in (2.4). We also collect some lemmas concerning functions on [0,1] which were used in the previous sections.

LEMMA 6.1. *If $\psi_n\in H^+(\varphi_n)$ and $(\varphi_n,\psi_n)\to(\varphi,\psi)$ in $\mathfrak{C}$, then $\psi\in H^+(\varphi)$. Moreover, if $I(\varphi,\psi)=\infty$, then $\lim_{n\to\infty}I(\varphi_n,\psi_n)=I(\varphi,\psi)=\infty$.*

PROOF. Obviously, $\psi\in AC_0^+[0,1]$ because $\|\psi_n-\psi\|\to 0$ as $n\to\infty$. To show $\psi\in H^+(\varphi)$, suppose $\varphi(t)>0$ at a point $t\in[0,1]$. Then there exists an interval $(t-\gamma,t+\gamma)$ such that $\varphi_n(s)>0$ and thus $\dot\psi_n(s)=1$ for all $s\in(t-\gamma,t+\gamma)$. Since ψ_n is a linear function with slope one in $(t-\gamma,t+\gamma)$, ψ is also a linear function with slope one. Hence $\dot\psi(t)=1$. Similarly, for t where $\varphi(t)<0$, $\dot\psi(t)=0$. It is also obvious that $0\le\dot\psi(t)\le 1$ throughout $[0,1]$ because $0\le\dot\psi_n(t)\le 1$. Hence $\psi\in H^+(\varphi)$.

To show that $\lim_{n\to\infty}I(\varphi_n,\psi_n)=\infty$ if $I(\varphi,\psi)=\infty$, first consider the case that $\psi\notin H^+(\varphi)$. Then by what we just proved, $\psi_n\notin H^+(\varphi_n)$ from some n on, thus $I(\varphi_n,\psi_n)=\infty$ when n is large. If $\psi\in H^+(\varphi)$ and $I(\varphi,\psi)=\infty$, then $\int_0^1|\dot\varphi(t)|^2\,dt=\infty$ and $\lim_{n\to\infty}\int_0^1|\dot\varphi_n(t)|^2\,dt=\infty$ because $\|\varphi_n-\varphi\|\to 0$ and $\varphi\to\int_0^1|\dot\varphi(t)|^2\,dt$ is lower semicontinuous. This completes the proof. □

LEMMA 6.2. *If $\psi_n\in AC_0^+[0,1]$ and $\|\psi_n-\psi\|\to 0$ as $n\to\infty$, then*

$$\lim_{n\to\infty}\int_0^1 f(t)\dot\psi_n(t)\,dt=\int_0^1 f(t)\dot\psi(t)\,dt$$

for any bounded measurable function f.

PROOF. We only need to take f to be indicator functions χ_A where A is a measurable set. It obviously holds if A is an interval.

Let A be an open set in $[0,1]$. Then $A=\bigcup_{i=1}^\infty A_i$ where A_i's are disjoint open intervals. Then, since

$$\int_{A_i}\dot\psi(t)\,dt\le m(A_i)\quad\text{and}\quad\sum_i m(A_i)\le 1,$$

for each n,

$$\lim_{n\to\infty}\int_A \dot\psi_n(t)\,dt = \lim_{n\to\infty}\sum_{i=1}^{\infty}\int_{A_i}\dot\psi_n(t)\,dt$$
$$= \sum_{i=1}^{\infty}\lim_{n\to\infty}\int_{A_i}\dot\psi_n(t)\,dt$$
$$= \sum_{i=1}^{\infty}\int_{A_i}\dot\psi(t)\,dt = \int_A \dot\psi(t)\,dt$$

by the Lebesgue dominated convergence theorem.

For a measurable set A and $\varepsilon > 0$, let G be an open set containing A with $m(G\backslash A) \le \varepsilon$. Then

$$\limsup_{n\to\infty}\int_A \dot\psi_n(t)\,dt \le \lim_{n\to\infty}\int_G \dot\psi_n(t)\,dt = \int_G \dot\psi(t)\,dt \le \int_A \dot\psi(t)\,dt + \varepsilon$$

and

$$\liminf_{n\to\infty}\int_A \dot\psi_n(t)\,dt \ge \lim_{n\to\infty}\int_G \dot\psi_n(t)\,dt - \varepsilon = \int_G \dot\psi(t)\,dt - \varepsilon \ge \int_A \dot\psi(t)\,dt - \varepsilon.$$

This completes the proof. □

THEOREM 6.3. *$I(\varphi, \psi)$ is a lower semicontinuous function on $\mathfrak{C}$.*

PROOF. Let $\|\varphi_n - \varphi\| \to 0$ and $\|\psi_n - \psi\| \to 0$ in $\mathfrak{C}$. If $I(\varphi,\psi) = \infty$, then $\lim_{n\to\infty} I(\varphi_n, \psi_n) = \infty$ by Lemma 6.1. We therefore can assume that $I(\varphi, \psi) < \infty$, $\psi \in H^+(\varphi)$ and $\psi_n \in H^+(\varphi_n)$. Trivially,

$$\left(\varphi_n(t) - \int_0^t b_{\varphi_n,\psi_n}(s)\,ds\right) - \left(\varphi(t) - \int_0^t b_{\varphi,\psi}(s)\,ds\right) \to 0 \quad \text{uniformly in } t \in [0,1].$$

Since

$$\begin{aligned} I(\varphi,\psi) &= \tfrac{1}{2}\int_0^1 |\dot{\bar\varphi}(t) - \bar b_{\varphi,\psi}(t)|^2\,dt \\ &\quad + \tfrac{1}{2}\int_{b_1^-(\varphi(t))>b_1^+(\varphi(t)) \text{ or } \varphi_1(t)\neq 0} |\dot\varphi_1(t) - b_{1\varphi,\psi}(t)|^2\,dt \\ &\quad + \tfrac{1}{2}\int_{\varphi_1(t)=0,\, b_1^+(\varphi(t))\ge b_1^-(\varphi(t))} (b_1^{+2}(\varphi(t))\dot\psi(t) + b_1^{-2}(\varphi(t))(1-\dot\psi(t)))\,dt, \end{aligned}$$

it follows because $f \to \int_0^1 |\dot f(t)|^2\,dt$ is lower semicontinuous that the first two terms on the right-hand side are lower semicontinuous. The third term is continuous because of Lemma 6.2. Hence $I(\cdot,\cdot)$ is lower semicontinuous. □

The following two lemmas are easy and we omit their proofs.

LEMMA 6.4. *For any* $(\varphi, \psi) \in \mathfrak{C}$ *with* $\psi \in H^+(\varphi)$, *there exists a sequence* (φ_n, ψ_n), $\psi_n \in H^+(\varphi_n)$, *such that* φ_n *and* ψ_n *are piecewise linear,* $(\varphi_n, \psi_n) \to (\varphi, \psi)$ *and* $I(\varphi_n, \psi_n) \to I(\varphi, \psi)$ *as* $n \to \infty$.

LEMMA 6.5. *Let* φ *be a continuous piecewise linear function with* $m(t\colon \varphi(t) = 0) = 0$. *If* φ_n *is a sequence of continuous functions such that* $\lim_{n\to\infty} \|\varphi_n - \varphi\| = 0$, *then* $\lim_{n\to\infty} \|\psi_n - \psi\| = 0$ *for any* $\psi_n \in H^+(\varphi_n)$ *and* $\psi \in H^+(\varphi)$.

REMARK 6.6. The assumption $m(t\colon \varphi(t) = 0) = 0$ in Lemma 6.5 is necessary. Counter examples can be easily constructed by letting $\varphi_n = \varphi = 0$. Also, the piecewise linearity of φ is not necessary. But this is all we need.

LEMMA 6.7. *Let* f *be a real-valued function on* R^d *with* $f(x) = f^+(x)$ *if* $x_1 > 0$ *and* $f(x) = f^-(x)$ *if* $x_1 \le 0$ *where* $f^+(x)$ *and* $f^-(x)$ *are bounded and continuous. Then the function*

$$(\varphi, \psi) \to \int_0^1 f_{\varphi,\psi}(t)\,dt$$

is continuous on the set $\{(\varphi, \psi), \varphi \in C_{x^0}([0,1], R^d), \psi \in H^+(\varphi)\}$.

PROOF. Let $(\varphi_n, \psi_n) \to (\varphi, \psi)$ in $\mathfrak{C}$ with $\psi_n \in H^+(\varphi_n)$, $\psi \in H^+(\varphi)$. Let $E = \{t\colon \varphi_1(t) \neq 0\}$. Then

$$\begin{aligned}
&\left|\int_E f_{\varphi_n,\psi_n}(t)\,dt - \int_E f_{\varphi,\psi}(t)\,dt\right| \\
&\quad= \left|\int_{E\cap\{|\varphi|\ge\delta\}} (f_{\varphi_n,\psi_n}(t) - f_{\varphi,\psi}(t))\,dt + \int_{E\cap\{|\varphi|<\delta\}} (f_{\varphi_n,\psi_n}(t) - f_{\varphi,\psi}(t))\,dt\right| \\
&\quad\le \int_{E\cap\{|\varphi|\ge\delta\}} |f(\varphi_n(t)) - f(\varphi(t))|\,dt + m(0 < |\varphi| < \delta)\|f\| \to 0 \quad \text{as } \delta \to 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\lim_{n\to\infty} \left|\int_{E^c} (f^+(\varphi_n(t))\dot\psi_n(t) - f^+(\varphi(t))\dot\psi(t))\,dt\right| \\
&\quad= \lim_{n\to\infty} \left|\int_{E^c} (f^+(\varphi_n(t)) - f^+(\varphi(t)))\dot\psi_n(t)\,dt \right.\\
&\qquad\left. + \int_{E^c} f^+(\varphi(t))(\dot\psi_n(t) - \dot\psi(t))\,dt\right| \\
&\quad\le \lim_{n\to\infty} \int_{E^c} |f^+(\varphi_n(t)) - f^+(\varphi(t))|\dot\psi_n(t)\,dt \\
&\qquad + \left|\int_{E^c} f^+(\varphi(t))(\dot\psi_n(t) - \dot\psi(t))\,dt\right| \\
&\quad= 0
\end{aligned}$$

because of the uniform continuity of f^+ on a compact set and Lemma 6.2.

Similarly,

$$\lim_{n\to\infty}\int_{E^c} f^-(\varphi_n(t))(1-\dot\psi_n(t))\,dt = \int_{E^c} f^-(\varphi(t))(1-\dot\psi(t))\,dt.$$

The proof is now complete. □

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