

## ASYMPTOTICS OF THE DISTRIBUTION OF THE INTEGRAL OF THE ABSOLUTE VALUE OF THE BROWNIAN BRIDGE FOR LARGE ARGUMENTS

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The distribution of the integral of the absolute value of the Brownian bridge was expressed by Cifarelli and independently by Johnson and Killeen in the form of a series. Rice obtained the corresponding probability density by numerical integration. Here we determine the exact tail asymptotics of this distribution, as well as the exact asymptotics of its density function for the large values of the argument.

**1. Introduction.** Let  $B(s)$ ,  $0 < s \leq t$ , be the Brownian motion,

$$(1) \quad \sigma(\lambda, t, x) = \text{Prob} \left\{ \int_0^t |B(s)| ds < \lambda | B(t) = x \right\},$$

and  $F(p, q, x)$  denote the Laplace–Stieltjes transform of  $\sigma_1(\lambda, t, x) = 1/\sqrt{2\pi t} \exp(-x^2/2t) \sigma(\lambda, t, x)$  with respect to  $\lambda, t$  correspondingly. A direct application of the classical result of Kac (1949) [see also Rosenblatt (1951)], yields

$$(2) \quad \begin{aligned} F(p, q, x) &= \int_0^{+\infty} \int_0^{+\infty} \exp(-p\lambda - qt) d_\lambda \sigma_1(\lambda, t, x) dt \\ &= -(2p)^{-1/3} \frac{Ai(2^{1/3} p^{-2/3} (q + px))}{Ai'(2^{1/3} p^{-2/3} q)}, \end{aligned}$$

where  $Ai(z)$  is the first Airy function,  $p$  and  $q$  are in the right half-plane, and the fractional powers are taken as their principal values.

In the following we will consider analytical continuations of (2) into the left  $p$ -half-plane with a cut along a certain ray.

Let  $\tilde{B}(s)$ ,  $0 < s \leq t$ , be the Brownian bridge and

$$(3) \quad \tilde{\sigma}(\lambda, t) = \text{Prob} \left( \int_0^t |\tilde{B}(s)| ds < \lambda \right).$$

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The following proposition is a consequence of Shepp's (1982) result and the scaling property. For the convenience of the reader we outline here a direct proof.

PROPOSITION 1.1. *The Laplace–Stieltjes transform  $\tilde{F}$  of  $1/\sqrt{2\pi t} \sigma(\lambda, t, 0)$  is given by*

$$(4) \quad \tilde{F}(p, q) = -(2p)^{-1/3} \frac{Ai(2^{1/3}p^{-2/3}q)}{Ai'(2^{1/3}p^{-2/3}q)}.$$

For the proof, apply (2) with  $x = 0$ .

Notice that  $\tilde{F}(p, q)$  is in fact the Laplace transform of  $\tilde{f}(\lambda, t)/\sqrt{2\pi t}$ ; see (6).

PROPOSITION 1.2. *The distribution function  $\bar{\sigma}(\lambda, t) = \sigma(\lambda, t, 0)$  and its density function  $\tilde{f}(\lambda, t)$  are given by the following double Laplace inversions:*

$$(5) \quad \bar{\sigma}(\lambda, t) = \frac{\sqrt{2\pi t}}{(2\pi i)^2} \int_{a-i\infty}^{a+i\infty} dq e^{qt} \int_{b-i\infty}^{b+i\infty} \frac{1}{p} \tilde{F}(p, q) e^{p\lambda} dp,$$

$$(6) \quad \tilde{f}(\lambda, t) = \frac{\sqrt{2\pi t}}{(2\pi i)^2} \int_{a-i\infty}^{a+i\infty} dq e^{qt} \int_{b-i\infty}^{b+i\infty} \tilde{F}(p, q) e^{p\lambda} dp$$

with any  $a > 0, b > 0$ .

PROOF. The proof follows via standard manipulations with Laplace's integrals.

The factor  $1/p$  in (5) stems from a well-known property of the Laplace transform: if  $f(x) \doteq \tilde{f}(p)$ , then  $\int_0^x f(t) dt \doteq \frac{1}{p}\tilde{f}(p)$ .  $\square$

Cifarelli (1975) expressed  $\bar{\sigma}(\lambda, 1)$  in one form of a series. Johnson and Killeen (1983), independently, obtained for  $\bar{\sigma}(\lambda, 1)$  a different expansion. In their work they used the results of Shepp (1982). Rice (1982) numerically inverted the integral in (6) in the case  $t = 1$  and by numerical observations suggested approximate asymptotic formulas for  $\tilde{f}(\lambda, 1)$  for small and large values of  $\lambda$ . In the present paper we use the saddlepoint method for integrals to obtain exact asymptotic expressions for  $\bar{\sigma}(\lambda, t)$  and  $\tilde{f}(\lambda, t)$  for large  $\lambda$ 's. The results are given in Theorems 4 and 5. The proposed method readily leads to asymptotic expansions of any order.

## 2. Transformation of the contours.

REMARK. All fractional powers of complex numbers are taken as their principal values.

The main results of this section are Theorems 1 and 2.

LEMMA 2.1.

$$(7) \quad \frac{\exp(\pi i/3) Ai(\exp(2\pi i/3)\xi)}{Ai'(\exp(2\pi i/3)\xi)} - \frac{\exp(-\pi i/3) Ai(\exp(-2\pi i/3)\xi)}{Ai'(\exp(-2\pi i/3)\xi)}$$

$$= \frac{-2i/\pi}{Ai'^2(\xi) + Bi'^2(\xi)},$$

where  $Bi(z)$  is the second Airy function.

PROOF. The proof follows from direct computation, making use of the following identities [see Olver (1974), Chapter 11]:

$$Ai(z \exp(\pm 2\pi i/3)) = \frac{1}{2} \exp(\pm \pi i/3) [Ai(z) \mp iBi(z)],$$

differentiation of which yields

$$Ai'(z \exp(\pm 2\pi i/3)) = \frac{1}{2} \exp(\mp \pi i/3) [Ai'(z) \mp iBi'(z)],$$

and use the expression for the Wronskian of the Airy's equation,

$$W\{Ai(z), Bi(z)\} = 1/\pi.$$

LEMMA 2.2. Let  $\operatorname{Re} q > 0$ . Then

$$(8) \quad \lim_{\varepsilon \rightarrow 0^+} \tilde{F}(i\varepsilon, q) = 1/\sqrt{2q}.$$

PROOF. Apply the following asymptotics for  $Ai(z)$  and  $Ai'(z)$  [see Olver (1974), Chapter 11]:

$$Ai(z) = \frac{1}{2} \pi^{-1/2} z^{-1/4} e^{-\zeta} [1 + O(1/\zeta)], \text{ in } |\arg z| \leq \pi - \delta,$$

$$Ai'(z) = -\frac{1}{2} \pi^{-1/2} z^{1/4} e^{-\zeta} [1 + O(1/\zeta)], \text{ in } |\arg z| \leq \pi - \delta.$$

THEOREM 1. Let  $\xi(\rho, \theta) = 2^{1/3} \rho^{-2/3} \exp(-2i\theta/3)q$ , where  $\theta = \arg q$  and  $\operatorname{Re} q > 0$ . Then

$$(9) \quad \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{1}{p} \tilde{F}(p, q) e^{p\lambda} dp$$

$$= \frac{1}{\sqrt{2q}} - \frac{2^{-1/3}}{\pi^2} \int_0^{+\infty} \frac{\rho^{-4/3} \exp(-i\theta/3) \exp(-\rho e^{i\theta}\lambda) d\rho}{Ai'^2(\xi) + Bi'^2(\xi)}.$$

PROOF. One can see that for any  $b > 0$ ,

$$(10) \quad \int_{b-i\infty}^{b+i\infty} \frac{1}{p} \tilde{F}(p, q) e^{p\lambda} dp = \lim_{\varepsilon \rightarrow 0} \int_{b-i\infty}^{b+i\infty} \frac{1}{p - i\varepsilon} \tilde{F}(p, q) e^{p\lambda} dp.$$

For any fixed  $\theta \in (-\pi/2, \pi/2)$ , and  $\delta \geq 0$ ,  $0 < r < \varepsilon < b < R$ , we define a closed contour  $C_\delta$  in the complex  $p$ -plane as follows:

$$\begin{aligned}
 C_\delta &= C_b \cup C_R^\delta \cup C_\delta^+ \cup C_r^\delta \cup C_\delta^- \quad \text{where} \\
 C_b &= \{p \mid \operatorname{Re} p = b \text{ and } |p| \leq R\}, \\
 C_R^\delta &= \{p \mid \operatorname{Re} p \leq b, |p| = R \text{ and } |\arg p - \theta| < \pi - \delta\}, \\
 C_r^\delta &= \{p \mid \operatorname{Re} p \leq b, |p| = r \text{ and } |\arg p - \theta| < \pi - \delta\}, \\
 C_\delta^\pm &= \{p \mid \arg p = \theta \pm \pi \mp \delta \text{ and } r \leq |p| \leq R\}.
 \end{aligned}$$

Let  $D_\delta$  denote the domain bounded by  $C_\delta$  and  $\bar{D}_\delta$  its closure. By making use of the basic properties of Airy functions, including the fact that all zeroes of  $Ai'(z)$  are real and negative [see Olver (1974), Chapter 11], one can show that the integrand at the right of (10) is analytical in  $D_\delta$  and continuous in  $\bar{D}_\delta$ , except the pole at  $p = i\varepsilon$ .

By the residue theorem,

$$(11) \quad \int_{C_\delta} \frac{1}{p - i\varepsilon} \tilde{F}(p, q) e^{p\lambda} dp = 2\pi i e^{i\varepsilon\lambda} \tilde{F}(i\varepsilon, q).$$

For the contour integral in (11) we have

$$(12) \quad \int_{C_\delta} = \int_{C_b} + \int_{C_R^\delta} + \int_{C_r^\delta} + \int_{C_\delta^+} + \int_{C_\delta^-}$$

By taking  $\delta \rightarrow 0$  in (11) and (12) we get

$$(13) \quad \int_{C_b} + \int_{C_R^0} + \int_{C_r^0} + \int_{C_0^+} + \int_{C_0^-} = 2\pi i \tilde{F}(i\varepsilon, q) e^{i\varepsilon\lambda}$$

By (7) with  $\xi(\rho, \theta) = 2^{1/3} \rho^{-2/3} \exp(-2i\theta/3)q$  and an obvious reparametrization,

$$(14) \quad \int_{C_0^+} + \int_{C_0^-} = \int_r^R \frac{\exp(-i\theta/3) \exp(2\rho)^{-1/3}}{\rho e^{i\theta} + i\varepsilon} \frac{-2i/\pi \exp(-\rho e^{i\theta}\lambda)}{Ai'^2(\xi) + Bi'^2(\xi)} e^{i\theta} d\rho.$$

By using the asymptotic estimate (9) for  $Ai(z)$ , we can show that  $\int_{C_R^0} \rightarrow 0$  when  $R \rightarrow \infty$  and  $\int_{C_r^0} \rightarrow 0$  when  $r \rightarrow 0$  and  $\int_{C_b} \rightarrow \int_{b-i\infty}^{b+i\infty}$  when  $R \rightarrow \infty$ ; that is,

$$(15) \quad \int_{b-i\infty}^{b+i\infty} = - \int_0^{+\infty} \frac{\exp(-i\theta/3) e^{i\theta} (2\rho)^{-1/3} 2i/\pi \exp(-\rho e^{i\theta}\lambda)}{\rho e^{i\theta} + i\varepsilon} \frac{d\rho}{Ai'^2(\xi) + Bi'^2(\xi)} + 2\pi i \tilde{F}(i\varepsilon, q) e^{i\varepsilon\lambda}.$$

Taking here  $\varepsilon \rightarrow 0$  completes the proof.  $\square$

**THEOREM 2.** For any  $a > 0$ , let  $z(\rho, \theta) = 2^{1/3}a\rho^{-2/3}\exp(-2i\theta/3)(1 + i \tan \theta)$ . Then

$$(16) \quad \begin{aligned} \tilde{\sigma}(\lambda, t) = 1 - \frac{a\sqrt{t}}{2^{5/6}\pi^{5/2}} \int_{-\pi/2}^{\pi/2} d\theta \exp(at(1 + i \tan \theta)) \sec^2 \theta \\ \times \int_0^{+\infty} \frac{\rho^{-4/3} \exp(-i\theta/3) \exp(-\rho e^{i\theta}\lambda) d\rho}{Ai'^2[z(\rho, \theta)] + Bi'^2[z(\rho, \theta)]}. \end{aligned}$$

**PROOF.** Apply to (9) the inverse Laplace transform with respect to  $q$  with the reparametrization  $q = a \sec \theta e^{i\theta} = a(1 + i \tan \theta)$  and refer to (5).

**3. Asymptotic expansions.** In this section we apply the saddlepoint method [see de Bruijn (1961)].

**LEMMA 3.1.** The double integral, which corresponds to the repeated integral in Theorem 2, converges absolutely. Therefore

$$(17) \quad \begin{aligned} \tilde{\sigma}(\lambda, t) = 1 - \frac{a\sqrt{t}}{2^{5/6}\pi^{5/2}} \int_{-\pi/2}^{+\pi/2} \int_0^{+\infty} (\exp at(1 + i \tan \theta)) \sec^2 \theta \\ \times \frac{\rho^{-4/3} \exp(-i\theta/3) \exp(-\rho e^{i\theta}\lambda) d\rho d\theta}{Ai'^2[z(\rho, \theta)] + Bi'^2[z(\rho, \theta)]}. \end{aligned}$$

The proof follows from straightforward estimates for the integrand [see Olver (1974), Chapter 11] and Fubini's theorem.  $\square$

The representation (17) holds for any  $a > 0$ . We will show that the choice

$$(18) \quad a = 18\lambda^2/t^4$$

provides a suitable saddlepoint at

$$(19) \quad (\rho, \theta) = (12\lambda/t^3, 0),$$

and thus Laplace's method for integrals is applicable.

**NOTATION.** For an independent variable  $z$ ,  $\zeta = \frac{2}{3}z^{3/2}$ ; if  $z$  is a function of  $\rho$  and  $\theta$ , then  $\zeta(\rho, \theta) = \frac{2}{3}z^{3/2}(\rho, \theta)$ , where  $z(\rho, \theta) = 2^{1/3}a\rho^{-2/3}\exp(-2i\theta/3)(1 + i \tan \theta)$  as defined in Theorem 2. In Olver [(1974), Chapter 11], the following facts are proved:

$$Ai'(z) = -\frac{1}{2}\pi^{-1/2}z^{1/4}e^{-\zeta} [1 + O(1/\zeta)] \quad \text{in } |\arg z| \leq \pi - \delta,$$

$$Bi'(z) = \pi^{-1/2}z^{1/4}e^{\zeta} [1 + O(1/\zeta)] \quad \text{in } |\arg z| \leq \pi/3 - \delta.$$

From these asymptotics the lemma follows.

**LEMMA 3.2.**  $[Ai'^2(z) + Bi'^2(z)]^{-1} = \pi z^{-1/2}e^{-2\zeta} [1 + O(1/\zeta)]$ , in  $|\arg z| \leq \pi/3 - \delta$ .

**THEOREM 3.** *For a fixed  $t > 0$  and  $\lambda \rightarrow +\infty$  we have*

$$(20) \quad \begin{aligned} \tilde{\sigma}(\lambda, t) &= 1 - \frac{3\sqrt{2}}{2} \frac{\lambda}{(\pi t)^{3/2}} \\ &\times \int_{-\pi/2}^{+\pi/2} \int_0^{+\infty} \left[ 1 + \frac{\rho \exp(-i\theta/2) \cos^{3/2} \theta}{\lambda^2} O(1) \right] \\ &\times 1/\rho \sec^2 \theta (1 + i \tan \theta)^{-1/2} \exp(-\Phi(\rho, \theta) \lambda^2/t^3) d\rho d\theta, \end{aligned}$$

where  $\Phi(\rho, \theta) = -18(1 + i \tan \theta) + 12\rho e^{i\theta} + 12/\rho e^{-i\theta} (1 + i \tan \theta)^{3/2}$ .

**PROOF.** By Lemmas 3.1 and 3.2 we get

$$(21) \quad \begin{aligned} \tilde{\sigma}(\lambda, t) &= 1 - \frac{\alpha\sqrt{t}}{2^{5/6}\pi^{5/2}} \int_{-\pi/2}^{+\pi/2} \int_0^{+\infty} \{1 + O[1/\zeta(\rho, \theta)]\} \\ &\times \exp(at(1 + i \tan \theta)) \sec^2 \theta \exp(-i\theta/3) \\ &\times \rho^{-4/3} \exp(-\rho e^{i\theta}\lambda) \pi z^{-1/2}(\rho, \theta) \exp(-2\zeta(\rho, \theta)) d\rho d\theta. \end{aligned}$$

Apply in this integral the change of the variable  $\rho = 12\lambda/t^3\rho'$  and the substitution  $\alpha = 18\lambda^2/t^4$ , and return to the original variable  $\rho$ .  $\square$

The following elementary lemma we give without proof.

**LEMMA 3.3.** *The function  $\operatorname{Re} \Phi(\rho, \theta)$  attains its absolute minimum  $\Phi_{\min} = 6$  at  $(\rho, \theta) = (1, 0)$ .*

Lemma 3.3 sets the stage for a routine application of Laplace’s method for integrals; see, for example, de Bruijn (1961).

The following elementary inequality will be used in the next theorem:

$$(22) \quad \rho \cos \theta + \frac{1}{\rho} \frac{\cos(\theta/2)}{\cos^{3/2} \theta} > \frac{1}{2} \left( \rho \cos \theta + \frac{1}{\rho} \frac{\sqrt{2}/2}{\cos^{3/2} \theta} \right) + \frac{\sqrt{2}/2}{\cos^{1/2} \theta},$$

where  $\rho > 0$  and  $-\pi/2 < \theta < \pi/2$ .

**THEOREM 4.** *For any fixed  $t > 0$  and  $\lambda \rightarrow +\infty$ ,*

$$(23) \quad \tilde{\sigma}(\lambda, t) = 1 - \frac{t\sqrt{t}}{\sqrt{6\pi} \cdot \lambda} \exp(-6\lambda^2/t^3)[1 + o(1)].$$

**PROOF.** Let  $\varepsilon > 0$ ,  $S_\varepsilon = (1 - \varepsilon, 1 + \varepsilon) \times (-\varepsilon, \varepsilon)$  and  $D = (0, +\infty) \times (-\pi/2, \pi/2)$ .

We consider the double integral in (20) in the form of the sum

$$(24) \quad \int_D = \int_{S_\varepsilon} + \int_{D \setminus S_\varepsilon}.$$

From this, by substituting the Taylor expansion of  $\Phi$  at the saddlepoint,

$$(25) \quad \begin{aligned} \Phi(\rho, \theta) &= 6 + 12(\rho - 1)^2 + 6i\theta(\rho - 1) \\ &+ \frac{3}{2}\theta^2 + [(\rho - 1)^2 + \theta^2]^{3/2} O(1), \end{aligned}$$

and other obvious Taylor expansions in the integrand, we obtain

$$\begin{aligned}
 \int_{S_\varepsilon} &= \int_{S_\varepsilon} \left[ 1 + \frac{\exp(-i\theta/2)\cos^{3/2}\theta}{\lambda^2} O(1) \right] \\
 &\times [1 + (\rho - 1)O(1)][1 + \theta O(1)] \\
 (26) \quad &\times \exp\left(-\left\{6 + 12(\rho - 1)^2 + 6i\theta(\rho - 1) \right. \right. \\
 &\quad \left. \left. + \frac{3}{2}\theta^2 + O(1)[(\rho - 1)^2 + \theta^2]^{3/2}\right\}\lambda^2/t^3\right) d\rho d\theta.
 \end{aligned}$$

Let  $S_{\varepsilon\lambda} = (-\varepsilon\lambda/t^{3/2}, \varepsilon\lambda/t^{3/2}) \times (-\varepsilon\lambda/t^{3/2}, \varepsilon\lambda/t^{3/2})$ .

The change of variables  $r = \sqrt{\lambda^2/t^3}(\rho - 1)$ ,  $s = \sqrt{\lambda^2/t^3}\theta$  and standard estimates yield

$$\begin{aligned}
 \int_{S_\varepsilon} &= \frac{t^3}{\lambda^2} \exp\left(\frac{-6\lambda^2}{t^3}\right) [1 + o(1)] \int_{S_{\varepsilon\lambda}} \exp\left(-12r^2 - 6isr - \frac{3}{2}s^2\right) dr ds \\
 (27) \quad &= \frac{t^3}{\lambda^2} \exp\left(\frac{-6\lambda^2}{t^3}\right) [1 + o(1)] \frac{\pi}{3\sqrt{3}}.
 \end{aligned}$$

It remains to estimate the integral  $\int_{D \setminus S_\varepsilon}$ .

Routine estimates with  $\operatorname{Re} \Phi(\rho, \theta) = -18 + 12\rho \cos \theta + \frac{12}{\rho} \frac{\cos(\theta/2)}{\cos^{3/2}\theta}$  show

$$(28) \quad \left| \int_{D \setminus S_\varepsilon} \right| < \exp(-(6 + \delta)\lambda^2/t^3) O(1),$$

with some  $\delta > 0$ .

To obtain the term  $O(1)$  in the last formula, we used (22).

The substitution of (27) and (28) in (24) and (20) yields (23).

**THEOREM 5.** *For any fixed  $t > 0$  and  $\lambda \rightarrow +\infty$ ,*

$$(29) \quad \tilde{f}(\lambda, t) = \frac{2\sqrt{6}}{\sqrt{\pi}t^{3/2}} \exp\left(\frac{-6\lambda^2}{t^3}\right) [1 + o(1)].$$

**PROOF.** By the differentiation of (17) on  $\lambda$  we get

$$\begin{aligned}
 \tilde{f}(\lambda, t) &= \frac{\alpha\sqrt{t}}{2^{5/6}\pi^{5/2}} \int_{-\pi/2}^{+\pi/2} \int_0^{+\infty} \rho \exp(at(1 + i \tan \theta)) \\
 (30) \quad &\times \sec^2\theta \frac{\rho^{-1/3} \exp(2i\theta/3) \exp(-\rho e^{i\theta}\lambda) d\rho d\theta}{Ai'^2(z) + Bi'^2(z)}
 \end{aligned}$$

and proceed similarly to the proof of (23).  $\square$

COROLLARY 1. *The logarithmic asymptotics of  $\tilde{f}(\lambda, t)$  as  $\lambda \rightarrow \infty$ , is as follows:*

$$(31) \quad -\ln \tilde{f}(\lambda, t) \sim 6\lambda^2/t^3.$$

*In particular, for  $t = 1$ ,*

$$(32) \quad -\ln \tilde{f}(\lambda, 1) \sim 6\lambda^2.$$

**4. Remarks on related work.** From numerical observations Rice (1982) suggested that for large  $\lambda$ 's approximately  $\tilde{f}(\lambda, 1) \approx 4.1 \exp(-6.3\lambda^2)$ . According to (29), the corresponding rigorous statement should be  $\tilde{f}(\lambda, 1) \approx 2.76 \dots \exp(-6\lambda^2)$ . Indeed, in the interval  $1 \leq \lambda \leq 1.5$ , on which Rice based his observations, the ratio of the logarithms of these two expressions is pretty close to 1.

In connection with the subject of the present paper, one should mention results concerning the closely related functional  $\int_0^t |B(s)| ds$ .

Kac (1946) determined the Laplace–Stieltjes transform of the distribution function of this functional. Takács (1993) determined the distribution function itself. In Borell (1975), in a context of some other results, the logarithmic tail asymptotics was obtained.

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