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ON THE CAUCHY PROBLEM FOR PARABOLIC SPDEs IN HÖLDER CLASSES

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We study Cauchy's problem for certain second-order linear parabolic stochastic differential equation (SPDE) driven by a cylindrical Brownian motion. Considering its solution as a function with values in a probability space and using the methods of deterministic partial differential equations, we establish the existence and uniqueness of a strong solution in Hölder classes.

1. Introduction. We consider the second-order linear parabolic SPDE of the type

$$\begin{cases} \partial_t u = 1/2\, a^{ij}\partial_{ij}u + b^i\partial_i u + cu + f + (h\,u + g)\dot{W}, & \text{in } \mathbf{R}^d_T, \\ u(0, x) = 0, & \text{in } \mathbf{R}^d, \end{cases} \tag{1}$$

where $\mathbf{R}^d_T = [0, T] \times \mathbf{R}^d$, W is a cylindrical Wiener process in some Hilbert space Y. The coefficients a^{ij}, b^i, c and f are real-valued functions, a^{ij} is deterministic, while h, g are Y-valued. The matrix $A = (a^{ij})$ is assumed to be symmetric and nonnegative. An important example of (1) is the Zakai equation [see Zakai (1969), Rozovskii (1990)]. It arises in the nonlinear filtering problem. Assume that the signal process X_t is a diffusion process defined by the It$_o$ equation,

$$X_t = X_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dw_s,$$

where w is a one-dimensional Wiener process and X_0 has a density function $p(x)$. The observation process is given by

$$Z_t = \int_0^t h(X_s)\, ds + \bar{w}_t,$$

where $\bar{w}$ is a Wiener process independent of w. Then for every function ψ such that $\mathbf{E}|\psi(X_t)|^2 < \infty$, the optional mean square estimate for $\psi(X_t)$, $t \in [0, 1]$, given the past of the observations $\mathscr{F}_t^Z = \sigma(Z_s, s \le t)$, is of the form

$$\hat{\psi}_t = \frac{\mathbf{E}_{\tilde{\mathbf{P}}}[\psi(X_t)\zeta_t|\mathscr{F}_t^Z]}{\mathbf{E}_{\tilde{\mathbf{P}}}[\zeta_t|\mathscr{F}_t^Z]},$$

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where $\zeta_t = \exp\{\int_0^t h(X_s)\,dZ_s - 1/2\int_0^t |h(X_s)|\,ds\}$ and $d\tilde{\mathbf{P}} = \zeta(1)^{-1}d\mathbf{P}$. Under some assumptions, one can show that

$$\mathbf{E}_{\tilde{\mathbf{P}}}[\psi(X_t)\zeta_t|\mathcal{F}_t^Z] = \int v(t,x)\psi(x)\,dx,$$

where $v(t,x)$, referred to as unnormalized filtering density function, is a solution of the Zakai equation

$$dv = [1/2(\sigma^2 v)_{xx} - (bv)_x]\,dt + hv\,dZ_{t,} \qquad t > 0,\ v(0,x) = p(x).$$

So, for $u(t,x) = v(t,x) - p(x)$ we have

$$\begin{aligned} du &= [1/2(\sigma^2 u)_{xx} - (bu)_x + 1/2(\sigma^2 p)_{xx} - (bp)_x]\,dt \\ &\quad + (hu + hp)\,dZ_{t,} \qquad t > 0,\ u(0,x) = 0. \end{aligned} \tag{2}$$

Since Z is a Wiener process with respect to $\tilde{\mathbf{P}}$, (2) is obviously a particular case of (1).

The general Cauchy problem (correlated noise case in the nonlinear filtering problem),

$$\begin{cases} \partial_t u = 1/2\,a^{ij}\,\partial_{ij}u + b^i\partial_i u + cu + f + (\sigma^i u_{x_i} + hu + g)\dot{W}, & \text{in } \mathbf{R}^d_T, \\ u(0,x) = 0, & \text{in } \mathbf{R}^d, \end{cases} \tag{3}$$

has been studied by many authors. When the matrix $(a^{ij} - 2\sigma^i\sigma^j)$ is uniformly nondegenerate there exists a complete theory in Sobolev spaces $W^{n,2}(\mathbf{R}^d)$ [see Pardoux (1975), Krylov and Rozovskii (1977), Rozovskii (1990), Da Prato and Zabczyk (1992) and references therein] and in the spaces of Bessel potentials $H^p_s(\mathbf{R}^d)$ [see Krylov (1996)].

Equation (1) in Hölder classes was considered first in Rozovskii (1975) regarding the unknown function as a deterministic one but taking values in a probability space. The results in Rozovskii (1975) were not sharp. In this article we adopt the same point of view and use the methods of deterministic PDEs [see Gilbarg and Trudinger (1983), Friedman (1964), Ladyzhenskaja, Solonnikov and Uralteseva (1968)]. Using the fundamental solution of the heat equation we represent a solution of (1) in a convenient form and derive the Hölder estimates for the equation with coefficients independent of space variables. Our main results are contained in Section 4 (see Theorems 19, 18, 17). By standard methods, we obtain a priori interior Schauder estimates for the general SPDE. The existence and uniqueness result then follows by continuity arguments. We show (see Theorem 19 below) that for $(a^{ij}), b^i, c, f \in C^\beta$ and $h, g \in C^{1+\beta}$ there exists a unique strong solution $u \in C^{2+\beta}$ of (1). So we generalize the corresponding results for the deterministic parabolic Cauchy problem [see Friedman (1964), Mikulevicius and Pragarauskas (1992)]. In Mikulevicius and Rozovskii (1998) the uniqueness and existence of a weak (soft) $C^{2+\beta}$-solution of (3) was proved when $a^{ij}, b^i, c, \sigma^i, h, g$ are deterministic C^β-functions.

We finish this section by introducing several notations to be used throughout the paper.

Let $L_p(\Omega, Y, \mathbf{P}) = L_p(\Omega, Y, \mathcal{F}, \mathbf{P})$, $p \in [1, \infty]$ be the space of Hilbert space Y-valued random variables X on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with finite norm $\|X\|_p = \|X\|_{p,Y} = (\mathbf{E}|X|_Y^p)^{1/p}$, $\|X\|_{\infty,Y} = \operatorname{ess\,sup}_\omega |X(\omega)|_Y$. If $Y = \mathbf{R}$ we write simply $L_p(\Omega, \mathbf{P})$, $|X|_p$.

For an open subset $D \subseteq \mathbf{R}^d$, denote $D_T = [0, T] \times D$; $\bar{D}_T = [0, T] \times \bar{D}$.

Let $B^p_{\text{loc}}(D_T, Y)$ be the space of locally bounded $L_p(\Omega, Y, \mathcal{F}, \mathbf{P})$-valued $\mathbf{F}$-adapted functions g on D_T, that is, for each compact subset $K \subset D$ $\|g\|_{0,p;K_T} = \sup_{K_T}(\mathbf{E}|g(t,x)|_Y^p)^{1/p} < \infty$, and for each t and $x g(t,x) \in L_p(\Omega, \mathcal{F}_t, Y, \mathbf{P})$, where $\mathbf{F} = (\mathcal{F}_t)$ is an increasing right continuous filtration of σ-subalgebras of $\mathcal{F}$. Let $B^p(D_T, Y) = \{g \in B^p_{\text{loc}}(D_T, Y)\colon \|g\|_{0,p} = \|g\|_{0,p;T} = \|g\|_{0,p;D_T} = \sup_{D_T}(\mathbf{E}|g(t,x)|_Y^p)^{1/p} < \infty\}$.

For $L_p(\Omega, Y, \mathcal{F}, \mathbf{P})$-valued function u on D_T, we denote its partial derivatives in $L_p(\Omega, Y, \mathcal{F}, \mathbf{P})$-sense $\partial_i u = \partial_{x_i} u = \partial u/\partial x_i$, $\partial^2_{ij} u = \partial^2_{x_i x_j} u = \partial^2 u/\partial x_i \partial x_j$, etc.; $\partial u = \partial_x u = (\partial_1 u, \dots, \partial_d u) =$ gradient of u with respect to x.

Let $C^{m,p}(D_T, Y) = \{g \in B^p_{\text{loc}}(D_T, Y)$: g is m times continuously differentiable in x as $L_p(\Omega, Y, \mathcal{F}, \mathbf{P})$-valued function and its derivatives in $L_p(\Omega, Y, \mathcal{F}, \mathbf{P})$-sense $\partial^k g = \partial^k_x g \in B^p_{\text{loc}}(D_T, Y)$ for each $k \le m\}$.

$C^{m,p}(\bar{D}_T, Y)$: the set of functions g in $C^{m,p}(D_T, Y)$ all of whose derivatives in $L_p(\Omega, Y, \mathcal{F}, \mathbf{P})$-sense of order less than or equal to m have continuous extensions to $\bar{D}_T$ and finite norm $\|g\|_{m,p} = \sum_{k \le m} \|\partial^k g\|_{0,p}$.

For $\beta \in (0, 1)$, $C^{m+\beta,p}(\bar{D}_T, Y)$ is the set of all $g \in C^{m,p}(\bar{D}_T, Y)$ with finite norm,

$$\|g\|_{m+\beta,p} = \|g\|_{m+\beta,p;T} = \|g\|_{m+\beta,p;D_T} = \|g\|_{m,p} + [g]_{m+\beta,p},$$

where $[g]_{m+\beta,p} = \sup_{t,x \ne y}(\mathbf{E}|\partial^m g(t,x) - \partial^m g(t,y)|_Y^p)^{1/p}/|x-y|^\beta$.

If $Y = \mathbf{R}$ we omit Y in the definition of these spaces and write simply $|\cdot|$ instead of $\|\cdot\|$.

$C = C(\cdot, \dots, \cdot)$, $c = c(\cdot, \dots, \cdot)$ denotes constants depending only on quantities appearing in parentheses. In a given context the same letter will (generally) be used to denote different constants depending on the same set of arguments.

2. Auxiliary results. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with right continuous filtration of σ-algebras $\mathbf{F} = (\mathcal{F}_t)_{t \ge 0}$. Let W be a cylindrical Wiener process in a separable Hilbert space Y. This means that we have a family of continuous martingales $W_t(v)$, $v \in Y$, such that

$$\langle W(v), W(v')\rangle_t = t\,(v, v')_Y \qquad \forall\, v, v' \in Y.$$

For an $\mathbf{F}$-adapted Y-valued function f such that $\int_0^t |f_s|_Y^2\, ds < \infty$ $\mathbf{P}$-a.s. for all t, we can define Itô's stochastic integral denoted

$$\int_0^t f_s\, dW_s = \int_0^t f_s \dot{W}_s\, ds.$$

It is a real-valued local martingale such that $\langle \int_0^\bullet f_s\, dW_s\rangle_t = \int_0^t |f_s|_Y^2\, ds$.

2.1. *Estimates of stochastic integrals.*

LEMMA 1. *Let* (μ_s) *be a measurable family of* σ*-finite measures on a measurable space* $(A, \mathscr{A})$. *Let* f, g *be* Y*-valued* $\mathscr{F}_t \otimes \mathscr{A}$*-adapted functions on* $[0, T] \times \Omega \times A$ *such that* $\int_0^T |\int_A g(s,a)\,\mu_s(da)|_Y^2\, ds < \infty$ *and* $\int_0^T \int_A |f(s,a)|_Y |\mu_s|\,(da)\, ds < \infty$ $\mathbf{P}$-a.s. *Then for each* $p \geq 2$ *there exist* C *independent of* T *such that*

$$\left|\int_0^T \int_A g(s,a)\mu_s(da)\, \dot{W}_s\, ds\right|_p \leq C \sup_{s,a} \|g(s,a)\|_p \left(\int_0^T |\mu_s|(A)^2\, ds\right)^{1/2}, \tag{4}$$

$$\left\|\int_0^T \int_A f(s,a)\mu_s(da)\, ds\right\|_p \leq \sup_{s,a} \|f(s,a)\|_p \int_0^T |\mu_s|(A)\, ds.$$

PROOF. If $p \geq 2$ we have by Minkowsky inequality $|\int_A g(s,a)\,\mu_s(da)|_Y^2 \leq (\int_A |g(s,a)|_Y |\,\mu_s|(da))^2$ and the second inequality in (4). Using Doob's and again Minkowsky's inequalities, we obtain

$$\begin{aligned}\left|\int_0^T \int_A g(s,a)\mu_s(da)\, \dot{W}_s\, ds\right|_p &\leq C\left|\int_0^T \left|\int_A g(s,a)\,\mu_s(da)\right|_Y^2 ds)^{1/2}\right|_p \\ &\leq C\left|\int_0^T \left(\int_A |g(s,a)|_Y\, |\mu_s|(da)\right)^2 ds)^{1/2}\right|_p \\ &\leq C\left(\int_0^T |\int_A |g(s,a)|_Y |\mu_s|(da)|_p^2 ds\right)^{1/2} \\ &\leq C\left(\int_0^T \left(\int_A \|g(s,a)\|_p\, |\mu_s|(da)\right)^2 ds\right)^{1/2}. \qquad \square\end{aligned}$$

COROLLARY 2. *Let the assumptions of Lemma* 1 *be satisfied. Assume that there is a nonnegative* σ*-finite measure* da *on* $(A, \mathscr{A})$ *such that* $\mu_s(da) = \rho(s,a)\, da$. *Then*

$$\begin{aligned}\left|\int_0^T \int_A g(s,a)\mu_s(da)\, \dot{W}_s\, ds\right|_p &\leq C\left(\int_0^T \left(\int_A \|g(s,a)\|_p \rho(s,a)\, da\right)^2 ds\right)^{1/2} \\ &\leq C \sup_{s,a} \|g(s,a)\|_p \int_A \left(\int_0^T \rho(s,a)^2\, ds\right)^{1/2} da.\end{aligned}$$

We will need some estimates for singular stochastic integrals. Assume we are given two deterministic functions $H_{t,s}^{(m)}(x)$, $m = 1, 2$, $s < t$, $x \in \mathbf{R}^d$. For $\beta \in (0,1)$ we will need the following assumptions $\mathbf{A}(m)(m = 1, 2,)$:

(a) *For all* t,

$$\int_0^t \left(\int \left|H_{t,s}^{(m)}(y)\right|(|y|^\beta \wedge 1)\, dy\right)^m ds < \infty.$$

(b) *There is a constant* C_1 *such that for all* t, x,

$$\left(\int_0^t \left|H_{t,s}^{(m)}(x)\right|^m ds\right)^{1/m} \leq C_1|x|^{-d}.$$

(c) *There is a constant* C_2 *such that for all* $t,\ x$,

$$\left(\int_0^t |\partial_x H_{t,s}(x)|^m ds\right)^{1/m} \leq C_2|x|^{-d-1}.$$

(d) *For each* $\gamma \in (0,1)$ *there is a constant* C_3 *such that for all* $t, \delta > 0, |x| \leq \gamma\delta$,

$$\int_0^t \left|\int_{|x+y|\geq\delta} H_{t,s}^{(m)}(y)\,dy\right|^m ds \leq C_3.$$

If **A** (1) is satisfied, we can define the operator on $C^{\beta,p}(\mathbf{R}_T^d)$,

$$\mathscr{H}^1 f(t,x) = \int_0^t \int H_{t,s}^{(1)}(x-y)(f(s,y)-f(s,x))\,dy\,ds. \tag{5}$$

If **A** (2) is satisfied, we can define an operator on $C^{\beta,p}(\mathbf{R}_T^d,\ Y)$,

$$\mathscr{H}^2 f(t,x) = \int_0^t \int H_{t,s}^{(2)}(x-y)(f(s,y)-f(s,x))\,dy\,\dot{W}_s\,ds. \tag{6}$$

LEMMA 3. *Let* $\mathbf{A}(m)$, $m = 1,2$ *be satisfied. Then* $\mathscr{H}^i f \in C^{\beta,p}(\mathbf{R}_T^d)$, $m = 1,2$, *and there is a constant* C *independent of* T *such that*

$$[\mathscr{H}^i f]_{\beta,\,p;T} \leq C(C_1+C_2+C_3)[f]_{\beta,\,p;T}.$$

PROOF. (i) Estimate of $[\mathscr{H}^1 f]_{\beta,\,p}$. Fix any $x, \bar{x}, t$. Writing $\delta = |x-\bar{x}|$, $\xi = 1/2(x+\bar{x})$, we consequently obtain by subtraction

$$\mathscr{H}^1 f(t,x) - \mathscr{H}^1 f(t,\bar{x}) = I_1 + I_2 + I_3 + I_4,$$

where the integrals I_i, $i = 1,2,3,4$, are given by

$$I_1 = \int_0^t \int_{B_\delta(\xi)} H_{t,s}^{(1)}(x-y)[f(s,y)-f(s,x)]\,dy\,ds,$$

$$I_2 = -\int_0^t \int_{B_\delta(\xi)} H_{t,s}^{(1)}(\bar{x}-y)[f(s,y)-f(s,\bar{x})]\,dy\,ds,$$

$$I_3 = \int_0^t \int_{B_\delta(\xi)^c} H_{t,s}^{(1)}(\bar{x}-y)[f(s,\bar{x})-f(s,x)]\,dy\,ds,$$

$$I_4 = \int_0^t \int_{B_\delta(\xi)^c} (H_{t,s}^{(1)}(x-y) - H_{t,s}^{(1)}(\bar{x}-y))[f(s,y)-f(s,x)]\,dy\,ds.$$

By Lemma 1,

$$|I_1|_p + |I_2|_p \le C[f]_{\beta,\,p} \int_0^t \int_{B_{3\delta/2}(x)} \left| H^{(1)}_{t,\,s}(x-y) \right| |x-y|^\beta \, dy\, ds$$

$$\le CC_1 \int_{B_{3\delta/2}(x)} |x-y|^{-d+\beta} \, dy \le CC_1 \delta^\beta [f]_{\beta,\,p}.$$

Applying Lemmas 1 and 4 again,

$$|I_3|_p \le C[f]_{\beta,\,p} \delta^\beta \int_0^t \left| \int_{B_\delta(\xi)^c} H^{(1)}_{t,\,s}(\bar{x}-y)\, dy \right| ds \le CC_3 \delta^\beta [f]_{\beta,\,p}.$$

If $\hat{x}$ is an arbitrary point on the segment joining x and $\bar{x}$ and $|\xi - y| \ge \delta$, then $\frac{3}{2}|\xi - y| \ge |\hat{x} - y| \ge \frac{1}{2}|\xi - y|$. Therefore by Lemma 4,

$$|I_4|_p \le C\delta\, [f]_{\beta,\,p} \int_0^1 \int_0^t \int_{|\xi-y|\ge\delta} \left| \partial_x H^{(1)}_{t,\,s}(r\bar{x} + (1-r)x - y) \right| |x-y|^\beta dy\, ds\, dr$$

$$\le CC_2 \delta\, [f]_{\beta,\,p} \int_{|\xi-y|\ge\delta} |\xi - y|^{-d-1+\beta}\, dy \le CC_2 \delta^\beta [f]_{\beta,\,p}.$$

(ii) Estimate of $[H^2 f]_{\beta,\,p}$. Denoting $\delta = |x - \bar{x}|$, $\xi = \frac{1}{2}(x + \bar{x})$ we obtain by subtraction $H^2 f(t,x) - H^2 f(t,\bar{x}) = I_1 + I_2 + I_3 + I_4$, where

$$I_1 = \int_0^t \int_{B_\delta(\xi)} H^{(2)}_{t,\,s} f\,(x-y)[f(s,y) - f(s,x)\,]\, dy\, \dot{W}\, ds,$$

$$I_2 = -\int_0^t \int_{B_\delta(\xi)} H^{(2)}_{t,\,s}(\bar{x}-y)[f(s,y) - f(s,\bar{x})\, dy\, \dot{W}\, ds,$$

$$I_3 = \int_0^t \int_{B_\delta(\xi)^c} H^{(2)}_{t,\,s}(\bar{x}-y)[f(s,\bar{x}) - f(s,x)]\, dy\, \dot{W}\, ds,$$

$$I_4 = \int_0^t \int_{B_\delta(\xi)^c} (H^{(2)}_{t,\,s}(x-y) - H^{(2)}_{t,\,s}(\bar{x}-y))[f(s,y) - f(s,x)]\, dy \dot{W}\, ds.$$

By Lemmas 1 and 4,

$$|I_1|_p + |\,I_2|_p \le C[f]_{\beta,\,p} \left(\int_0^t \left(\int_{B_{3\delta/2}(x)} \left| H^{(2t)}_{t,\,s}(x-y) \right| |x-y|^\beta \, dy \right)^2 ds \right)^{1/2}$$

$$\le C[f]_{\beta,\,p} \int_{B_{3\delta/2}(x)} \left(\int_0^t H^{(2)}_{t,\,s}(x-y)^2 ds \right)^{1/2} |x-y|^\beta \, dy$$

$$\le C[f]_{\beta,\,p} C_1 \int_{B_{3\delta/2}(x)} |x-y|^{-d+\beta}\, dy \le CC_1 [f]_{\beta,\,p}\, \delta^\beta.$$

Again by Lemmas 1 and 4,

$$|I_3|_p \le C\delta^\beta [f]_{\beta,\,p} \left(\int_0^t \left(\int_{\partial B_\delta(\xi)} H^{(2)}_{t,\,s}(\bar{x}-y)\, dy \right)^2 ds \right)^{1/2}$$

$$\le CC_3 \delta^\beta [f]_{\beta,\,p}.$$

If $\hat{x}$ is an arbitrary point on the segment joining x and $\bar{x}$ and $|\xi - y| \geq \delta$, then $\frac{3}{2}|\xi - y| \geq |\hat{x} - y| \geq \frac{1}{2}|\xi - y|$. So by Lemmas 1 and 4 we achieve the estimate

$$\begin{aligned}|I_4|_p &\leq C\delta\,[f]_{\beta,\,p} \int_0^1 \left(\int_0^t \left(\int_{|\xi-y|\geq\delta} |\partial_x H^{(2)}_{t,\,s}(r\bar{x} + (1-r)x - y)| \right.\right.\\ &\qquad\left.\left. |x-y|^\beta dy \right)^2 ds \right)^{1/2} dr \\ &\leq C\delta[f]_{\beta,\,p} \int_0^1 \int_{|\xi-y|\geq\delta} \left(\int_0^t |\partial_x H^{(2)}_{t,\,s}(r\bar{x} + (1-r)x - y)|^2\,ds \right)^{1/2} \\ &\qquad |x-y|^\beta\,dy \\ &\leq C\delta\,[f]_{\beta,\,p} \int_{|\xi-y|\geq\delta} |\xi - y|^{-d-1+\beta}\,dy \leq CC_2\delta^\beta[f]_{\beta,\,p}. \qquad \square\end{aligned}$$

2.2. *Inequalities for the fundamental solution of heat equation.* Consider a heat equation in $(0, T) \times \mathbf{R}^d$,

$$\partial_t u = \tfrac{1}{2}\, a^{ij}_t\, \partial^2_{ij} u - \lambda u, \tag{7}$$

where $a = (a^{ij}_t)$, $t \in (0, T)$ is a measurable nonnegative symmetric matrix and $\lambda \geq 0$. The summation convention that repeated indices indicate summation from 0 to d is followed here as it will throughout. It will be assumed that there exist $\lambda_0, K > 0$ such that

$$|a| \leq K,\, a^{ij}_t\, \xi_i \xi_j \geq \lambda_0 |\xi|^2 \qquad \forall \xi = (\xi_1, \ldots, \xi_d) \in \mathbf{R}^d. \tag{8}$$

Let $A_{t,\,s} = (\int_s^t a^{ij}_r\, dr)_{1\leq i,\,j\leq d}$. The function

$$G^\lambda_{t,\,s}(x) = \frac{1}{(2\pi)^{d/2}(\det\, A_{t,\,s})^{1/2}} \exp\{-1/2\,(A^{-1}_{t,\,s}x, x) - \lambda\,(t-s)\}, \qquad s < t,$$

satisfies (7) for each x, $t > s$ dt-a.e. Obviously, $G^\lambda_{t,\,s}(x) = \exp -\lambda\,(t-s) G^0_{t,\,s}(x)$. Define

$$\begin{aligned} B^h_{t,\,s}(x) &= G^0_{t+h,\,s}(x) - G^0_{t,\,s}(x) \\ &= \int_t^{t+h} \frac{1}{2} a^{ij}_r \partial^2_{ij} G^0_{r,\,s}(x)\, dr. \end{aligned} \tag{9}$$

REMARK 1. *Let* (8) *be satisfied. Then*:

(a) *For $k \geq 0$ there exists a constant $C = C(\lambda_0, K, k, m, d)$ such that for each $t > s$, x,*

$$|\partial^k G^\lambda_{t,\,s}(x)| \leq C(t-s)^{-(d+k)/2} \exp\{-c|x|^2/(t-s)\}. \tag{10}$$

(b) *For $k = 0, 1$ there exists a constant $C = C(\lambda_0, K, k, m, d)$ such that for each $s < t$, $h > 0$,*

(11)
$$
\begin{aligned}
|\partial^k B^h_{t,s}(x)| &\le C\int_t^{t+h} (r-s)^{-(d+k+2)/2}\exp\{-c|x|^2/(r-s)\}\,dr\\
&\le C|x|^{-(d+k)}\big(F(|x|/\sqrt{t-s}) - F(|x|/\sqrt{t+h-s})\big),
\end{aligned}
$$

where $F(s)=\int_0^s r^{d+k-1}\exp\{-cr^2\}\,dr$.

REMARK 2. *Let* (8) *hold. Then*:

(a) *For* $k\ge 0, m\ge 1$, $m(d+k)>2$ *there exist a constant* $C=C(\lambda_0,K,k,m,d)$ *such that for all* $t\ge 0$,

(12)
$$
\begin{aligned}
\int_0^t |\partial^k G^\lambda_{t,r}(x)|^m\,dr &\le C|x|^{-m(d+k)+2}F(|x|/\sqrt{t})\\
&\le C|x|^{-m(d+k)+2},
\end{aligned}
$$

where $F(s)=\int_s^\infty r^{m(d+k)-3}\exp\{-cr^2\}\,dr$.

(b) *For* $k=0,1$ *there exists a constant* $C=C(\lambda_0,K,k,m,d)$ *such that for each* $s<t$, $h>0$,

(13)
$$
\int_0^t |\partial^k B^h_{t,s}(x)|^m\,ds \le C\min\{|x|^{-(d+k)m+1}h^{1/2},\ |x|^{-m(d+k)}h\}.
$$

PROOF. Indeed, we have by (10),

$$
\begin{aligned}
\int_0^t |\partial^k G^\lambda_{t,r}(x)|^m dr &\le C\int_0^t (t-s)^{-m(d+k)/2}\exp\{-c|x|^2/(t-s)\}\,ds\\
&\le C|x|^{-m(d+k)+2}F\big(|x|/\sqrt{t}\big)
\end{aligned}
$$

and (12) follows.

Now for $k=0,1,2$,

$$
\begin{aligned}
&\int_0^t |\partial^k B^h_{t,s}(x)|^m\,ds\\
&\quad\le C|x|^{-m(d+k)}\int_0^t\Big(F\big(|x|/\sqrt{t-s}\big)-F\big(|x|/\sqrt{t+h-s}\big)\Big)^m\,ds\\
&\quad\le C\min\bigg\{|x|^{-(d+k)m+1}\int_0^t\bigg(\frac{1}{\sqrt{t-s}}-\frac{1}{\sqrt{t+h-s}}\bigg)\,ds,\ |x|^{-m(d+k)}h\bigg\}
\end{aligned}
$$

and (13) follows. □

LEMMA 4. *Let* (8) *hold. Then*:

(a) *For* $m\ge 0$, $k\ge(2/m+1-d)\vee 1$, $\gamma\in(0,1)$ *there exist* $C=C(\lambda_0,K,k,\gamma,m,d)$ *such that for each* $s<t, a>0, |x|\le\gamma a$,

$$\begin{aligned}(14)\qquad &\int_s^t \left| \int_{|x+y|\geq a} \partial^k G^\lambda_{t,\,r}(y)\,dy \right|^m dr \\ &\quad = \int_s^t \left| \int_{|x+y|\leq a} \partial^k G^\lambda_{t,r}(x-y)\,dy \right|^m dr \leq C a^{2-mk}.\end{aligned}$$

(b) *For* $m = 1, 2$, $k = 0, 1$, $\gamma \in (0,1)$ *there exist* $C = C(\lambda_0, K, k, \gamma, m, d)$ *such that for each* $s < t$, $a > 0$, $|x| \leq \gamma a$,

$$\begin{aligned}(15)\qquad &\int_s^t \left| \int_{|x+y|\geq a} \partial^k B^h_{t,r}(y)\,dy \right|^m dr \\ &\quad = \int_s^t \left| \int_{|x+y|\leq a} \partial^k B^h_{t,r}(y)\,dy \right|^m dr \\ &\quad \leq C \min\{a^{1-km} h^{1/2},\ a^{-km} h\}.\end{aligned}$$

PROOF. (a) If $k \geq 1$, $\int \partial^k G^\lambda_{t,r}(y)\,dy = 0$. So,

$$(16)\qquad \left| \int_{|x+y|\leq a} \partial^k G^\lambda_{t,\,r}(y)\,dy \right| = \left| \int_{|x+y|\geq a} \partial^k G^\lambda_{t,\,r}(y)\,dy \right|.$$

Denoting $B(r)$ the right side of (16) we have by (12) (Remark 2),

$$\begin{aligned}(17)\qquad B(r) &\leq \left(\int_{|y|\leq a(1+\gamma)} |\partial^k G^\lambda_{t,\,r}(y)|\,dy \right)^m \\ &\leq CF((1+\gamma)a/\sqrt{t-r})^m (t-r)^{-mk/2},\end{aligned}$$

where $F(s) = \int_0^s \rho^{d-1} \exp\{-c\rho^2\}\,d\rho$. On the other hand, it follows from (16) and (12) (Remark 2),

$$\begin{aligned}(18)\qquad B(r) &\leq \left(\int_{|y|\geq (1-\gamma)a} |\partial^k G^\lambda_{t,\,r}(y)|\,dy \right)^m \\ &\leq C\tilde{F}((1-\gamma)a/\sqrt{t-r})^m (t-r)^{-mk/2},\end{aligned}$$

where $\tilde{F}(b) = F(\infty) - F(b)$. Let $\bar{F}(b) = \min\{\tilde{F}((1-\gamma)b),\ F((1+\gamma)b)\}$. So by (17), (18)

$$B(r) \leq C\bar{F}(a/\sqrt{t-r})^m (t-r)^{-mk/2}.$$

Thus,

$$\int_0^t B(r)\,dr \leq C \int_s^t \bar{F}(a/\sqrt{t-r})^m (t-r)^{-mk/2}\,dr.$$

Introducing a new variable of integration $\bar{r} = a/\sqrt{t-r}$ we have

$$\int_s^t B(r)\,dr \leq C a^{2-mk} \int_0^\infty \bar{F}(\bar{r})^m \bar{r}^{mk-3}\,d\bar{r}.$$

Since $\bar{F}(b) \leq Cb^d$, if $b \leq 1$, and $\bar{F}(b) \leq Ce^{-cb^2}$ for large b, the inequality follows.

(b) Since $\int \frac{1}{2} a_r^{ij} \partial^k \partial_{ij}^2 G_{r,s}^0(y)\,dy = 0$, the first equality in (15) follows. Denote

$$
\begin{aligned}
D(r) &= \left| \int_{|x+y|\le a} \tfrac{1}{2} a_r^{ij} \partial^k \partial_{ij}^2 G_{r,s}^0(y)\,dy \right| \\
&= \left| \int_{|x+y|\ge a} \tfrac{1}{2} a_r^{ij} \partial^k \partial_{ij}^2 G_{r,s}^0(y)\,dy \right|.
\end{aligned} \tag{19}
$$

By Remark 1 and using the first equality, we have

$$
\begin{aligned}
D(r) &\le C \int_{|y|\le(1+\gamma)a} (r-s)^{-(d+k+2)/2} \exp\{-c|y|^2/(r-s)\}\,dy \\
&\le C\bar{F}(a/\sqrt{r-s})(r-s)^{-(k+2)/2},
\end{aligned}
$$

where $\bar{F}(s) = \int_0^{(1+\gamma)s} \rho^{d-1} \exp(-c\rho^2)\,d\rho$.

On the other hand, by Remark 1 and the second equality in (19),

$$
\begin{aligned}
D(r) &\le C \int_{|y|\ge(1-\gamma)a} (r-s)^{-(d+k+2)/2} \exp\{-c|y|^2/(r-s)\}\,dy \\
&= C\tilde{F}(a/\sqrt{r-s})(r-s)^{-(k+2)/2},
\end{aligned}
$$

where $\tilde{F}(s) = \int_{(1-\gamma)s}^{\infty} \rho^{d-1} \exp(-c\rho^2)\,d\rho$. Let $F(s) = \min\{\bar{F}(s),\ \tilde{F}(s)\}$. Then

$$
\begin{aligned}
\left| \int_{|x+y|\ge a} \partial^k B_{t,r}^h(y)\,dy \right| = \left| \int_{|x+y|\le a} \partial^k B_{t,r}^h(y)\,dy \right| \\
\times C \int_t^{t+h} F(a/\sqrt{r-s})(r-s)^{-(k+2)/2}\,dr \\
\le Ca^{-k}[G(a/\sqrt{t-s}) - G(a/\sqrt{t+h-s})],
\end{aligned}
$$

where $G(s) = \int_0^s F(u)u^{k-1}\,du$ is a bounded and boundedly continuously differentiable function. Then for $m = 1, 2$,

$$
\begin{aligned}
&\int_0^t [G(a/\sqrt{t-s}) - G(a/\sqrt{t+h-s})]^m\,ds \\
&\qquad \le C\min\{ah^{1/2},\ h\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_s^t \left| \int_{|x+y|\ge a} \partial^k B_{t,r}^h(y)\,dy \right|^m dr &= \int_s^t \left| \int_{|x+y|\le a} \partial^k B_{t,r}^h(y)\,dy \right|^m dr \\
&\le C\min\{a^{1-km}h^{1/2},\ a^{-km}h\}. \qquad \square
\end{aligned}
$$

3. Linear equation with constant coefficients. Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a complete probability space with right continuous filtration of σ-algebras $\mathbf{F} = (\mathscr{F}_t)_{0\le t}$. Let W be a cylindrical Wiener process in a separable Hilbert space Y. This means that we have a family of continuous martingales $W_t(v)$, $v \in Y$, such that

$$
\langle W(v), W(v')\rangle_t = t\,(v, v')_Y \qquad \forall\, v, v' \in Y.
$$

For an $\mathbf{F}$-adapted Y-valued function f such that $\int_0^t |f_s|_Y^2\, ds < \infty$ $\mathbf{P}$-a.s. for all t, we can define Itô's stochastic integral, denoted

$$\int_0^t f_s\, dW_s = \int_0^t f_s \dot{W}_s ds.$$

It is a real-valued local martingale such that $\langle \int_0^{\bullet} f_s\, dW_s \rangle_t = \int_0^t |f_s|_Y^2\, ds$.

We start with the equation

$$\begin{cases} \partial_t u = 1/2\, a_t^{ij}\, \partial_{ij} u - \lambda u + f + g\dot{W}, & \text{in } D_T, \\ u(0, \cdot) = 0, & \text{in } D, \end{cases} \tag{20}$$

where a_t^{ij} are measurable deterministic functions on $[0, T]$, $\lambda \geq 0$ and $a^{ij} = a^{ji}$ for all $i,\ j$. The summation convention that repeated indices indicate summation from 0 to d is followed here as it will be throughout. It will be assumed in this section that the following condition holds.

A1. There exist $\lambda_0, K > 0$ such that

$$|a| \leq K, a_t^{ij} \xi_i \xi_j \geq \lambda_0 |\xi|^2 \quad \text{for each } \xi = (\xi_1, \ldots, \xi_d) \in \mathbf{R}^d \text{ and } t \in [0, T].$$

DEFINITION 5. Let $f \in B_{\text{loc}}^p(D_T)$, $g \in B_{\text{loc}}^p(D_T, Y)$. We say that (20) holds for $u \in C^{2,p}(D_T)$ or $u \in C^{2,p}(D_T)$ is a solution of (20) if for each $(t, x) \in D_T$ $\mathbf{P}$-a.s.,

$$u(t, x) = \int_0^t (\tfrac{1}{2} a_s^{ij}\, \partial_{ij}^2 u(s, x) - \lambda u(s, x) + f(s, x))\, ds + \int_0^t g(s, x)\, \dot{W}_s\, ds, \tag{21}$$

where $\int_0^t g(s, x)\, \dot{W}_s ds = \int_0^t g(s, x)\, dW_s$.

3.1. *Representation formula.* As a part of the preparation for the regularity and existence considerations, we will derive some representation formulas.

Let

$$G_{t,s}^{\lambda}(x) = \frac{1}{(2\pi)^{d/2} (\det A_{t,s})^{1/2}} \exp\{-1/2 (A_{t,s}^{-1} x, x) - \lambda (t - s)\}, \qquad s < t,$$

where $A_{t,s} = (\int_s^t a_r^{ij}\, dr)_{1 \leq i,\, j \leq d}$. The verification of the following relations for $s < t$ is straightforward:

$$\begin{aligned} \partial_t G_{t,s}^{\lambda}(x) &= \tfrac{1}{2} a_t^{ij}\, \partial_{ij}^2\, G_{t,s}^{\lambda}(x) - \lambda G_{t,s}^{\lambda}(x), \ dt\text{-a.e.}, \qquad t > s, \\ \partial_s G_{t,s}^{\lambda}(x) &= -\tfrac{1}{2}\, a_s^{ij}\, \partial_{ij}^2 G_{t,s}^{\lambda}(x) + \lambda G_{t,s}^{\lambda}(x), \ ds\text{-a.e.}, \qquad s < t. \end{aligned} \tag{22}$$

The following statement follows easily by the Itô formula, (22) and (21).

LEMMA 6. *Assume* (20) *holds for* $u \in C^{2,p}(D_T)$, $f \in B^p_{\mathrm{loc}}(D_T)$, $g \in B^p_{\mathrm{loc}}(D_T, Y)$. *Then for* $r < t$, $x, y \in D$, $t > 0$, $\varphi \in C^2(D)$ *we have* $\mathbf{P}$-*a.s.*,

$$
\begin{aligned}
&G^\lambda_{t,r}(x,y)u(r,y)\varphi(y)\\
&\quad=\int_0^r\Big[G^\lambda_{t,s}(x,y)f(s,y)\varphi(y)+G^\lambda_{t,s}(x,y)g(s,y)\varphi(y)\dot W_s\\
&\qquad+1/2\,a^{ij}_s\{\partial_{y_j}(\partial_i u(s,y)G^\lambda_{t,s}(x,y)\varphi(y))\\
&\qquad-\partial_{y_i}(u(s,y)\partial_{y_j}G^\lambda_{t,s}(x,y)\varphi(y))-\partial_{y_i}(u(s,y)G^\lambda_{t,s}(x,y)\partial_j\varphi(y))\\
&\qquad\quad+2u(s,y)\partial_{y_j}G^\lambda_{t,s}(x,y)\partial_i\varphi(y)+u(s,y)G^\lambda_{t,s}(x,y)\partial^2_{ij}\varphi(y)\}\Big]\,ds.
\end{aligned}
\tag{23}
$$

Here we derive a representation of a solution to (20) in D.

LEMMA 7. *Assume* (20) *holds for* $u \in C^{2,p}(D_T)$, $f \in B^p_{\mathrm{loc}}(D_T)$, $g \in B^p_{\mathrm{loc}}(D_T, Y)$. *Then for* $\varphi \in C^\infty_0(D)$ *such that* $\varphi(x) = 1$ *and* $t \ge 0$ $\mathbf{P}$-*a.s.*,

$$
\begin{aligned}
u(t,x)&=\int_0^t\int G^\lambda_{t,s}(x-y)(f(s,y)\varphi(y)\,dy\,ds\\
&\quad+\int_0^t\int G^\lambda_{t,s}(x-y)g(s,y)\varphi(y)\,dy\,\dot W_s\,ds\\
&\quad+1/2\int_0^t\int a^{ij}_s u(s,y)\{G^\lambda_{t,s}(x-y)\partial^2_{ij}\varphi(y)\\
&\quad-2\partial_j G^\lambda_{t,s}(x-y)\partial_i\varphi(y)\}\,dy\,ds.
\end{aligned}
$$

PROOF. Applying Lemma 6 to u, f, g, φ and integrating with respect to y, we get

$$
\begin{aligned}
\int G^\lambda_{t,r}(x-y)u(r,y)\varphi(y)\,dy=\int\int_0^r&[G^\lambda_{t,s}(x-y)f(s,y)\varphi(y)\\
&+G^\lambda_{t,s}(x-y)g(s,y)\varphi(y)\dot W_s\\
&+1/2\,a^{ij}_s u(s,y)\{G^\lambda_{t,s}(x-y)\partial^2_{ij}\varphi(y)\\
&-2\partial_j G^\lambda_{t,s}(x-y)\partial_i\varphi(y)\}]\,ds\,dy
\end{aligned}
\tag{24}
$$

for each $r < t$. We see immediately that

$$
\begin{aligned}
&\int_0^r\int G^\lambda_{t,s}(x-y)g(s,y)\varphi(y)\,dy\dot W_s\,ds\\
&\quad=\int\int_0^r G^\lambda_{t,s}(x-y)g(s,y)\varphi(y)\,\dot W_s\,ds\,dy.
\end{aligned}
\tag{25}
$$

Indeed, since $\varphi \in C_0^\infty(D)$ and $g \in B^p_{\mathrm{loc}}(D_T, Y)$ we have

$$(26)\quad \begin{aligned} &\sup_{s\le T, y} |g(s, y)\varphi(y)|^2_{2, Y} + \sup_{s\le T}\int |g(s, y)\varphi(y)|^2_{2,Y}\,dy \\ &\qquad \le \sup_{s\le T, y} |g(s, y)\varphi(y)|^2_{p, Y} + \sup_{s\le T}\int |g(s, y)\varphi(y)|^2_{p, Y}\,dy < \infty. \end{aligned}$$

Let $M_r(y) = \int_0^r G^\lambda_{t,s}(x-y)g(s, y)\varphi(y)\,\dot W_s\,ds$, and $N_r = \int_0^r \int G^\lambda_{t,s}(x-y)g(s, y)$ $\varphi(y)\,dy\dot W_s ds$. Inequality (26) guarantees that for each $yN_r, M_r(y)$, $\bar N_r = \int M_r(y)\,dy$ are well defined. We have

$$\begin{aligned} \mathbf{E}M_r(y)^2 &= \mathbf{E}\int_0^r |G^\lambda_{t,s}(x-y)g(s, y)\varphi(y)|^2_Y\,ds \\ &= \int_0^r G^\lambda_{t,s}(x-y)^2\mathbf{E}|g(s, y)|^2_Y\varphi(y)^2\,ds, \\ \mathbf{E}N_r^2 &= \mathbf{E}\int_0^r \int\Big|\int G^\lambda_{t,s}(x-y)g(s, y)\varphi(y)\,dy\Big|^2_Y\,ds \\ &= \int_0^r \int\int G^\lambda_{t,s}(x-y)\,G^\lambda_{t,s}(x-\bar y)\mathbf{E}\big(g(s,\bar y),\ g(s, y)\big)_Y \\ &\qquad\times \varphi(\bar y)\varphi(y)\,dy\,d\bar y\,ds. \end{aligned}$$

Since $M_r(y)$ has a compact support in y and $G^\lambda_{t,s}(x-y)$ is uniformly bounded for $s \in [0, r]\bar N_r$ is well defined. Also,

$$\begin{aligned} \mathbf{E}M_r(y)M_r(\bar y) &= \int_0^r G^\lambda_{t,s}(x-y)\,G^\lambda_{t,s}(x-\bar y)\mathbf{E}\big(g(s,\bar y),\ g(s, y)\big)_Y \\ &\qquad\times\varphi(\bar y)\varphi(y)\,ds, \\ \mathbf{E}\bar N_r^2 &= \mathbf{E}\Big(\int M_r(y)\,dy\Big)^2 \\ &= \int\int\int_0^r G^\lambda_{t,s}(x-y)\,G^\lambda_{t,s}(x-\bar y)\mathbf{E}\big(g(s,\bar y),\ g(s, y)\big)_Y \\ &\qquad\times\varphi(\bar y)\varphi(y)\,ds\,dy\,d\bar y, \\ \mathbf{E}\Big(\int M_r(y)\,dy\,N_r\Big) &= \int\int_0^r\int G^\lambda_{t,s}(x-y)\,G^\lambda_{t,s}(x-\bar y)\mathbf{E}\big(g(s,\bar y),\ g(s, y)\big)_Y \\ &\qquad\times\varphi(\bar y)\varphi(y)\,d\bar y\,ds\,dy. \end{aligned}$$

So, $\mathbf{E}(N_r - \bar N_r)^2 = 0$ and (25) holds. It means we can interchange integrals in (24). Passing to the limit as $r \to t$, we obtain the representation of $u(t, x)$. □

From this statement follows easily the uniqueness result for the Cauchy problem,

$$(27)\quad \begin{cases} \partial_t u = 1/2\,a_t^{ij}\partial_{ij}u - \lambda u + f + g\dot W, & \text{in } \mathbf{R}^d_T, \\ u(0, x) = 0, & \text{in } \mathbf{R}^d. \end{cases}$$

THEOREM 8. *Assume* (27) *holds for* $u \in C^{2,p}(\mathbf{R}_T^d) \cap B^p(\mathbf{R}_T^d)$, $f \in B^p(\mathbf{R}_T^d)$, $g \in B^p(\mathbf{R}_T^d, Y)$, $p \geq 2$. *Then for each* $(t, x) \in \mathbf{R}_T^d$ *we have* $\mathbf{P}$-*a.s.*,

$$u(t, x) = \int_0^t \int G^\lambda_{t,s}(x - y) f(s, y)\, dy\, ds + \int_0^t \int G^\lambda_{t,s}(x - y) g(s, y)\, dy\, \dot{W}_s\, ds.$$

PROOF. Let $\phi \in C_0^\infty(\mathbf{R}^d)$ be such that $0 \leq \phi \leq 1$, $\phi(x) = 1$, if $|x| \leq 1$, $\phi(x) = 0$, if $|x| \geq 2$. Define $\varphi_\varepsilon(x) = \phi(x\varepsilon)$, $\varepsilon > 0$ and apply Lemma 7. For $|x| \leq \varepsilon^{-1}$ we have, $\mathbf{P}$-a.s.,

$$\begin{aligned} u(t, x) = &\int_0^t \int G^\lambda_{t,s}(x - y) f(s, y)\varphi_\varepsilon(y)\, dy\, ds \\ &+ \int_0^t \int G^\lambda_{t,s}(x - y) g(s, y)\varphi_\varepsilon(y)\, dy\, \dot{W}_s\, ds \\ &+ 1/2 \int_0^t \int a_s^{ij} u(s, y)\{G^\lambda_{t,s}(x - y)\partial^2_{ij}\varphi_\varepsilon(y) \\ &- 2\partial_j G^\lambda_{t,s}(x - y)\partial_i\varphi_\varepsilon(y)\}\, dy\, ds. \end{aligned}$$

Using the boundedness of f, g, u we obtain the representation by passing to the limit, as $\varepsilon \to 0$. □

3.2. *Interior Hölder estimates.* Introduce the operators R^λ: $B^p(\mathbf{R}_T^d) \to B^p(\mathbf{R}_T^d)$, $\tilde{R}^\lambda$: $B^p(\mathbf{R}_T^d, Y) \to B^p(\mathbf{R}_T^d)$ defined by stochastic integrals,

$$R^\lambda f = R^\lambda f(t, x) = \int_0^t \int G^\lambda_{t,s}(x - y) f(s, y)\, dy\, ds,\ (t, x) \in \mathbf{R}_T^d,\quad f \in B^p(\mathbf{R}_T^d),\ p \geq 2,$$

$$\tilde{R}^\lambda g = \tilde{R}^\lambda g(t, x) = \int_0^t \int G^\lambda_{t,s}(x - y) g(s, y)\, dy\, \dot{W}_s ds,\ (t, x) \in \mathbf{R}_T^d,\quad g \in B^p(\mathbf{R}_T^d, Y),\ p \geq 2.$$

Indeed, for each $(t, x) \in \mathbf{R}_T^d$ applying Lemma 1 with $\mu_s(da) = G^\lambda_{t,s}(x - y)\, dy$, we have

$$|\tilde{R}^\lambda g(t, x)|_p \leq C \sup_{s,y} \|g(s, y)\|_p \left(\int_0^T \mu_s(\mathbf{R}^d)^2\, ds\right),$$

$$|R^\lambda f(t, x)|_p \leq C \sup_{s,y} |f(s, y)|_p \int_0^T \mu_s(\mathbf{R}^d)\, ds,$$

and obviously $\mu_s(\mathbf{R}^d) = \int G^\lambda_{t,s}(y)\, dy \leq C \int \exp\{-c|y|^2\}\, dy < \infty$.

Now we show that Hölder continuity in the $L_p(\Omega, \mathscr{F}, \mathbf{P})$-sense of f, g implies differentiability and Hölder continuity in the $L_p(\Omega, \mathscr{F}, \mathbf{P})$-sense of $R^\lambda f$ and $\tilde{R}^\lambda g$.

LEMMA 9. *Let* $f \in C^{\beta,p}(\mathbf{R}^d_T)$, $g \in C^{1+\beta,p}(\mathbf{R}^d_T, Y)$, $p \geq 2$. *Then* $R^\lambda f$, $\tilde{R}^\lambda g \in C^{2,p}(\mathbf{R}^d_T)$, *and for each* (t,x) $\mathbf{P}$-a.s.,

$$\partial^2_{ij} R^\lambda f(t,x) = \int_0^t \int \partial^2_{ij} G^\lambda_{t,s}(x-y)[f(s,y)-f(s,x)]\,dy\,ds,$$

$$\partial^2_{ij} \tilde{R}^\lambda g(t,x) = \int_0^t \int \partial_j G^\lambda_{t,s}(x-y)[\partial_i g(s,y)-\partial_i g(s,x)]\,dy\,\dot{W}_s\,ds,$$

$$\begin{aligned}\partial_i R^\lambda f(t,x) &= \int_0^t \int \partial_i G^\lambda_{t,s}(x-y)[f(s,y)-f(s,x)]\,dy\,ds\\ &= \int_0^t \int \partial_i G^\lambda_{t,s}(x-y) f(s,y)\,dy\,ds,\end{aligned}$$

$$\begin{aligned}\partial_i \tilde{R}^\lambda g(t,x) &= \int_0^t \int G^\lambda_{t,s}(x-y)[\partial_i g(s,y)-\partial_i g(s,x)]\,dy\,\dot{W}_s\,ds\\ &\quad + \int_0^t \partial_i g(s,x)\,\dot{W}_s\,ds = \int_0^t \int G^\lambda_{t,s}(x-y)\partial_i g(s,y)\,dy\,\dot{W}_s\,ds.\end{aligned}$$

PROOF. For $r < t$ we define

$$A(t,r,x) = \int_0^r \int G^\lambda_{t,s}(x-y) f(s,y)\,dy\,ds,$$

$$B(t,r,x) = \int_0^r \int G^\lambda_{t,s}(x-y) g(s,y)\,dy\,\dot{W}_s\,ds.$$

By virtue of the estimates (10) for each $k \geq 0$,

$$\partial^k A(t,r,x) = \int_0^r \int \partial^k G^\lambda_{t,s}(x-y) f(s,y)\,dy\,ds,$$

$$\partial^k B(t,r,x) = \int_0^r \int \partial^k G^\lambda_{t,s}(x-y) g(s,y)\,dy\,\dot{W}_s\,ds.$$

Since for $k \geq 1$, $\int \partial^k G^\lambda_{t,s}(x-y)\,dy = 0$ we see immediately that for each i, j,

$$\partial^k A(t,r,x) = \int_0^r \int \partial^k G^\lambda_{t,s}(x-y)(f(s,y)-f(s,x))\,dy\,ds,$$

$$\begin{aligned}\partial^2_{ij} B(t,r,x) = \int_0^r \int \partial_j G^\lambda_{t,s}(x-y)(\partial_i g(s,y)\\ - \partial_i g(s,x))\,dy\,\dot{W}_s\,ds.\end{aligned}$$

Also, obviously,

$$\begin{aligned}\partial_i B(t,r,x) &= \int_0^r \int \partial_i G^\lambda_{t,s}(x-y)\,g(s,y)\,dy\,\dot{W}_s\,ds\\ &= \int_0^r \int G^\lambda_{t,s}(x-y)(\partial_i g(s,y)-\partial_i g(s,x))\,dy\,\dot{W}_s\,ds\\ &\quad + \int_0^r \partial_i g(s,x)\,\dot{W}_s\,ds.\end{aligned}$$

Consider

$$\tilde{A}(t, v, x) = \int_v^t \int \partial_{ij}^2 G_{t,s}^{\lambda}(x-y)[f(s,y) - f(s,x)]\, dy\, ds,$$

$$\tilde{B}(t, v, x) = \int_v^t \int \partial_j G_{t,s}^{\lambda}(x-y)[\partial_i g(s,y) - \partial_i g(s,x)]\, dy\, \dot{W}_s\, ds.$$

By Lemmas 1 and 4 these functions are well defined and $|\tilde{A}(t,v,x)|_p \le C(t-v)^{\beta/2}$, $|\tilde{B}(t,v,x)|_p \le C(t-v)^{\beta/2}$.

Thus the first two equalities follow by uniform convergence:

$$\sup_x |A(t,r,x) - R^{\lambda} f(t,x)|_p + |B(t,r,x) - \tilde{R}^{\lambda} g(t,x)|_p \to 0 \quad \text{as } r \to t,$$

$$|\partial_{ij}^2 A(t,r,x) - \tilde{A}(t,0,x)|_p \le |\tilde{A}(t,r,x)|_p \le C(t-r)^{\beta/2} \to 0 \quad \text{as } r \to t,$$

$$|\partial_{ij}^2 B(t,r,x) - \tilde{B}(t,0,x)|_p \le |\tilde{B}(t,r,x)|_p \le C(t-r)^{\beta/2} \to 0 \quad \text{as } r \to t.$$

Similarly we prove that $\partial A(t,r,x)$, $\partial B(r,t,x)$ converge uniformly to the right-hand sides of the last two equalities of this lemma. So the statement follows. □

Now we can prove that the formula for u given in Theorem 8 defines a solution of the Cauchy problem (27).

THEOREM 10. *Let $f \in C^{\beta, p}(\mathbf{R}_T^d)$, $g \in C^{1+\beta, p}(\mathbf{R}_T^d, Y)$, $p \ge 2$. Then $u = R^{\lambda} f + \tilde{R}^{\lambda} g$ is a solution of the Cauchy problem* (27).

PROOF. Denote for $s < t$,

$$J(t,s,x) = \int G_{t,s}^{\lambda}(x-y) f(s,y)\, dy,$$

$$\tilde{J}(t,s,x) = \int G_{t,s}^{\lambda}(x-y) g(s,y)\, dy.$$

Then for $s < t$, $k \ge 0$ we have $\mathbf{P}$-a.s.,

$$\partial^k J(t,s,x) = \int \partial^k G_{t,s}^{\lambda}(x-y) f(s,y)\, dy,$$

$$\partial^k \tilde{J}(t,s,x) = \int \partial^k G_{t,s}^{\lambda}(x-y) g(s,y)\, dy.$$

By the estimates (12), Remark 1,

$$\begin{aligned} \partial_{ij}^2 J(t,s,x) &= \int \partial_{ij}^2 G_{t,s}^{\lambda}(x-y)[f(s,y) - f(s,x)]\, dy\, ds, \\ \partial_{ij}^2 \tilde{J}(t,s,x) &= \int \partial_j G_{t,s}^{\lambda}(x-y)[\partial_i g(s,y) - \partial_i g(s,x)]\, dy. \end{aligned} \tag{28}$$

By Lemma 1, and 4 and (28),

$$\begin{aligned}(29)\qquad &\int_s^t |\partial^2 J(r,s,x)|_p\,dr \le C|f|_{\beta,p}(t-s)^{\beta/2},\\ &\int_s^t \|\partial^2_{ij}\tilde J(r,s,x)\|_p^2\,dr \le C[\partial g]^2_{\beta,p}(t-s)^{\beta}.\end{aligned}$$

Then for each $s<t$,

$$J(t,s,x) = f(s,x) + \int_s^t (1/2a_r^{ij}\partial^2_{ij}J(r,s,x) - \lambda J(r,s,x))\,dr,$$

$$\tilde J(t,s,x) = g(s,x) + \int_s^t (1/2a_r^{ij}\partial^2_{ij}\tilde J(r,s,x) - \lambda \tilde J(r,s,x))\,dr.$$

Since $u(t,x) = \int_0^t J(t,s,x)\,ds + \int_0^t \tilde J(t,s,x)\,\dot W\,ds$, we have for each t **P**-a.s.,

$$\begin{aligned}u(t,x) = &\int_0^t f(s,x)\,ds + \int_0^t g(s,x)\dot W_s\,ds\\ &+\int_0^t\int_s^t 1/2a_r^{ij}\partial^2_{ij}J(r,s,x)\,dr\,ds + \int_0^t\int_s^t 1/2a_r^{ij}\partial^2_{ij}\tilde J(r,s,x)\,dr\,\dot W ds\\ &-\lambda\int_0^t\int_s^t J(r,s,x)\,dr\,ds - \lambda\int_0^t\int_s^t \tilde J(r,s,x)\,dr\,\dot W_s\,ds.\end{aligned}$$

By (29) and stochastic Fubini's theorem,

$$\begin{aligned}u(t,x) = &\int_0^t f(s,x)\,ds + \int_0^t g(s,x)\,\dot W_s\,ds\\ &+\int_0^t 1/2a_r^{ij}\int_0^r \partial^2_{ij}J(r,s,x)\,ds\,dr + \int_0^t 1/2a_r^{ij}\int_0^r \partial^2_{ij}\tilde J(r,s,x)\,\dot W ds\,dr\\ &-\lambda\int_0^t\int_0^r J(r,s,x)\,ds\,dr - \lambda\int_0^t\int_0^r \tilde J(r,s,x)\,\dot W ds\,dr.\end{aligned}$$

Now the statement follows immediately by (28) and Lemma 9. □

The following two lemmas are crucial for the Cauchy problem and interior Hölder estimates.

LEMMA 11. *Let $f \in C^{\beta,p}(\mathbf{R}^d_T)$, $g \in C^{1+\beta,p}(\mathbf{R}^d_T, Y)$, $p \ge 2$. Then $R^\lambda f, \tilde R^\lambda g \in C^{2+\beta,p}(\mathbf{R}^d_T)$ and*

$$\begin{aligned}[R^\lambda f]_{2+\beta,p} &\le C[f]_{\beta,p},\\ [\tilde R^\lambda g]_{2+\beta,p} &\le C[\partial g]_{\beta,p},\\ |R^\lambda f|_{0,p} &\le (1/\lambda\wedge T)\,|f|_{0,p},\\ |\tilde R^\lambda g|_{0,p} &\le C(1/\sqrt{\lambda}\wedge\sqrt{T})\|g\|_{0,p}.\end{aligned}$$

PROOF. (i) Estimate of $|\tilde{R}^\lambda g|_{0,p}$, $|R^\lambda f|_{0,p}$. By Lemma 1,

$$|\tilde{R}^\lambda g(t,x)|_p \le C\left(\int_0^t \left(\int G^\lambda_{t,s}(x-y)|g(s,y)|_p\,dy\right)^2 ds\right)^{1/2}$$

$$\le C|g|_{0,p}\left(\int_0^t \exp(-2\lambda(t-s))\,ds\right)^{1/2},$$

$$|R^\lambda f(t,x)|_p \le \int_0^t \int G^\lambda_{t,s}(x-y)|f(s,y|_p\,dy\,ds$$

$$\le |f|_{0,p}\int_0^t \exp(-\lambda(t-s))\,ds.$$

(ii) Estimates of $[R^\lambda f]_{2+\beta,p}$, $[\tilde{R}^\lambda g]_{\beta,p}$. Denoting $w = \partial^2_{ij}R^\lambda f$, $\tilde{w} = \partial^2_{ij}\tilde{R}^\lambda g$, we have by Lemma 9 for any x, t,

$$w(t,x) = \int_0^t \int \partial^2_{ij}G^\lambda_{t,s}(x-y)[f(s,y)-f(s,x)]\,dy\,ds,$$

$$\tilde{w}(t,x) = \int_0^t \int \partial_j G^\lambda_{t,s}(x-y)[\partial_i g(s,y)-\partial_i g(s,x)]\,dy\,\dot{W}_s\,ds.$$

Now the statement follows by Lemma 3, Remark 2 and Lemma 4. □

LEMMA 12. *Let $B' = B_R(x_0)$, $B = B_{2R}(x_0)$ be two concentric balls, $f \in C^{\beta,p}(\mathbf{R}^d_T)$, $g \in C^{1+\beta,p}(\mathbf{R}^d_T, Y)$, $p \ge 2$, be such that $f(t,x) = 0$, $g(t,x) = 0$ if $x \notin B$, $0 \le t \le T$. Then*

$$|\partial^2 R^\lambda f|_{0,p;B'_T} \le C(R^\beta[f]_{\beta,p;B_T} + |f|_{0,p;B'_T}),$$

$$|\partial^2 \tilde{R}^\lambda g|_{0,p;B'_T} \le C(R^\beta[\partial g]_{\beta,p;B_T} + |\partial g|_{0,p;B'_T}).$$

PROOF. By Lemma 9 for $(t,x) \in B'_T$ we have $\partial^2 R^\lambda f(t,x) = I_1 + I_2$, where

$$I_1 = \int_0^t \int_B \partial^2_{ij}G^\lambda_{t,s}(x-y)(f(s,y)-f(s,x))\,dy\,ds,$$

$$I_2 = -\int_0^t \int_{B^c} \partial^2_{ij}G^\lambda_{t,s}(x-y)f(s,x)\,dy\,ds.$$

By Lemmas 1 and 4,

$$|I_1|_p \le CR^\beta[f]_{\beta,p;B_T},$$

$$|I_2|_p \le C|f|_{0,p;B'}\int_0^t \left|\int_{B^c} \partial^2_{ij}G^\lambda_{t,s}(x-y)\,dy\right| ds \le C|f|_{0,p;B'}.$$

By Lemmas 9 for $(t,x) \in B'_T$ we have $\partial^2 \tilde{R}^\lambda g(t,x) = I_1 + I_2$, where

$$I_1 = \int_0^t \int_B \partial_j G^\lambda_{t,s}(x-y)(\partial_i g(s,y)-\partial_i g(s,x))\,dy\,\dot{W}_s\,ds,$$

$$I_2 = -\int_0^t \int_{B^c} \partial_j G^\lambda_{t,s}(x-y)\partial_i g(s,x)\,dy\,\dot{W}_s\,ds.$$

By Lemmas 1 and 4,

$$|I_1|_p \le C[\partial g]_{\beta,\, p;\, B_T}\left(\int_0^t \left(\int_{|x-y|\le 3R} \left|\partial_j G^{\lambda}_{t,\, s}(x-y)\right| |x-y|^{\beta} dy\right)^2 ds\right)^{1/2}$$
$$\le C R^{\beta}[\partial g]_{\beta,\, p;\, B_T},$$
$$|I_2|_p \le C[\partial g]_{0,\, p;\, B'_T}\left(\int_0^t \left(\int_{B^c} \partial_j G^{\lambda}_{t,s}(x-y)\, dy\right)^2 ds\right)^{1/2} \le C[\partial g]_{0,\, p;\, B'_T}. \qquad \square$$

Using the interior representation formula and the last two lemmas, we prove now the basic result needed for the interior Hölder estimates.

LEMMA 13. *Let* (20) *hold for* $u \in C^{2,\, p}(D_T)$, $f \in C^{\beta, p}(D_T)$, $g \in C^{\beta,\, p}(D_T, Y)$ *and* $p \ge 2$. *Let* $B^1 = B_R(x_0)$, $B^2 = B_{2R}(x_0)$ *be concentric balls in* D, $\bar{B}_2 \subset D$. *Then*

$$|\partial^2 u|_{0,\, p;\, B^1_T} \le C\Big(R^{-2}|u|_{0,\, p;\, B^2_T} + R^{\beta}[f]_{\beta,\, p;\, B^2_T} + |\, f\,|_{0,\, p;\, B^2_T} + R^{\beta}[\partial g]_{\beta,\, p;\, B^2_T} + \|\partial g\|_{0,\, p;\, B^2_T} + R^{-1+\beta}[g]_{\beta,\, p;\, B^2_T} + R^{-1}\|g\|_{0,\, p;\, B^2_T}\Big),$$

$$[\partial^2 u]_{\beta,\, p;\, B^1_T} \le C\Big(R^{-2-\beta}|u|_{0,\, p;\, B^2_T} + [f]_{\beta,\, p;\, B^2_T} + R^{-\beta}|\, f\,|_{0,\, p;\, B^2_T} + [\partial g]_{\beta,\, p;\, B^2_T} + R^{-\beta}\|\partial g\|_{0,\, p;\, B^2_T} + R^{-1}[g]_{\beta,\, p;\, B^2_T} + R^{-1-\beta}\|g\|_{0,\, p;\, B^2_T}\Big).$$

PROOF. Fix $\phi \in C_0^{\infty}(\mathbf{R}^d)$ such that $0 \le \phi \le 1$, $\phi(x) = 1$, if $|x| \le 3/2$, $\phi(x) = 0$, if $|x| \ge 2$. Define $\varphi(x) = \phi((x - x_0)/R)$ for some $x_0 \in D$. Obviously for each $k \ge 0$,

$$|\partial^k \varphi(x)| \le C(k) R^{-k}. \tag{30}$$

By Lemma 7 for each $(t, x) \in B^1_T$ $\mathbf{P}$-a.s.,

$$\begin{aligned} u(t, x) = &\int_0^t \int G^{\lambda}_{t,\, s}(x-y) f(s, y)\varphi(y)\, dy\, ds \\ &+ \int_0^t \int G^{\lambda}_{t,\, s}(x-y) g(s, y)\varphi(y)\, dy\, \dot{W}_s\, ds \\ &+ 1/2 \int_0^t \int a^{ij}_s u(s, y)\{G^{\lambda}_{t,\, s}(x-y)\partial^2_{ij}\varphi(y) \\ &- 2\partial_j G^{\lambda}_{t,\, s}(x-y)\partial_i \varphi(y)\}\, dy\, ds. \end{aligned}$$

Denote the last integral $E(t, x)$. We have by estimates on B_1 for each $k \ge 0$,

$$\partial^k E(t, x) = 1/2 \int_0^t \int a^{ij}_s u(s, y)\big\{\partial^k G^{\lambda}_{t,s}(x-y)\partial^2_{ij}\varphi(y) - 2\partial^k \partial_j G^{\lambda}_{t,s}(x-y)\partial_i \varphi(y)\big\}\, dy\, ds.$$

So by Lemma 1, (30) and Lemma 4 for each $x \in B_1$, $k \geq 2$,

$$|\partial^k E(t,x)|_p \leq C\,|u|_{0,\,p;\,B_T^2} \int_0^t \int_{R/2 \leq |x-y| \leq 3R} \big(|\partial^k G^\lambda_{t,\,s}(x-y)| R^{-2} + |\partial^k \partial_j G^\lambda_{t,\,s}(x-y)| R^{-1}\big) \times dy\,ds \leq CR^{-k} |u|_{0,\,p;\,B_T^2}.$$

We have obviously

$$[\partial^2 E]_{\beta,\,p} \leq CR^{-2-\beta} |u|_{0,\,p;\,B_T^2}.$$

Now the inequalities follow by Lemmas 11 and 12. Indeed, denoting $w = \partial^2 R^\lambda (f\varphi)$, $\tilde{w} = \partial^2 \tilde{R}^\lambda (g\varphi)$ we have

$$\begin{aligned}
[w]_{\beta,\,p;\,B_T^1} &\leq C[f\varphi]_{\beta,\,p;\,B_T^2} \leq C([f]_{\beta,\,p;\,B_T^2} + R^{-\beta} |f|_{0,\,p;\,B_T^2}),\\
|w|_{0,\,p;\,B_T^1} &\leq C(R^\beta [f\varphi]_{\beta,\,p;\,B_T^2} + |f\varphi|_{0,\,p;\,B_T^2})\\
&\leq C(R^\beta [f]_{\beta,\,p;\,B_T^2} + |f|_{0,\,p;\,B_T^2}),\\
[\tilde{w}]_{\beta,\,p;\,B_T^1} &\leq C[\partial(g\varphi)]_{\beta,\,p;\,B_T^2}\\
&\leq C([\partial g]_{\beta,\,p;\,B_T^2} + R^{-\beta} \|\partial g\|_{0,\,p;\,B_T^2} + R^{-1}[g]_{\beta,\,p;\,B_T^2} + R^{-1-\beta} \|g\|_{0,\,p;\,B_T^2}),\\
|\tilde{w}|_{0,\,p;\,B_T^1} &\leq C(R^\beta [\partial(g\varphi)]_{\beta,\,p;\,B_T^2} + |\partial(g\varphi)|_{0,\,p;\,B_T^2})\\
&\leq CR^\beta [\partial g]_{\beta,\,p;\,B_T^2} + \|\partial g\|_{0,\,p;\,B_T^2} + R^{-1+\beta}[g]_{\beta,\,p;\,B_T^2} + R^{-1}\|g\|_{0,\,p;\,B_T^2}.
\end{aligned}$$

Now the statement follows. □

For $x, y \in D$ let us write $d_x = \operatorname{dist}(x, \partial D) \wedge 1$, $d_{x,\,y} = d_x \wedge d_y$. We define for $u \in C^{m,\,p}(D_T, Y)$, $C^{m+\beta,\,p}(D_T, Y)$ the following quantities which are analogous of the global Hölder seminorms and norms:

$$\begin{aligned}
\|u\|^*_{m,p} = \|u\|^*_{m,\,p;\,D_T} &= \|u\|_{0,\,p;\,D_T} + [u]^*_{m,\,p;\,D_T}\\
&= \|u\|_{0,\,p;\,D_T} + \sup_{D_T} d_x^m |\partial^m u(t,x)|_p,\\
\|u\|^*_{m+\beta,p} = \|u\|^*_{m+\beta,\,p;\,D_T} &= \|u\|^*_{m,\,p} + [u]^*_{m+\beta,p},
\end{aligned}$$

where

$$\begin{aligned}
[u]^*_{m+\beta,p} &= [\partial^m u]^*_{\beta,p}\\
&= \sup_{t,\,x\neq y} \frac{\|\partial^m u(t,x) - \partial^m u(t,y)\|_p d_{x,y}^{m+\beta}}{|x-y|^\beta}.
\end{aligned}$$

If $Y = \mathbf{R}$, we write simply $|\cdot|^*$ instead of $\|\cdot\|^*$. We note that $\|u\|^*_{m,\,p;\,D_T}$, $\|u\|^*_{m+\beta,\,p;\,D_T}$ are norms on the subspaces of $C^{m,\,p}(D_T)$, $C^{m+\beta,\,p}(D_T)$, respectively, for which they are finite. It is convenient here to also introduce the quantities

$$\|g\|^{(k)}_{m,\,p;\,D_T} = \sum_{j=0}^{m} \sup_{D_T} d_x^{j+k}\|\partial^j g(t,x)\|_p,$$

$$\|g\|^{(k)}_{m+\beta,\,p;\,D_T} = \|g\|^{(k)}_{m,\,p;\,D_T} + \sup_{t,x\neq y} d_{x,y}^{m+k+\beta}\|\partial^m g(t,x) - \partial^m g(t,y)\|_p/|x-y|^\beta$$

$$= \|g\|^{(k)}_{m,\,p;\,D_T} + [g]^{(k)}_{m,\,p;\,D_T}.$$

If $Y = \mathbf{R}$, we use $|\cdot|^{(k)}$ instead of $\|\cdot\|^{(k)}$ again.

We notice also that if $D = \mathbf{R}^d$, then $d_x = 1$ for each $x \in D$ and $\|u\|^*_{m+\beta,\,p} = \|u\|_{m+\beta,\,p}$ $\|u\|^*_{m,\,p} = \|u\|_{m,\,p}$,$[u]^*_{m+\beta,\,p} = [u]_{m+\beta,\,p}$, $\|g\|^{(k)}_{m,\,p} = \|g\|_{m,\,p}$, $\|g\|^{(k)}_{m+\beta,\,p} = \|g\|_{m+\beta,\,p}$.

THEOREM 14. *Let* (20) *hold for* $u \in C^{2,\,p}(D_T)$, $f \in C^{\beta,\,p}(D_T)$, $g \in C^{1+\beta,\,p}(D_T, Y)$, $p \geq 2$. *Then*

$$|u|^*_{2+\beta,\,p;\,D_T} \leq C(|u|_{0,\,p;D_T} + |\,f\,|^{(2)}_{\beta,\,p;\,D_T} + \|g\|^{(1)}_{1+\beta,\,p;\,D_T}),$$

where $C = C(\lambda_0, K)$ *and* λ_0, K *are constants in Assumption* A1.

PROOF. For $x \in D$, $R = \frac{1}{3}d_x$, $B^1 = B_R(x)$, $B^2 = B_{2R}(x)$, we have by Lemma 13,

$$\begin{aligned} d_x^2|\partial^2 u(t,x)|_p &\leq (3R)^2|\partial^2 u|_{0,\,p;\,B^1_T} \\ &\leq C\big(|u|_{0,\,p;\,B^2_T} + R^2|f|_{0,\,p;\,B^2_T} + R^{2+\beta}[f]_{\beta,\,p;\,B^2_T} \\ &\quad + R^{2+\beta\beta}[\partial g]_{\beta,\,p;\,B^2_T} + R^2\|\partial g\|_{0,\,p;\,B^2_T} + R^{1+\beta}[g]_{\beta,\,p;\,B^2_T} \\ &\quad + R\|g\|_{0,\,p;\,B^2_T}\big) \\ &\leq C(|u|_{0,\,p;\,B^2_T} + |f|^{(2)}_{\beta,\,p;\,B^2_T} + \|g\|^{(1)}_{1+\beta,\,p;\,D_T}). \end{aligned}$$

Hence we obtain

$$|u|^*_{2,\,p;\,D_T} \leq C(|u|_{0,\,p;\,D_T} + |\,f\,|^{(2)}_{\beta,\,p;\,D_T} + \|g\|^{(1)}_{1+\beta,\,p;\,D_T}).$$

To estimate $[u]^*_{2+\beta,\,p;\,D_T}$ we let $x,\ y \in D$ with $d_x \leq d_y$. Then by Lemma 13,

$$\begin{aligned} &d_{x,\,y}^{2+\beta}|\partial^2 u(t,x) - \partial^2 u(t,y)|_p/|x-y|^\beta \\ &\quad\leq (3R)^{2+\beta}[\partial^2 u]_{\beta,\,p;\,B^1_T} \\ &\qquad + 3^\beta(3R)^2(|\partial^2 u(t,x)|_p + |\partial^2 u(t,y)|_p) \\ &\quad\leq C\big(|u|_{0,\,p;\,B^2_T} + R^{2+\beta}[f]_{\beta,\,p;\,B^2_T} + R^2|\,f\,|_{0,\,p;\,B^2_T} \\ &\qquad + R^{2+\beta}[\partial g]_{\beta,\,p;\,B^2_T} + R^2\|\partial g\|_{0,\,p;\,B^2_T} + R^{1+\beta}[g]_{\beta,\,p;\,B^2_T} + R\|g\|_{0,\,p;\,B^2_T}\big) \\ &\qquad + 6[u]^*_{2,\,p;\,D_T} \leq C(|u|_{0,\,p;\,D_T} + |\,f\,|^{(2)}_{\beta,p;D_T} + \|g\|^{(1)}_{1+\beta,\,p;\,D_T}). \end{aligned}$$

The estimate now follows. □

We finish this section with a result concerning a regularity of $R^\lambda f$, $\tilde{R}^\lambda g$ in time variable.

LEMMA 15. *Let* $f \in C^{\beta,p}(\mathbf{R}^d_T)$, $g \in C^{1+\beta,p}(\mathbf{R}^d_T, Y)$, $p \geq 2$, $k \leq 1$. *Then there exists a constant* $C = C(\lambda_0, \lambda, K, k, \beta, T)$ *such that for each* $(t, x) \in \mathbf{R}^d_T$, $h > 0, t + v \leq T$,

$$|R^\lambda f(\cdot + v, \cdot) - R^\lambda f(\cdot, \cdot)|_{1+\beta,\, p; T-v} + |\tilde{R}^\lambda g(\cdot + v, \cdot) - \tilde{R}^\lambda g(\cdot, \cdot)|_{1+\beta,\, p; T-v}$$
$$\leq C v^{1/2}(|f|_{\beta,\, p; T} + \|g\|_{1+\beta,\, p; T}).$$

PROOF. Since $R^\lambda f = e^{-\lambda t} R^0 f$, $\tilde{R}^\lambda g = e^{-\lambda t}\tilde{R}^0 g$, it is enough to consider the case $\lambda = 0$. If $k = 0$, the inequality follows immediately by Theorem 10 and Lemma 11. If $k = 1$, we use the representations of $\partial R^0 f$, $\partial \tilde{R}^0 g$ given by Lemma 9. According to it,

$$\begin{aligned} w(t, x) &= \partial R^0 f(t + v, x) - \partial R^0 f(t, x) \\ &= \int_0^t \int (\partial G^0_{t+v,\, s}(x - y) - \partial G^0_{t,\, s}(x - y))(f(s, y) - f(s, x))\, dy\, ds \\ &\qquad + \int_t^{t+v} \int \partial G^0_{t+v,\, s}(x - y)\, (f(s, y) - f(s, x)) dy\, ds\, A_1(t, x) \\ &\qquad + A_2(t, x). \end{aligned}$$

Also by Lemmas 9 and 1,

$$\begin{aligned} \tilde{w}(t, x) &= \partial \tilde{R}^0 g(t + v, x) - \partial \tilde{R}^0 g(t, x) \\ &= \int_0^t \left(\int (G^0_{t+v,\, s}(x - y) - G^0_{t,\, s}(x - y))\, (\partial g(s, y) - \partial g(s, x)\right) dy\, ds \\ &\qquad + \int_t^{t+v} \int G^0_{t+v,\, s}(x - y)\, (\partial g(s, y) - \partial g(s, x))\, dy\, ds \\ &\qquad + \int_t^{t+v} \partial g(s, x)\, \dot{W}\, ds \\ &= B_1(t, x) + B_2(t, x) + B_3(t, x). \end{aligned}$$

Applying Lemmas 3 and 4 and Remark 2, we obtain

$$[A_1]_{\beta,\, p} \leq C v^{1/2} [f]_{\beta,\, p; T},$$
$$[B_1]_{\beta,\, p} \leq C v^{1/2} [\partial g]_{\beta,\, p; T}.$$

For $k = 0, 1, 2$, $m = 1, 2$, we have for all t, v (see Remark 1),

$$\begin{aligned}\int_t^{t+v} |\partial^k G^\lambda_{t+v,\,s}(x)|^m\,ds \\ \le C|x|^{-m(d+k)+m-2}\int_t^{t+v}(t+v-s)^{-(2-m)/2}\,ds \\ \le C|x|^{-m(d+k)+m-2}v^{m/2}.\end{aligned} \tag{31}$$

Also, for $k = 0, 1$, $m = 1, 2$, $|x| \le \gamma a$ we have

$$\begin{aligned}&\int_t^{t+v}\left|\int_{|x+y|\ge a}\partial^k G^\lambda_{t+v,\,s}(y)\,dy\right|^m ds\\ &\le C\int_t^{t+v}\left|\int_{|y|\ge(1-\gamma)a}(t+v-s)^{-(d+k)/2}\exp\{-c|x^2|/(t+v-s)\}\,dy\right|^m ds\\ &\le C\int_t^{t+v}(t+v-s)^{-mk/2}\,ds \le Cv^{m/2}.\end{aligned} \tag{32}$$

By Lemma 3 and inequalities (31), (32),

$$[A_2]_{\beta,p} \le Cv^{1/2}[f]_{\beta,p;T},$$
$$[B_2]_{\beta,\,p} \le Cv^{1/2}[\partial g]_{\beta,\,p;\,T}.$$

Estimating directly,

$$[B_3]_{\beta,\,p} \le Cv^{1/2}[\partial g]_{\beta,\,p;\,T}.$$ □

4. Linear equation with variable coefficients. Throughout this section we consider the equation

$$\begin{cases}\partial_t u = 1/2\,a^{ij}\,\partial_{ij}u + b^i\partial_i u + cu - \lambda u + f + (hu+g)\dot W, & \text{in } D_T,\\ u(0,x) = 0, & \text{in } D,\end{cases} \tag{33}$$

where $\lambda \ge 0$ and $a = (a^{ij})$ is symmetric. It will be assumed in this section that a^{ij} are deterministic measurable locally bounded functions on D_T, and b^i, c are $L_\infty(\Omega, \mathbf{P})$-valued, $\mathbf{F}$-adapted locally bounded functions on D_T, h is $L_\infty(\Omega, Y, \mathbf{P})$-valued $\mathbf{F}$-adapted locally bounded function on D_T.

DEFINITION 16. Let $f \in B^p_{\mathrm{loc}}(D_T)$, $g \in B^p_{\mathrm{loc}}(D_T, Y)$. We say that (33) holds for $u \in C^{2,\,p}(D_T)$ or u is a solution of (33) if for each $(t,x) \in D_T$ we have $\mathbf{P}$-a.s.,

$$\begin{aligned}u(t,x) = \int_0^t\big[&1/2\,a^{ij}(s,x)\,\partial_{ij}u(s,x) + b^i(s,x)\partial_i u(s,x) + c(s,x)\,u(s,x)\\ &- \lambda u(s,x) + f(s,x) + (h(s,x)\,u(s,x) + g(s,x))\dot W_s\big]\,ds.\end{aligned}$$

Our first objective now is the derivation of the Schauder interior estimates for the solutions of (33).We introduce the following additional interior seminorms and norms in the spaces $C^{m,p}(Y, D_T)$, $C^{m+\beta,p}(Y, D_T)$. For τ a real number and m a nonnegative integer we define

$$[V]^{(\tau)}_{m,p;D_T} = \sup_{D_T} d_x^{m+\tau} \|\partial^m V(t,x)\|_p,$$

$$[V]^{(\tau)}_{m+\beta,p;D_T} = \sup_{t,\, x\neq y} d_{x,y}^{\tau+m+\beta} \frac{\|\partial^m V(t,x) - \partial^m V(t,y)\|_p}{|x-y|^\beta},\ 0 < \beta \le 1,$$

$$\|V\|^{(\tau)}_{m,p;D_T} = [V]^{(\tau)}_{0,p;D_T} + [V]^{(\tau)}_{m,p;D_T},$$

$$\|V\|^{(\tau)}_{m+\beta,p;D_T} = \|V\|^{(\tau)}_{m,p;D_T} + [V]^{(\tau)}_{m+\beta,p;D_T}.$$

In particular, we notice

$$\|V\|^{(1)}_{1+\beta,p;D_T} = \|\partial V\|^{(2)}_{\beta,p;D_T} + \|V\|^{(1)}_{0,p;D_T}. \tag{34}$$

Indeed,

$$\|V\|^{(1)}_{1+\beta,p;D_T} = \|V\|^{(1)}_{0,p;D_T} + \|\partial V\|^{(2)}_{0,p;D_T} + [\partial V]^{(2)}_{\beta,p;D_T}.$$

If $Y = \mathbf{R}$ we write $|\cdot|$ instead of $\|\cdot\|$. Here again we notice that if $D = \mathbf{R}^d$ ($d_x = 1$ in this case) we have simply the usual global Hölder norms.

REMARK 3. *Let $F \in C^{\beta,\infty}(D_T)$, $G \in C^{\beta,p}(Y, D_T)$. It is easy to verify that*

$$\begin{aligned} [FG]^{(\tau+\tau')}_{\beta,p;D_T} &\le [F]^{(\tau)}_{\beta,\infty;D_T}[G]^{(\tau')}_{0,p;D_T} + [F]^{(\tau)}_{0,\infty;D_T}[G]^{(\tau')}_{\beta,p;D_T}, \\ [FG]^{(\tau+\tau')}_{0,p;D_T} &\le [F]^{(\tau)}_{0,\infty;D_T}[G]^{(\tau')}_{0,p;D_T}, \\ \|FG\|^{(\tau+\tau')}_{\beta,p;D_T} &\le |F|^{(\tau)}_{\beta,\infty;D_T}\|G\|^{(\tau')}_{\beta,p;D_T}. \end{aligned} \tag{35}$$

Let $\varepsilon \in (0, 1/2)$. Fix $\phi \in C_0^\infty(\mathbf{R}^d)$ such that $0 \le \phi \le 1$, $\phi(x) = 1$, if $|x| \le 1$ and $\phi(x) = 0$, if $|x| \ge 2$. For $z \in D$ define $\eta^z = \eta^{z,\varepsilon}(x) = \phi((x-z)/\varepsilon d_z)$. So $\eta^z(x) = 0$, if $x \notin B^z = B_{2\varepsilon d_z}(z)$.

REMARK 4. (a) *One can notice easily that for each integer m there exists $C = C(m, d)$ such that for $\tau \ge 0$, $\varepsilon > 0$,*

$$\sup_{x\neq y\in D} d^\tau_{x,y} \frac{|\partial^m \eta^z(x) - \partial^m \eta^z(y)|}{|x-y|^\beta} \le C(1+2\varepsilon)^\tau d_z^{\tau-m-\beta}\varepsilon^{-m-\beta}. \tag{36}$$

In particular,

$$[\partial^m \eta^z]^{(m)}_{\beta,\infty;D_T} \le C\varepsilon^{-m-\beta}. \tag{37}$$

(b) *Let $U \in C^{2+\beta,p}(Y, D_T)$. Then there exists a constant $C = C(\varepsilon, \beta, d)$ such that*

$$|U|^*_{2+\beta,p;D_T} \le 2\sup_{z\in D} |\eta^z U|^*_{2+\beta,p;D_T} + C|U|_{0,p;D_T}. \tag{38}$$

Indeed, let $x, y \in D$ and $d_x = d_{x,y}$. If $|x-y| \geq \frac{1}{4}\varepsilon d_x$, then

$$d_{x,y}^{2+\beta}\frac{|\partial^2 U(t,x) - U(t,y)|_p}{|x-y|^\beta} \leq 4^\beta/\varepsilon^\beta\, |U|^*_{2,p;D_T}.$$

If $|x-y| < \frac{1}{4}\varepsilon d_x$, we take $z = x$. So

$$d_{x,y}^{2+\beta}\frac{|\partial^2 U(t,x) - \partial^2 U(t,y)|_p}{|x-y|^\beta} = d_{x,y}^{2+\beta}\frac{|\partial^2(\eta^z U(t,x) - \eta^z U(t,y))|_p}{|x-y|^\beta},$$

and the statement follows using interpolation inequalities.

4.1. *Schauder estimates.* We now establish the basic Schauder interior estimates.

THEOREM 17. *Let D be an open subset of $\mathbf{R}^n$. Assume that $f \in B^p_{\mathrm{loc}}(D_T)$, $g \in B^p_{\mathrm{loc}}(D_T, Y)$, $p \geq 2$, and the coefficients satisfy the following conditions. There are positive constants λ_0, K such that*

$$a^{ij}\xi_i\xi_j \geq \lambda_0|\xi|^2 \qquad \forall (t,x) \in D_T,\ \xi \in \mathbf{R}^n$$

and

$$|a|^{(0)}_\beta,\ |b^i|^{(1)}_{\beta,\infty;D_T},\ |c|^{(2)}_{\beta,\infty;D_T},\ \|h\|^{(1)}_{1+\beta,\infty;D_T} \leq K,\ |f|^{(2)}_{\beta,p;D_T} + \|g\|^{(1)}_{1+\beta,p;D_T} < \infty.$$

Then if (33) *holds for u and $|u|^*_{2+\beta,p;D_T} < \infty$, we have the estimate*

$$|u|^*_{2+\beta,p;D_T} \leq C(|u|_{0,p;D_T} + |\,f\,|^{(2)}_{\beta,p;D_T} + \|g\|^{(1)}_{1+\beta,mp;D_T}),$$

where $C = C(d, \lambda_0, K, \beta)$.

PROOF. Let $u^z = \eta^z u$. Then we have

$$\text{(39)} \qquad \begin{cases} \partial_t u^z = 1/2 a^{ij}(t,z)\partial^2_{ij}u^z - \lambda u^z + f^z + g^z\dot{W}, & \text{in } D_T, \\ u^z(0,x) = 0, & \text{in } D, \end{cases}$$

where

$$\begin{aligned} f^z(t,x) &= 1/2(a^{ij}(t,x) - a^{ij}(t,z))\partial^2_{ij}u(t,x)\eta^z(x) + b^i(t,x)\partial_i u(t,x)\eta^z(x) \\ &\quad + c(t,x)u(t,x)\eta^z(x) + f(t,x)\eta^z(x) - u(t,x)\partial^2_{ij}\eta^z(x)a^{ij}(t,x) \\ &\quad - a^{ij}(t,x)\partial_i u(t,x)\partial_j\eta^z(x) = \sum_{l=1}^{6} A^l(t,x) \end{aligned}$$

and

$$g^z(t,x) = h(t,x)u(t,x)\eta^z(x) + g(t,x)\eta^z(x) = \tilde{A}^1(t,x) + \tilde{A}^2(t,x).$$

Thus, we have by Theorem 14 applied to (39),

$$\text{(40)} \qquad |u^z|^*_{2+\beta,p;D_T} \leq C(|u|_{0,p;D_T} + |f^z|^{(2)}_{\beta,p;D_T} + \|g^z\|^{(1)}_{1+\beta,p;D_T}).$$

We notice here, that from (35) and (37) follows

$$
\begin{aligned}
|A^2(t,x)|^{(2)}_{\beta,p;D_T} &\le |\partial_i u|^{(1)}_{\beta,p;D_T} |\eta^z b_i|^{(1)}_{\beta,\infty;D_T} \\
&\le |\partial_i u|^{(1)}_{\beta,p;D_T} |b_i|^{(1)}_{\beta,\infty;D_T} |\eta^z|^{(0)}_{\beta,\infty;D_T} \le C\varepsilon^{-\beta} |\partial_i u|^{(1)}_{\beta,p;D_T}.
\end{aligned} \tag{41}
$$

By the same arguments and (36),

$$
\begin{aligned}
&|A^3|^{(2)}_{\beta,p;D_T} \le |c|^{(2)}_{\beta,\infty;D_T} |\eta^z|^{(0)}_{\beta,\infty;D_T} |u|^{(0)}_{\beta,p;D_T} \le C\varepsilon^{-\beta} |u|^{(0)}_{\beta,p;D_T}, \\
&|A^4|^{(2)}_{\beta,p;D_T} \le |f|^{(2)}_{\beta,p;D_T} |\eta^z|^{(0)}_{\beta,\infty;D_T}, \\
&|A^5|^{(2)}_{\beta,p;D_T} \le |u|^{(0)}_{\beta,p;D_T} |a^{ij}|^{(0)}_{\beta,\infty;D_T} |\partial^2_{ij}\eta^z|^{(2)}_{\beta,\infty;D_T} \le C\varepsilon^{-2-\beta} |u|^{(0)}_{\beta,p;D_T}, \\
&|A^6|^{(2)}_{\beta,p;D_T} \le |\partial_i u|^{(1)}_{\beta,p;D_T} |a^{ij}|^{(0)}_{\beta,\infty;D_T} |\partial_j \eta^z|^{(1)}_{\beta,\infty;D_T} \le C\varepsilon^{-1-\beta} |\partial u|^{(1)}_{\beta,p;D_T}.
\end{aligned} \tag{42}
$$

If $|x-z| \le 2\varepsilon d_z$, then $d_x \ge (1-2\varepsilon) d_z$ and

$$
|a(t,x) - a(t,z)| \le \frac{2\varepsilon}{(1-2\varepsilon)} |a|^{(0)}_{\beta,\infty;D_T}.
$$

Thus by (35) and (36) there exists $C = C(\varepsilon)$ such that

$$
|A^1|^{(2)}_{\beta,p;D_T} \le \frac{2\varepsilon}{(1-2\varepsilon)} [\partial^2 u]^{(2)}_{\beta,p;D_T} + C\varepsilon^{-\beta} |\partial^2 u|^{(2)}_{0,p;D_T}. \tag{43}
$$

According to (34),

$$
\|\tilde{A}^l\|^{(1)}_{1+\beta,p;D_T} = \|\tilde{A}^l\|^{(1)}_{0,p;D_T} + \|\partial \tilde{A}^l\|^{(2)}_{\beta,p;D_T}, \qquad l = 1,2.
$$

Obviously $\sum_l \|\tilde{A}^l\|^{(1)}_{0,p;D_T} \le \|h\|^{(1)}_{0,\infty;D_T} |u|_{0,p;D_T} + \|g\|^{(1)}_{0,p:D_T}$. By (35) and (36),

$$
\begin{aligned}
\|\partial \tilde{A}^1\|^{(2)}_{\beta,p;D_T} &\le |u|^{(0)}_{\beta,p;D_T} \|\partial h\|^{(2)}_{\beta,\infty;D_T} |\eta^z|^{(0)}_{\beta,\infty;D_T} \\
&\quad + |\partial u|^{(1)}_{\beta,p;D_T} \|h\|^{(1)}_{\beta,\infty;D_T} |\eta^z|^{(0)}_{\beta,\infty;D_T} \\
&\quad + |u|^{(0)}_{\beta,p;D_T} \|h\|^{(1)}_{\beta,\infty;D_T} |\partial \eta^z|^{(1)}_{\beta,\infty;D_T} \\
&\le C(\varepsilon^{-1-\beta} |u|^{(0)}_{\beta,p;D_T} + \varepsilon^{-\beta} |\partial u|^{(1)}_{\beta,p;D_T}), \\
\|\partial \tilde{A}^2\|^{(2)}_{\beta,p;D_T} &\le \|\partial g\|^{(2)}_{\beta,p;D_T} |\eta^z|^{(0)}_{\beta,\infty;D_T} + \|g\|^{(1)}_{\beta\bullet p;D_T} |\partial \eta^z|^{(1)}_{\beta,\infty;D_T} \\
&\le C\varepsilon^{-1-\beta} \|g\|^{(1)}_{1+\beta,p;D_T}.
\end{aligned} \tag{44}
$$

Now choosing ε such that $4\varepsilon/(1-2\varepsilon) < 1/2$ we have the desired estimate by (38), (40), (43), (41), (42), (44). □

If $D = \mathbf{R}^d$ we have in (33) the usual Cauchy problem and Theorem 17 gives an a priori estimate of its solution. Now we will solve this problem in Hölder spaces.

4.2. *Cauchy problem.* Consider (33) for $D = \mathbf{R}^d$, that is, we have

$$(45)\quad \begin{cases} \partial_t u = 1/2\, a^{ij}\, \partial_{ij} u + b^i \partial_i u + cu - \lambda u + f + (hu + g)\dot{W}, & \text{in } \mathbf{R}^d_T, \\ u(0, x) = 0, & \text{in } \mathbf{R}^d, \end{cases}$$

where $\lambda \geq 0$, $a = (a^{ij})$ is symmetric and positive. We assume that a is a deterministic locally bounded function on D_T and b^i, c are $L_\infty(\Omega, \mathbf{P})$-valued, $\mathbf{F}$-adapted locally bounded functions on D_T, h is a $L_\infty(\Omega, Y, \mathbf{P})$-valued $\mathbf{F}$-adapted locally bounded function on D_T.

THEOREM 18. *Assume that $f \in B^p_{\text{loc}}(D_T)$, $g \in B^p_{\text{loc}}(D_T, Y)$, $p \geq 2$ and the coefficients of* (45) *satisfy the following conditions. There are positive constants λ_0, K such that*

$$a^{ij} \xi_i \xi_j \geq \lambda_0 |\xi|^2 \qquad \forall (t, x) \in D_T, \xi \in \mathbf{R}^n$$

and

$$|a|_{\beta, \infty; T}, |b^i|_{\beta, \infty; T}, |c|_{\beta, \infty; D_T}, \|h\|_{1+\beta, \infty; T} \leq K, \ |f|_{\beta, p; T} + \|g\|_{1+\beta, p; T} < \infty.$$

Then if (45) *holds for u and $|u|_{2+\beta, p; T} < \infty$, we have the estimate*

$$|u|_{2+\beta, p; T} \leq C(|f|_{\beta, p; T} + \|g\|_{1+\beta, p; T}),$$

where $C = C(\lambda_0, K, d, \beta, T, p)$. Moreover, there is a constant $C = C(\lambda_0, K, k, d, \beta, T, p, \lambda)$ such that for each $t, v > 0$, $t + v \leq T$, $x \in \mathbf{R}^d$,

$$|u(\cdot + v, \cdot) - u(\cdot, \cdot)|_{1+\beta, p; T} \leq C v^{1/2} (|f|_{\beta, p; T} + \|g\|_{1+\beta, p; T}).$$

PROOF. By Theorem 17 in Section 4.1,

$$(46)\qquad |u|_{2+\beta, p; T} \leq C(|u|_{0, p; T} + |f|_{\beta, p; T} + \|g\|_{1+\beta, p; T}),$$

where $C = C(\lambda_0, K, \beta, d, T, p)$. Fix an arbitrary $z \in \mathbf{R}^d$. Then

$$(47)\qquad \begin{cases} \partial_t u = 1/2 a^{ij}(t, z) \partial^2_{ij} u - \lambda u + f^z + g^z \dot{W}, & \text{in } \mathbf{R}^d_T, \\ u^z(0, x) = 0, & \text{in } \mathbf{R}^d, \end{cases}$$

where

$$f^z(t, x) = 1/2(a^{ij}(t, x) - a^{ij}(t, z)) \partial^2_{ij} u(t, x) + b^i(t, x) \partial_i u(t, x) + c(t, x) u(t, x) + f(t, x)$$

and

$$g^z(t, x) = h(t, x) u(t, x) + g(t, x).$$

Now by Theorem 10, Lemma 11 and (46),

$$|u|_{0, p; T} \leq C(\lambda^{-1} \vee \lambda^{-1/2})(|u|_{0, p; T} + |f|_{\beta, p; T} + \|g\|_{1+\beta, p; T}).$$

Thus, if $C(\lambda^{-1} \vee \lambda^{-1/2}) \leq 1/2$, that is, for $\lambda > M = M(\lambda_0, K, \beta, d.p, T)$,

$$|u|_{0, p; T} \leq C(|f|_{\beta, p; T} + \|g\|_{1+\beta, p; T}),$$

and the estimate of $|u|_{2+\beta,\,p;T}$ follows for $\lambda > M$ by (46). If $\lambda \le M$, it is enough to notice that $\tilde{u} = e^{-\gamma t}u$ is a solution of (45) for $\lambda + \gamma$. Since u satisfies (47) the last inequality of our statement follows by Lemma 15. □

Using standard continuity arguments [see Theorem 5.2 in Gilbarg and Trudinger (1983)], Theorems 10 and 18, we derive the existence and uniqueness result for the Cauchy problem (45).

THEOREM 19. *Assume that* $f \in B^p_{\mathrm{loc}}(D_T)$, $g \in B^p_{\mathrm{loc}}(D_T, Y)$, $p \ge 2$, *and the coefficients of* (45) *satisfy the following conditions. There is a positive constant* λ_0 *such that*

$$a^{ij}\xi_i\xi_j \ge \lambda_0|\xi|^2 \qquad \forall (t,x) \in D_T,\ \xi \in \mathbf{R}^n$$

and

$$|a|_{\beta,\infty;T},\ |b^i|_{\beta,\infty;T},\ |c|_{\beta,\infty;D_T},\ \|h\|_{1+\beta,\infty;T},\ |f|_{\beta,p;T},\ \|g\|_{1+\beta,p;T} < \infty.$$

Then there exists a unique solution $u \in C^{2+\beta,\,p}(\mathbf{R}^d_T)$ *of* (45). *Moreover,*

$$|u|_{2+\beta,\,p;T} \le C(|f|_{\beta,\,p;T} + \|g\|_{1+\beta,\,p;T})$$

and

$$|u(\cdot + s,\cdot) - u(\cdot,\cdot)|_{1+\beta,\,p;T-s} \le Cs^{1/2}(|f|_{\beta,\,p;T} + \|g\|_{1+\beta,\,p;T}), \qquad k \le 1,$$

where the constants C depend on λ_0, β, p, d, T ,λ *and the constant K bounding the norms* $|a|_{\beta,\infty;T}$, $|b^i|_{\beta,\infty;T}$, $|c|_{\beta,\infty;D_T}$, $\|h\|_{1+\beta,\infty;T}$.

PROOF. Let $Lu = 1/2\, a^{ij}\,\partial_{ij}u + b^i\partial_i u + cu - \lambda u$, $Mu = hu$, $\tau \in [0,1]$,

$$L_\tau u = \tau L u + (1-\tau)\Delta u, \qquad M_\tau u = \tau M u,$$

where Δ is Laplacian in x. We introduce the space $\tilde{C}^{2+\beta,\,p}(\mathbf{R}^d_T)$ of functions $u \in C^{2+\beta,\,p}(\mathbf{R}^d_T)$ such that for each (t,x) $\mathbf{P}$-a.s.,

$$u(t,x) = \int_0^t F(s,x)\,ds + \int_0^t G(s,x)\,\dot{W}_s\,ds,$$

where $F \in C^{\beta,\,p}(\mathbf{R}^d_T)$, $G \in C^{1+\beta,\,p}(\mathbf{R}^d_T,\,Y)$. It is a Banach space with respect to the norm

$$|u|^{\tilde{}}_{2+\beta,\,p;T} = |u|_{2+\beta,\,p;T} + |F|_{\beta,\,p;T} + \|G\|_{1+\beta,\,p;T}.$$

Let $\mathscr{V}^{\beta,\,p}$ be a Banach space of all pairs $l = (f,g)$, $f \in C^{\beta,\,p}(\mathbf{R}^d_T)$ $g \in C^{1+\beta,\,p}(\mathbf{R}^d_T, Y)$ with the norm

$$|l|_{\beta,\,p} = |f|_{\beta,\,p;T} + \|g\|_{1+\beta,\,p;T}.$$

Consider the mappings T_τ: $\tilde{C}^{2+\beta,\,p}(\mathbf{R}^d_T) \to \mathscr{V}^{\beta,\,p}$ defined by

$$u(t,x) = \int_0^t F(s,x)\,ds + \int_0^t G(s,x)\,\dot{W}_s\,ds \longmapsto (F - L_\tau u,\ G - M_\tau u).$$

Obviously, for some constant C independent of τ,

$$|T_\tau u|_{\beta,\,p} \le C|u|^{\tilde{}}_{2+\beta,\,p}.$$

On the other hand, there is a constant C independent of τ such that for all $u \in \tilde{C}^{2+\beta,p}(\mathbf{R}^d_T)$,

$$|u|^{\tilde{}}_{2+\beta,\,p} \le C|T_\tau u|_{\beta,p}. \tag{48}$$

Indeed,

$$\begin{aligned} u(t,x) &= \int_0^t F(s,x)\,ds + \int_0^t G(s,x)\,\dot{W}_s\,ds \\ &= \int_0^t (L_\tau u + (F - L_\tau u))\,ds + \int_0^t (M_\tau u + (G - M_\tau u))\,\dot{W}_s\,ds. \end{aligned}$$

By Theorem 18 there is a constant C independent of τ such that

$$\begin{aligned} |u|_{2+\beta,\,p} &\le C|T_\tau u|_{\beta,\,p} \\ &= C(|F - L_\tau u|_{\beta,\,p} + \|G - M_\tau u\|_{1+\beta,\,p}). \end{aligned} \tag{49}$$

Thus

$$\begin{aligned} |u|^{\tilde{}}_{2+\beta,p} &= |u|_{2+\beta,\,p} + |F|_{\beta,\,p} + \|G\|_{1+\beta,\,p} \le |u|_{2+\beta,\,p} \\ &\quad + |F - L_\tau u|_{\beta,\,p} + \|G - M_\tau u\|_{1+\beta,\,p} + |L_\tau u|_{\beta,\,p} + \|M_\tau u\|_{1+\beta,\,p} \\ &\le C(|u|_{2+\beta,\,p} + |F - L_\tau u|_{\beta,\,p} + \|G - M_\tau u\|_{1+\beta,\,p}) \\ &\le C(|F - L_\tau u|_{\beta,\,p} + \|G - M_\tau u\|_{1+\beta,\,p}) = C|T_\tau u|_{\beta,\,p} \end{aligned}$$

and (48) follows. According to Theorem 8 and Lemma 11, T_0 is an onto map. Now by Theorem 5.2 in Gilbarg and Trudinger (1983) all the T_τ are onto maps and the statement follows. □

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