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BROWNIAN-TIME PROCESSES: THE PDE CONNECTION AND THE HALF-DERIVATIVE GENERATOR

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We introduce a class of interesting stochastic processes based on Brownian-time processes. These are obtained by taking Markov processes and replacing the time parameter with the modulus of Brownian motion. They generalize the iterated Brownian motion (IBM) of Burdzy and the Markov snake of Le Gall, and they introduce new interesting examples. After defining Brownian-time processes, we relate them to fourth order parabolic partial differential equations (PDE's). We then study their exit problem as they exit nice domains in $\mathbb{R}^d$, and connect it to elliptic PDE's. We show that these processes have the peculiar property that they solve fourth order parabolic PDE's, but their exit distribution—at least in the standard Brownian time process case—solves the usual second order Dirichlet problem. We recover fourth order PDE's in the elliptic setting by encoding the iterative nature of the Brownian-time process, through its exit time, in a standard Brownian motion. We also show that it is possible to assign a formal generator to these non-Markovian processes by giving such a generator in the half-derivative sense.

0. Introduction. Let $B(t)$ be a one-dimensional Brownian motion starting at 0 and let $X^x(t)$ be an independent $\mathbb{R}^d$-valued continuous Markov process started at x, both defined on a probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$. We call the process $\mathbb{X}_B^x(t) \triangleq X^x(|B(t)|)$ a Brownian-time process (BTP). In the special case where X^x is a Brownian motion starting at x we call the process $\mathbb{X}_B^x(t)$ a Brownian-time Brownian motion (BTBM). Excursions-based Brownian-time processes (EBTP's) are obtained from BTP's by breaking up the path of $|B(t)|$ into excursion intervals—maximal intervals (r, s) of time on which $|B(t)| > 0$—and, on each such interval, we pick an independent copy of the Markov process $\mathbb{X}^x$ from a finite or an infinite collection. Brownian-time processes and their close cousins EBTP's may be regarded as canonical constructions for several famous as well as interesting new processes. To see this, observe that the following processes have the one-dimensional distribution $\mathbb{P}(\mathbb{X}_B^x(t) \in dy)$:

1. *Markov snake*—when $|B_t|$ increases we generate a new independent path. See Le Gall ([13–15]) for applications to the nonlinear PDE $\Delta u = u^2$.
2. Let $X^{x,1}(t), \ldots, X^{x,k}(t)$ be independent copies of $\mathbb{X}^x(t)$ starting from point x. On each excursion interval of $|B(t)|$ use one of the k copies chosen at random. When $x = 0$, X^x is a Brownian motion starting at 0, and when $k = 2$ this reduces to the iterated Brownian motion (IBM). See Burdzy [2, 3].

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We identify such a process by the abbreviation kEBTP and we denote it by $\mathbb{X}^{x,k}_{B,c}(t)$. Of course, when $k = 1$ we obtain a BTP.

3. Use an independent copy of X^x on each excursion interval of $|B(t)|$. This is the $k \to \infty$ limit of process 2 (for a rigorous statement and proof, see the Appendix). It is intermediate between IBM and the Markov snake. Here, we go forward on a new independent path only after $|B_t|$ reaches 0. This process is abbreviated as EBTP and is denoted by $\mathbb{X}^x_{B,c}(t)$.

In Sections 1 and 2 we connect $\mathbb{X}^x_B(t)$, $\mathbb{X}^{x,t}_{B,c}(t)$ and $\mathbb{X}^x_{B,c}(t)$ to new fourth order parabolic PDE's and to second and fourth order elliptic PDE's. As a special case of our results, we get the missing connection of the IBM of Burdzy to PDE's. There are of course other iterated processes that have been linked to different PDE's (see [8], [10] and the references therein), but none solves the IBM PDE. In Section 3 we show that, even though $\mathbb{X}^x_B(t)$ is not Markovian, we can still assign to it a "generator" in the half-derivative sense, which we therefore call the half-derivative generator.

In Section 1 the PDE connection is given by the following theorem.

THEOREM 0.1. *Let $\mathscr{T}_s f(x) = \mathbb{E}_{\mathbb{P}} f(\mathbb{X}^x(s))$ be the semigroup of the continuous Markov process $\mathbb{X}^x(t)$ and let $\mathscr{A}$ be its generator. Let f be a bounded measurable function in the domain of $\mathscr{A}$, with $D_{ij}f$ bounded and Hölder continous for all $l \le i, j \le d$. If $u(t,x) = \mathbb{E}_{\mathbb{P}} f(\mathbb{X}^{x,k}_{B,c}(t))$ for any $k \in \mathbb{N}$ [as stated before $\mathbb{X}^{x,1}_{B,c}(t) = \mathbb{X}^x_B(t)$], or if $u(t,x) = \mathbb{E}_{\mathbb{P}} f(\mathbb{X}^x_{B,c}(t))$, then u solves the PDE*

$$
\begin{aligned}
&\frac{\partial}{\partial t}u(t,x) = \frac{\mathscr{A}f(x)}{\sqrt{2\pi t}} + \frac{1}{2}\mathscr{A}^2 u(t,x), \qquad t > 0, x \in \mathbb{R}^d, \\
&u(0,x) = f(x), \qquad x \in \mathbb{R}^d,
\end{aligned} \tag{0.1}
$$

where the operator $\mathscr{A}$ acts on $u(t,x)$ as a function of x with t fixed. In particular, if $\mathbb{X}^x_B(t)$ is a BTBM and Δ is the standard Laplacian, then u solves

$$
\begin{aligned}
&\frac{\partial}{\partial t}u(t,x) = \frac{\Delta f(x)}{\sqrt{8\pi t}} + \frac{1}{8}\Delta^2 u(t,x), \qquad t > 0,\ x \in \mathbb{R}^d, \\
&u(0,x) = f(x), \qquad x \in \mathbb{R}^d.
\end{aligned} \tag{0.2}
$$

REMARK 0.1. The inclusion of the initial function $f(x)$ in the PDE's (0.1) and (0.2) is a reflection of the non-Markovian property of our BTP. Thus the role of f here is fundamentally different from its role in the standard Markov–PDE connection.

In Section 2, we focus on BTBM's and we take up the exit problem for $\mathbb{X}^x_B(t)$. Toward this end, let G be a bounded open subset of $\mathbb{R}^d$ with regular boundary ∂G. Each time, we start $\mathbb{X}^x_B(t)$ at a point $x \in G \cup \partial G$, and we let $\mathrm{T}^x_G := \inf\{t \ge 0; \mathbb{X}^x_B(t) \notin G\}$. Our first result says that if we look at the exit distribution of our iterated process, we solve the usual second order Dirichlet problem. This might seem surprising at first, but upon reflection, we see that the iterated nature of both our process $\mathbb{X}_B^x(t)$ and its exit time T_G^x "cancel" each other, and we are effectively reduced to the exit distribution of an ordinary Brownian motion. For a precise explanation of this phenomenon see the Proof of Theorem 0.2. In Theorem 0.4 we show how to "recover" the fourth order PDE in this elliptic setting.

THEOREM 0.2. *Let G and T_G^x be defined as above. If $u(x) = \mathbb{E}_{\mathbb{P}} f(\mathbb{X}_B^x(\mathrm{T}_G^x))$, then u satisfies the Dirichlet problem*

$$\begin{aligned} \Delta u(x) &= 0, \qquad x \in G, \\ u(x) &= f(x), \qquad x \in \partial G. \end{aligned} \tag{0.3}$$

The next result links the first exit time T_G^x of the BTP $\mathbb{X}_B^x(t)$ to fourth order PDE's.

THEOREM 0.3. *Let G and T_G^x be defined as above. If $u(x) = \mathbb{E}_{\mathbb{P}} \mathrm{T}_G^x$, then u satisfies*

$$\begin{aligned} \Delta^2 u(x) &= 8, \qquad x \in G, \\ u(x) &= 0, \qquad x \in \partial G. \end{aligned} \tag{0.4}$$

We now show how to "encode" the iterated nature of our BTBM process in a Brownian motion so as to recover a fourth order elliptic PDE. The idea is to look at the Brownian motion X^x evaluated at the iterated exit time T_G^x [the first exit time for the iterated process $X^x(|B(t)|)$], that is, $X^x(\mathrm{T}_G^x)$. Note that this is not the exit distribution of X^x (since $\mathrm{T}_G^x \neq \tau_G^x = \inf\{t \geq 0; X^x \notin G\}$ in general). The fact that T_G^x is not a stopping time with respect to the natural filtration of X^x makes it inconvenient to deal with directly, so we are led to the deterministic time that captures the desired properties of T_G^x, namely $\mathscr{T}^x = \mathbb{E}_{\mathbb{P}} \mathrm{T}_G^x$.

REMARK 0.2. If $x \in \partial G$, then $\mathrm{T}_G^x = 0$ a.s. $\mathbb{P}$ and so $\mathscr{T}^x = \mathbb{E}_{\mathbb{P}} \mathrm{T}_G^x = 0$. Of course, by Theorem 0.3, $\mathscr{T}^x$ satisfies (0.4). We are now ready to give the elliptic fourth order PDE connection to a Brownian motion at the expected value of the iterated exit time T_G^x.

THEOREM 0.4. *Assume that X^x is the outer Brownian motion in $\mathbb{X}_B^x(t) = X^x(|B(t)|)$, starting at x under $\mathbb{P}$, and let $\mathscr{T}^x = \mathbb{E}_{\mathbb{P}} \mathrm{T}_G^x$. Let $f \in C^4(\mathbb{R}^d; \mathbb{R})$ be biharmonic ($\Delta^2 f \equiv 0$), and assume polynomial growth for f and all of its partial derivatives of order $k \leq 4$. Then $u(x) = \mathbb{E}_{\mathbb{P}} f(X^x(\mathscr{T}^x))$ satisfies*

$$\begin{aligned} \Delta^2 u(x) &= 4\Delta f(x) + \alpha(x) + \beta(x), \qquad x \in G, \\ u(x) &= f(x), \qquad x \in \partial G, \end{aligned} \tag{0.5}$$

where

$$\alpha(x) = \nabla(\Delta f(x)) \cdot \nabla(\Delta \mathbb{E}_{\mathbb{P}}[\tau_G^x]^2) \quad \textit{and} \quad \beta(x) = 2 \sum_{\substack{1 \leq i, j \leq d \\ i \neq j}} D_{ij} \Delta f(x) D_{ij} \mathbb{E}_{\mathbb{P}}[\tau_G^x]^2.$$

In particular, if in addition to the above assumptions on f and its partial derivatives we assume that $\nabla(\Delta f(x)) = 0$, where ∇ is the usual gradient, then $u(x) = \mathbb{E}_{\mathbb{P}} f(X^x(\mathscr{T}^x))$ solves

$$
\begin{aligned}
\Delta^2 u(x) &= 4\Delta f(x), \qquad x \in G,\\
u(x) &= f(x), \qquad x \in \partial G.
\end{aligned}
\tag{0.6}
$$

REMARK 0.3. Comparing (0.5) and (0.6) with (0.2), we see that they all include the bi-Laplacian of u and the Laplacian of the function f. So that, also in the elliptic case (0.5), f plays a fundamentally different role than in the usual Brownian motion–PDE connection: it acts on $G \cup \partial G$, and not just on the boundary ∂G.

The following result attaches a formal generator to our BTP's, in the half-derivative sense. More precisely, we have the following theorem.

THEOREM 0.5. *Let X^x be the outer Markov process in our BTP, starting at $x \in \mathbb{R}^d$ under $\mathbb{P}$. Suppose that the generator $\mathscr{A}$ of X^x is given by a divergence form second order partial differential operator as in* (3.1). *Let $\mathscr{A}_t^*$ be the generator of the time-reversed Markov process $\{X^x(T-t); 0 \le t \le T\}$ and suppose that $C_0^2(\mathbb{R}^d;\mathbb{R}) \supset \mathbb{D}(\mathscr{A}) \cap \mathbb{D}(\mathscr{A}_t^*)$, where $\mathbb{D}(\mathscr{A})$ and $\mathbb{D}(\mathscr{A}_t^*)$ are the domains of $\mathscr{A}$ and $\mathscr{A}_t^*$, respectively. Finally, assume that condition* (3.6) *holds. If*

$$
\mathscr{A}_s^{1/2} f(x) \triangleq \lim_{t \searrow s} \frac{\mathbb{E}_{\mathbb{P}}[f(\mathbb{X}_B^x(t)) | \mathbb{X}_B^x(s)] - f(\mathbb{X}_B^x(s))}{(t-s)^{1/2}}, \qquad 0 < s \le t,
\tag{0.7}
$$

then $\mathscr{A}_s^{1/2} f(x)$ is given by

$$
\frac{1}{\sqrt{2\pi}} \left[\mathscr{A} f(\mathbb{X}_B^x(s)) + \frac{\int_0^\infty p(0,s;0,y) h(0,y;x,\mathbb{X}_B^x(s)) \mathscr{A}_y^* f(\mathbb{X}_B^x(s))\, dy}{\int_0^\infty p(0,s;0,y) h(0,y;x,\mathbb{X}_B^x(s))\, dy} \right],
\tag{0.8}
$$

where $p(s,t;x,y)$ and $h(s,t;x,y)$ are the transition densities (*with respect to Lebesgue measure*) *of $|B(t)|$ and $X(t)$, respectively. In particular, if $\mathscr{A} = \mathscr{A}_t^*$ for all t, then $\mathscr{A}_s^{1/2} f(x)$ is simply $\sqrt{2/\pi}\mathscr{A} f(\mathbb{X}_B^x(s))$.*

NOTATION. We alternate freely between the notations $X(t)$ and X_t for aesthetic reasons and for typesetting convenience.

1. Proof of Theorem 0.1. We first prove the theorem for the case of $u(t,x) = \mathbb{E}_{\mathbb{P}} f(\mathbb{X}_B^x(t))$ using the following generator computation:

$$
\mathbb{E}_{\mathrm{P}} f(\mathbb{X}_B^x(t)) = 2\int_0^\infty p_t(0,s) \mathscr{T}_s f(x)\, ds,
\tag{1.1}
$$

where $p_t(0,s)$ is the transition density of $B(t)$. Differentiating (1.1) with respect to t and putting the derivative under the integral, which is easily justified by the dominated convergence theorem, then using the fact that $p_t(0,s)$

satisfies the heat equation we have

$$\frac{\partial}{\partial t}\mathbb{E}_{\mathbb{P}} f(\mathbb{X}_B^x(t)) = 2\int_0^\infty \frac{\partial}{\partial t} p_t(0,s)\mathscr{T}_s f(x)\,ds$$
$$= \int_0^\infty \frac{\partial^2}{\partial s^2} p_t(0,s)\mathscr{T}_s f(x)\,ds.$$

We now integrate by parts twice, and we observe that the boundary terms always vanish at ∞ (as $s \nearrow \infty$) and we have $(\partial/\partial s)p_t(0,s) = 0$ at $s = 0$ but $p_t(0,0) > 0$. Thus,

$$\frac{\partial}{\partial t}\mathbb{E}_{\mathbb{P}} f(\mathbb{X}_B^x(t)) = -\int_0^\infty \frac{\partial}{\partial s} p_t(0,s)\frac{\partial}{\partial s}\mathscr{T}_s f(x)\,ds$$
$$= p_t(0,0)\mathscr{A} f(x) + \int_0^\infty p_t(0,s)\mathscr{A}^2\mathscr{T}_s f(x)\,ds.$$

Taking the application of $\mathscr{A}^2$ outside the integral using the conditions on f and $D_{ij}f$, and writing $u(t,x) = \mathbb{E}_{\mathbb{P}}\times\ f(\mathbb{X}_B^x(t))$ we have

$$\frac{\partial}{\partial t}u(t,x) = p_t(0,0)\mathscr{A} f(x) + \frac{1}{2}\mathscr{A}^2 u(t,x),$$

where, clearly, the operator $\mathscr{A}$ acts on $u(t,x)$ as a function of x with t fixed. Obviously, $u(0,x) = f(x)$, so that $u(t,x) = \mathbb{E}_{\mathbb{P}} f(\mathbb{X}_B^x(t))$ solves (0.1).

To prove the result for $\mathbb{X}_{B,c}^{x,k}(t)$ for $k \in \mathbb{N}\backslash\{1\}$, we show that $\mathbb{E}_{\mathbb{P}} f(\mathbb{X}_{B,c}^{x,k}(t)) = \mathbb{E}_{\mathbb{P}} f(\mathbb{X}_B^x(t))$. Toward this end, let $e^-(t)$ be the $|B(t)|$-excursion immediately preceding the excursion straddling t, $e(t)$; and condition on the event that we pick the jth copy of $\mathbb{X}^x$ on $e^-(t)$ (uniformly from among the k available independent copies of X^x), using the independence of the choice of the process $X^{x,j}$ on $e^-(t)$ from $B(t)$ and from the following choice of the X^x copy, on $e(t)$, to get

$$\mathbb{E}_{\mathbb{P}} f(\mathbb{X}_{B,c}^{x,k}(t)) = 2\sum_{j=1}^k \int_0^\infty p_t(0,s)\mathscr{T}_s f(x)\mathbb{P}[\text{we pick the } j\text{th copy on } e^-(t)]\,ds$$
$$= \frac{2}{k}\sum_{j=1}^k \int_0^\infty p_t(0,s)\mathscr{T}_s f(x)\,ds = 2\int_0^\infty p_t(0,s)\mathscr{T}_s f(x)\,ds$$
$$= \mathbb{E}_{\mathbb{P}} f(\mathbb{X}_B^x(t)).$$

Finally, to prove that $u(t,x) = \mathbb{E}_{\mathbb{P}} f(\mathbb{X}_{B,c}^x(t))$ solves (0.1), we use the fact (proven in the Appendix) that $\mathbb{X}_{B,c}^{x,k} \Longrightarrow \mathbb{X}_{B,c}^x$, for some subsequence $\mathbb{X}_{B,c}^{x,k}$. Following Skorohod's celebrated result, we may construct processes $Y_k \stackrel{\mathscr{L}}{=} \mathbb{X}_{B,c}^{x,k}$ and $Y \stackrel{\mathscr{L}}{=} \mathbb{X}_{B,c}^x$ on some probability space such that $Y_k \to Y$ as $k \to \infty$ a.s. uniformly in t on compact sets of $\mathbb{R}_+$. The result then follows since $\mathbb{E}_{\mathbb{P}} f(\mathbb{X}_{B,c}^{x,k}(t)) = \mathbb{E}_{\mathbb{P}} f(\mathbb{X}_B^x(t))$ for each k and since f is bounded and continuous. $\square$

2. Exit PDEs for $\mathbb{X}_B^x(t)$. Throughout this section the outer process X^x is always assumed to be a Brownian motion starting at x under $\mathbb{P}$, and G is a bounded open subset of $\mathbb{R}^d$ with regular boundary ∂G.

PROOF OF THEOREM 0.2. Let

$$\tau_G^x \triangleq \inf\{t \geq 0;\ X^x(t) \notin G\} \quad \text{and} \quad \sigma_B^x \triangleq \inf\{t \geq 0; |B(t)| = \tau_G^x\};$$

of course $\sigma_B^x = \mathrm{T}_G^x$. We then have

$$\begin{aligned} u(x) &= \mathbb{E}_{\mathbb{P}} f\big[\mathbb{X}_B^x(\mathrm{T}_G^x)\big] = \mathbb{E}_{\mathbb{P}} f\big[X^x(\tau_G^x| \, |B(\sigma_B^x)| = \tau_G^x\big]\mathbb{P}\big[|B(\sigma_B^x)| = \tau_G^x\big] \\ &= \mathbb{E}_{\mathbb{P}} f\big[X^x(\tau_G^x)\big], \end{aligned} \tag{2.1}$$

where the last equality in (2.1) follows from the obvious fact that

$$\mathbb{P}\big[|B(\sigma_B^x)| = \tau_G^x\big] = 1,$$

a fact which also clearly gives us the independence of the event $\{|B(\sigma_B^k)| = \tau_G^x\}$ from $X^x(\tau_G^x)$.

Now, $u(x) \triangleq \mathbb{E}_{\mathbb{P}} f[X^x(\tau_G^x)]$ is a harmonic function in G (since X^x is a Brownian motion starting at x under $\mathbb{P}$, and τ_G^x is its first exit time from G). It follows that $u(x)$ solves the Dirichlet problem (0.3). $\square$

We then prove the connection of the iterated exit time T_G^x to fourth order PDE's.

PROOF OF THEOREM 0.3. Let $u(x) = \mathbb{E}_{\mathbb{P}} \mathrm{T}_G^x$ and observe that

$$\begin{aligned} \mathrm{T}_G^x \triangleq \inf\{t \geq 0; \mathbb{X}_B^x(t) \notin G\} &= \inf\{t \geq 0; |B(t)| \notin [0, \tau_G^x)\} \\ &= \inf\{t \geq 0; B(t) \notin (-\tau_G^x, \tau_G^x)\}, \end{aligned} \tag{2.2}$$

where $\tau_G^x \triangleq \inf\{t \geq 0;\ X^x(t) \notin G\}$. Thus, conditioning on τ_G^x we easily get

$$u(x) = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[\mathrm{T}_G^x | \tau_G^x]] = \mathbb{E}_{\mathbb{P}}(\tau_G^x)^2. \tag{2.3}$$

However, from [9] and [11] we have that $u(x) = \mathbb{E}_{\mathbb{P}}(\tau_G^x)^2$ solves the equation

$$\Delta^2 u = 8, \tag{2.4}$$

for any smooth bounded domain G. Plainly, $u(x) = 0$ for $x \in \partial G$. We thus obtain (0.4) and this completes the proof. $\square$

We are now ready to prove Theorem 0.4.

PROOF OF THEOREM 0.4. Let $u(x) = \mathbb{E}_{\mathbb{P}} f(X^x_{\mathscr{T}^x})$, and let τ^x_G be the first exit time for the Brownian motion X^x. Itô's formula, applied twice, gives us

$$
\begin{aligned}
&f(X^x_{\mathscr{T}^s}) - f(x)\\
&\quad = \int_0^{\mathscr{T}^x} \nabla f(X^x_s)\cdot dX^x_s + \tfrac{1}{2}\int_0^{\mathscr{T}^x} \Delta f(X^x_s)\,ds\\
&\quad = \int_0^{\mathscr{T}^x} \nabla f(X^x_s)\cdot dX^x_s + \tfrac{1}{2}\mathscr{T}^x\Delta f(x) + \tfrac{1}{2}\int_0^{\mathscr{T}^x}\int_0^s \nabla(\Delta f(X^x_r))\cdot dX^x_r\,ds\\
&\qquad + \tfrac{1}{4}\int_0^{\mathscr{T}^x}\left[\int_0^s \Delta^2 f(X^x_r)\,dr\right]ds\\
&\quad = \int_0^{\mathscr{T}^x} \nabla f(X^x_s)\cdot dX^x_s + \tfrac{1}{2}\mathscr{T}^x\Delta f(x) + \tfrac{1}{2}\int_0^{\mathscr{T}^x}\int_0^s \nabla(\Delta f(X^x_r))\cdot dX^x_r\,ds\\
&\quad = \int_0^{\mathscr{T}^x} \nabla f(X^x_s)\cdot dX^x_s + \tfrac{1}{2}\mathscr{T}^x\Delta f(x)\\
&\qquad + \tfrac{1}{2}\int_0^{\mathscr{T}^x}\int_0^{\mathscr{T}^x} 1_{\{r<s\}}(r)\,\nabla(\Delta f(X^x_r))\,dX_r\,ds\\
&\quad = \int_0^{\mathscr{T}^x} \nabla f(X^x_s)\cdot dX^x_s + \tfrac{1}{2}\mathscr{T}^x\Delta f(x) + \tfrac{1}{2}\int_0^{\mathscr{T}^x}(\mathscr{T}^x - r)\nabla(\Delta f(X^x_r))\,dX_r,
\end{aligned}
$$

where we used the assumption that $\Delta^2 f \equiv 0$ to get the third equality. Now, taking expectations, we get that all the expectations involving stochastic integrals vanish. This is because we assumed that both $\nabla f(x)$ and $\nabla(\Delta f(x))$ have polynomial growth while the density of X^x_r has exponential decay, so that

$$
\mathbb{E}_{\mathbb{P}}\left[\int_0^{\mathscr{T}^x} |\nabla f(X^x_s)|^2\,ds\right] < \infty \quad \text{and} \quad \mathbb{E}_{\mathbb{P}}\left[\int_0^{\mathscr{T}^x} |(\mathscr{T}^x - r)\nabla(\Delta f(X^x_r))|^2 dr\right] < \infty.
$$

We then have

$$
\mathbb{E}_{\mathbb{P}} f(X^x_{\mathscr{T}^x}) - f(x) = \tfrac{1}{2}\mathscr{T}^x\Delta f(x). \tag{2.5}
$$

Applying the bi-Laplacian to both sides of (2.5), remembering that $u(x) = \mathbb{E}_{\mathbb{P}} f(X^x_{\mathscr{T}^x})$, that $\Delta^2 f \equiv 0$ and that $\mathscr{T}^x = \mathbb{E}_{\mathbb{P}}\mathrm{T}^x_G$ (by assumption) and invoking (2.3) and (2.4), we obtain

$$
\begin{aligned}
\Delta^2 u(x) &= \tfrac{1}{2}\Delta^2[\mathscr{T}^x\Delta f(x)] = \tfrac{1}{2}\Delta^2[\mathscr{T}^x]\Delta f(x)\\
&\qquad + \nabla(\Delta f(x))\cdot\nabla(\Delta[\mathscr{T}^x]) + \Delta(\nabla(\Delta f(x))\cdot\nabla[\mathscr{T}^x])\\
&= 4\Delta f(x) + \nabla(\Delta f(x))\cdot\nabla(\Delta[\mathscr{T}^x]) + \Delta(\nabla(\Delta f(x))\cdot\nabla[\mathscr{T}^x]) \qquad (2.6)\\
&= 4\Delta f(x) + \nabla(\Delta f(x))\cdot\nabla(\Delta[\mathscr{T}^x]) + 2\sum_{\substack{1\le i,j\le d\\ i\ne j}} D_{ij}\Delta f(x) D_{ij}[\mathscr{T}^x]\\
&= 4\Delta f(x) + \nabla(\Delta f(x))\cdot\nabla(\Delta\mathbb{E}_{\mathbb{P}}[\tau^x_G]^2) + 2\sum_{\substack{1\le i,j\le d\\ i\ne j}} D_{ij}\Delta f(x) D_{ij}\mathbb{E}_{\mathbb{P}}[\tau^x_G]^2, \quad x\in G,
\end{aligned}
$$

with the convention that $\sum_{i\ne j} D_{ij}\Delta f(x) D_{ij}\mathbb{E}_{\mathbb{P}}[\tau^x_G]^2 = 0$ if $d = 1$.

Finally, as stated in Remark 0.2, $\mathscr{T}^x = 0$ whenever $x \in \partial G$, and so $u(x) = \mathbb{E}_{\mathbb{P}} f(X^x(\mathscr{T}^x)) = f(x)$ for every $x \in \partial G$. $\square$

3. The half-derivative formal generator. In this section, we prove the formula for the half-derivative generator of our Brownian-time processes. We denote by $p(s,t;x,y)$ and $h(s,t;x,y)$ the transition densities (with respect to Lebesgue measure) of $|B(t)|$ and $X(t)$, respectively. We denote the generator of X by $\mathscr{A}$, and we assume that $X(0) = x_0$ is deterministic.

It is well known that, for each fixed but arbitrary $0 < T < \infty$, the time reversed process $X_T^* = \{X_T^*(t) \triangleq X(T-t); 0 \le t \le T\}$ is still Markovian; we denote its (time-dependent) generator by $\mathscr{A}_t^*$. We assume for simplicity that $\mathscr{A}$ is given by a divergence form second order partial differential operator

$$\mathscr{A} f = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left[g^{ij}(x) \frac{\partial}{\partial x_j} f \right], \tag{3.1}$$

where d is the space dimension and $g^{ij} \in C^2(\mathbb{R}^d;\mathbb{R})$ satisfies $c < g^{ij}(x) < c^{-1}$ for some positive constant c. From Aronson's inequality we have a constant c_1 such that

$$h(s,t;x,y) \le \frac{c_1}{(t-s)^{d/2}} \exp\left\{ -\frac{|x-y|^2}{c_1(t-s)} \right\}. \tag{3.2}$$

Moreover (see, e.g, [16] and [17])

$$\mathscr{A}_t^* f = \mathscr{A} f + 2 \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \log h(0,t;x_0,x) g^{ij}(x) \frac{\partial}{\partial x_j} f. \tag{3.3}$$

In particular, when $\mathscr{A} = \frac{1}{2}\Delta$, $\mathscr{A}_t^* = \frac{1}{2}\Delta + [(x_0 - x)/t]\nabla$.

We assume that, for every $f \in C_0^2(\mathbb{R}^d;\mathbb{R})$,

$$\lim_{t \searrow s} |t-s|^{-1} \left[\int h(s,t;x,y) f(y)\, dy - f(x) \right] = \mathscr{A} f(x), \tag{3.4}$$

$$\lim_{s \nearrow t} |t-s|^{-1} \left[\int \frac{h(0,s;x_0,y) h(s,t;y,x) f(y)}{h(0,t;x_0,x)}\, dy - f(x) \right] = \mathscr{A}_t^* f(x), \tag{3.5}$$

and without losing generality we assume that there is a constant $0 < c_2 < \infty$ such that

$$\frac{\partial}{\partial x_i} \log h(0,t;x_0,x) \le c_2 \frac{|x_0 - x| + c_2}{t^{c_2}}. \tag{3.6}$$

When $\mathscr{A}$ is the Laplacian, the above condition is easily satisfied. It is easy to deduce the following lemma.

LEMMA 3.1. *For any fixed $f \in C_0^2(\mathbb{R}^d;\mathbb{R})$ and $x \in \mathbb{R}^d$, there is a constant $0 < c_3 < \infty$ such that*

$$\sup_{s<t} \left\{ |t-s|^{-1} \left[\int \frac{h(0,s;x_0,y) h(s,t;y,x) f(y)}{h(0,t;x_0,x)}\, dy - f(x) \right] \right\} < c_3 t^{-c_2}. \tag{3.7}$$

PROOF. Since $\int h(0,s;x_0,y)h(s,t;y,x)\,dy = h(0,t;x_0,x)$,

$$\int \frac{h(0,s;x_0,y)h(s,t;y,x)f(y)}{h(0,t;x_0,x)}\,dy$$

is bounded by the same bound on f. Thus, when $s < 2^{-1}t$, (3.7) is true as $(t-s) > 2^{-1}t$. So it is sufficient to consider the case where $s \ge 2^{-1}t$. From the form of $\mathscr{A}_t^*$ in (3.3), it is easy to see that our time-reversed process has the following decomposition for fixed $t > 0$:

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}[X_T^*(T-s) - X_T^*(T-t)|X_t = x] \\
&\quad = \mathbb{E}_{\mathbb{P}}\biggl[\int_s^t \mathscr{A}f(X_r)\,dr\bigg|X_t = x\biggr] \\
&\qquad + 2\mathbb{E}_{\mathbb{P}}\biggl[\sum_{i,j}\int_s^t \frac{\partial}{\partial x_j}\log h(0,r;x_0,X_r)g^{ij}(X_r)\frac{\partial}{\partial x_i}f(X_r)\,dr\bigg|X_t = x\biggr] \\
&\quad \le (t-s)\|\mathscr{A}f\|_\infty + C\mathbb{E}_{\mathbb{P}}\biggl[\int_s^t \frac{|x_0 - X_r| + c_2}{t^{c_2}}\,dr\bigg|X_t = x\biggr] \\
&\quad \le (t-s)\|\mathscr{A}f\|_\infty + C\int_s^t s^{-c_2}\,dr,
\end{aligned}$$

where we used Aronson's inequality in the last step, and C is a constant depending on the C_1-norm of f, $|x_0 - x|$, c, c_1 and c_2. Dividing both sides of the last inequality by $(t-s)$ and noticing that $s > 2^{-1}t$, we get the lemma. □

We also have the following lemma.

LEMMA 3.2. *For all $f \in C_0^2(\mathbb{R}^d;\mathbb{R})$ the following convergence holds for almost every $y > 0$:*

$$\begin{aligned}
&\lim_{t\searrow s}\int_0^y \biggl\{(t-s)^{-1/2}p(s,t;y,z)\biggl[\int \frac{h(0,z;x_0,\eta)h(z,y;\eta,\xi)}{h(0,y;x_0,\xi)}f(\eta)\,d\eta - f(\xi)\biggr]\biggr\}\,dz \\
&\qquad = \frac{\mathscr{A}_y^* f(\xi)}{\sqrt{2\pi}}.
\end{aligned}$$

Moreover, there is a constant c_4 such that

$$\begin{aligned}
&\int_0^y \biggl\{(t-s)^{-1/2}p(s,t;y,z)\biggl[\int \frac{h(0,z;x_0,\eta)h(z,y;\eta,\xi)}{h(0,y;x_0,\xi)}f(\eta)\,d\eta - f(\xi)\biggr]\biggr\}\,dz \\
&\qquad \le c_4 y^{-c_2}.
\end{aligned}$$

PROOF. By the reflection principle, the transition density of the reflecting Brownian motion $|B(s)|$ is

$$p(s,t;y,z) = \frac{1}{\sqrt{2\pi(t-s)}}\biggl[\exp\biggl\{\frac{|y-z|^2}{2(t-s)}\biggr\} + \exp\biggl\{\frac{|y+z|^2}{2(t-s)}\biggr\}\biggr]. \tag{3.8}$$

By Lemma 3.1,

$$\begin{aligned}
&\int_0^y (t-s)^{-1/2}|y-z|\,p(s,t;y,z)\\
&\qquad\times \frac{\left[\int [h(0,z;x_0,\eta)h(z,y;\eta,\xi)/h(0,y;x_0,\xi)]f(\eta)\,d\eta - f(\xi)\right]}{|y-z|}\,dz\\
&\qquad\le C\int_0^y (t-s)^{-1/2}|y-z|\,p(s,t;y,z)\,y^{-c_2}\,dz\\
&\qquad\le C\int_0^y (t-s)^{-1}|y-z|\exp\left\{-\frac{|y-z|^2}{2(t-s)}\right\}y^{-c_2}\,dz\\
&\qquad= C\int_0^{y/\sqrt{t-s}} z\exp\left\{-\frac{z}{2}\right\}y^{-c_2}\,dz\\
&\qquad\le Cy^{-c_2},
\end{aligned}$$

where C is a generic constant that may vary from line to line. Now, we may write, for $z < y$,

$$(y-z)^{-1}\left[\int \frac{h(0,z;x_0,\eta)h(z,y;\eta,\xi)}{h(0,y;x_0,\xi)}f(\eta)\,d\eta - f(\xi)\right] = \mathscr{A}_y^* f(\xi) + o(y-z),$$

where $o(y-z) \to 0$ [as $(y-z) \to 0$] and $o(y-z) \le Cy^{-c_2}$. On the other hand,

$$\begin{aligned}
&\lim_{t\searrow s}\int_0^y (t-s)^{-1/2}|y-z|\,p(s,t;y,z)\left[\mathscr{A}_y^* f(\xi) + o(y-z)\right]dz\\
&\qquad= \lim_{t\searrow s}\int_0^y \frac{|y-z|}{\sqrt{2\pi}(t-s)}\exp\left\{-\frac{|y-z|^2}{2(t-s)}\right\}\left[\mathscr{A}_y^* f(\xi) + o(y-z)\right]dz\\
&\qquad= \lim_{t\searrow 0}\int_0^{y/\sqrt{t-s}} \frac{|z|}{\sqrt{2\pi}}\exp\left\{-\frac{|z|^2}{2}\right\}\left[\mathscr{A}_y^* f(\xi) + o(z\sqrt{t-s})\right]dz\\
&\qquad= \mathscr{A}_y^* f(\xi)\int_0^\infty \frac{|z|}{\sqrt{2\pi}}\exp\left\{-\frac{|z|^2}{2}\right\}dz\\
&\qquad= \frac{\mathscr{A}_y^* f(\xi)}{\sqrt{2\pi}}.
\end{aligned}$$

Thus we get the lemma. □

Similarly we have the following lemma.

LEMMA 3.3. *For all $f \in C_0^2(\mathbb{R}^d;\mathbb{R})$ the following convergence holds for almost every $y > 0$*:

$$\lim_{t\searrow s}\int_y^\infty (t-s)^{-1/2}p(s,t;y,z)\left[\int h(y,z;\xi,\eta)f(\eta)\,d\eta - f(\xi)\right]dz = \frac{\mathscr{A}f(\xi)}{\sqrt{2\pi}}.$$

Now, we easily have

$$\mathbb{P}(X(|B(s)|) \in d\xi) = \left[\int_0^\infty p(0, s; 0, y) h(0, y; x_0, \xi)\, dy\right] d\xi.$$

For $t > s$, we see that

$$\begin{aligned}
&\mathbb{P}(X(|B(s)|) \in d\xi, |B(t)| \geq |B(s)|, X(|B(t)|) \in d\eta) \\
&\quad = \left[\int_0^\infty \int_y^\infty p(0, s; 0, y) p(s, t; y, z) h(0, y; x_0, \xi) h(y, z; \xi, \eta)\, dz\, dy\right] d\xi\, d\eta,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}\big(X(|B(s)|) \in d\xi, |B(t)| < |B(s)|, X(|B(t)|) \in d\eta\big) \\
&\quad = \left[\int_0^\infty \int_0^y p(0, s; 0, y) p(s, t; y, z) h(0, y; x_0, \xi)\right. \\
&\qquad\qquad \left. \times \mathbb{P}[X(z) \in d\eta | X(y) \in d\xi]\, dz\, dy\right] d\xi \\
&\quad = \left[\int_0^\infty \int_0^y p(0, s; 0, y) p(s, t; y, z) h(0, y; x_0, \xi)\right. \\
&\qquad\qquad \left. \times \frac{h(0, z; x_0, \eta) h(z, y; \eta, \xi)}{h(0, y; x_0, \xi)}\, dz\, dy\right] d\xi\, d\eta \\
&\quad = \left[\int_0^\infty \int_0^y p(0, s; 0, y) p(s, t; y, z) h(0, z; x_0, \eta) h(z, y; \eta, \xi)\, dz\, dy\right] d\xi\, d\eta.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}[f[X(|B_t|)] \mid X(|B_s|) = \xi] \\
&\quad = \left\{\int_0^\infty p(0, s; 0, y) h(0, y; x_0, \xi)\, dy\right\}^{-1} \\
&\qquad \times \left\{\int \left[\int_0^\infty \int_y^\infty p(0, s; 0, y) p(s, t; y, z) h(0, y; x_0, \xi)\right.\right. \\
&\qquad\qquad\qquad \left. \times h(y, z; \xi, \eta)\, dz\, dy\right] f(\eta)\, d\eta \\
&\qquad\quad + \int \left[\int_0^\infty \int_0^y p(0, s; 0, y) p(s, t; y, z)\right. \\
&\qquad\qquad \left.\left. \times h(0, z; x_0, \eta) h(z, y; \eta, \xi)\, dz\, dy\right] f(\eta)\, d\eta\right\}
\end{aligned}$$

and so, to compute

$$\lim_{t \searrow s} (t - s)^{-1/2} \Big\{\mathbb{E}_{\mathbb{P}}\Big[f(X(|B_t|)) | X(|B_s|)\Big] - f(X(|B_s|))\Big\},$$

we observe that

$$
\begin{aligned}
&\lim_{t\searrow s}(t-s)^{-1/2}\{\mathbb{E}_{\mathbb{P}}[f[X(|B_t|)]\,|\,X(|B_s|)=\xi]-f(\xi)\}\\
&\quad=\lim_{t\searrow s}(t-s)^{-1/2}\left\{\int_0^\infty p(0,s;0,y)h(0,y;x_0,\xi)\,dy\right\}^{-1}\\
&\qquad\times\left\{\int\left[\int_0^\infty\int_y^\infty p(0,s;0,y)p(s,t;y,z)h(0,y;x_0,\xi)h(y,z;\xi,\eta)\,dz\,dy\right]f(\eta)\,d\eta\right.\\
&\qquad\quad+\int\left[\int_0^\infty\int_0^y p(0,s;0,y)p(s,t;y,z)h(0,z;x_0,\eta)h(z,y;\eta,\xi)\,dz\,dy\right]\\
&\qquad\qquad\left.\times f(\eta)\,d\eta-f(\xi)\right\}\\
&\quad=\lim_{t\searrow s}(t-s)^{-1/2}\left\{\int_0^\infty p(0,s;0,y)h(0,y;x_0,\xi)\,dy\right\}^{-1}\\
&\qquad\times\left\{\int\left[\int_0^\infty\int_y^\infty p(0,s;0,y)p(s,t;y,z)h(0,y;x_0,\xi)h(y,z;\xi,\eta)\,dz\,dy\right]f(\eta)\,d\eta\right.\\
&\qquad\quad+\int\left[\int_0^\infty\int_0^y p(0,s;0,y)p(s,t;y,z)h(0,z;x_0,\eta)h(z,y;\eta,\xi)\,dz\,dy\right]f(\eta)\,d\eta\\
&\qquad\qquad\left.-f(\xi)\int_0^\infty p(0,s;0,y)h(0,y;x_0,\xi)\,dy\right\}\\
&\quad=\lim_{t\searrow s}(t-s)^{-1/2}\left\{\int_0^\infty p(0,s;0,y)h(0,y;x_0,\xi)\,dy\right\}^{-1}\\
&\qquad\times\left\{\int\left[\int_0^\infty\int_y^\infty p(0,s;0,y)p(s,t;y,z)\right.\right.\\
&\qquad\qquad\left.h(0,y;x_0,\xi)h(y,z;\xi,\eta)\,dz\,dy\right]f(\eta)\,d\eta\\
&\qquad\quad+\int\left[\int_0^\infty\int_0^y p(0,s;0,y)p(s,t;y,z)h(0,z;x_0,\eta)h(z,y;\eta,\xi)\,dz\,dy\right]f(\eta)\,d\eta\\
&\qquad\qquad\left.-f(\xi)\int_0^\infty\int_0^\infty p(0,s;0,y)h(0,y;x_0,\xi)p(s,t;y,z)\,dy\,dz\right\}
\end{aligned}\tag{3.9}
$$

$$
\begin{aligned}
&\quad=\lim_{t\searrow s}\left\{\int_0^\infty p(0,s;0,y)h(0,y;x_0,\xi)\,dy\right\}^{-1}\\
&\qquad\times\left\{\int_0^\infty\int_y^\infty (t-s)^{-1/2}|y-z|\,p(0,s;0,y)p(s,t;y,z)h(0,y;x_0,\xi)\right.\\
&\qquad\quad\times|y-z|^{-1}\left[\int h(y,z;\xi,\eta)f(\eta)\,d\eta-f(\xi)\right]dz\,dy
\end{aligned}
$$

$$+\int_0^\infty \int_0^y (t-s)^{-1/2}|y-z|p(0,s;0,y)p(s,t;y,z)h(0,y;x_0,\xi)$$
$$\times |y-z|^{-1}\left[\int \frac{h(0,z;x_0,\eta)h(z,y;\eta,\xi)}{h(0,y;x_0,\xi)} f(\eta)d\eta - f(\xi)\right]dz\,dy\bigg\}.$$

It is easy to see by Lemma 3.3 that

$$\lim_{t\searrow s}\left\{\int_0^\infty p(0,s;0,y)h(0,y;x_0,\xi)\,dy\right\}^{-1}\left\{\int_0^\infty \int_y^\infty (t-s)^{-1/2}|y-z|p(0,s;0,y)\right.$$
$$\left.\times p(s,t;y,z)h(0,y;x_0,\xi)|y-z|^{-1}\left[\int h(y,z;\xi,\eta)f(\eta)\,d\eta - f(\xi)\right]dz\,dy\right\}$$
$$=\frac{1}{\sqrt{2\pi}}\mathscr{A}f(\xi).$$

So let us consider the last term in (3.10). From Aronson's inequality (3.2) and Lemma 3.2, when $|x_0-\xi|>0$,

$$\int_0^y (t-s)^{-1/2}|y-z|p(s,t;y,z)h(0,y;x_0,\xi)$$
$$\times |y-z|^{-1}\left[\int \frac{h(0,z;x_0,\eta)h(z,y;\eta,\xi)}{h(0,y;x_0,\xi)} f(\eta)\,d\eta - f(\xi)\right]dz$$
$$\le c_4 h(0,y;x_0,\xi)y^{-c_2}$$

is bounded in $(t-s,y)$ for fixed ξ, and we may pass to the limit through the integral over $\mathbb{R}_+$. Thus, the following half-derivative exists for every $s>0$ and is given by

$$\lim_{t\searrow s}(t-s)^{-1/2}\{\mathbb{E}_{\mathbb{P}}[f(X(|B_t|))|X(|B_s|)]-f(X(|B_s|))\}$$
$$=\frac{1}{\sqrt{2\pi}}\left[\mathscr{A}f(X(|B_s|))+\frac{\int_0^\infty p(0,s;0,y)h(0,y;x_0,X(|B_s|))\mathscr{A}_y^* f(X(|B_s|))dy}{\int_0^\infty p(0,s;0,y)h(0,y;x_0,X(|B_s|))dy}\right],$$

proving 0.5.

APPENDIX

We now rigorize and prove our claim in statement 1 of the Introduction that $\mathbb{X}_{B,c}^{x}(t)$ is the $k\to\infty$ limit of $\mathbb{X}_{B,c}^{x,k}(t)$. This is accomplished by showing weak convergence of the process $\{\mathbb{X}_{B,c}^{x,k}(t);0\le t<\infty\}$ to $\{\mathbb{X}_{B,c}^{x}(t);0\le t<\infty\}$. Without losing generality, we may assume that, for each $p>0$, there are positive constants $c_{1,p}$, $c_{2,p}$ and $c_{3,p}$ such that

$$\mathbb{P}\left[\sup_{a\le s\le t\le a+b}|X^{x,1}(t)-X^{x,1}(s)|^p > c_{1,p}b^{c_{2,p}p}\right] \le \exp\left\{-\frac{c_{3,p}}{b}\right\} \qquad \forall\, a,b\ge 0. \tag{A.1}$$

Clearly, (A.1) is true when X^x is a Brownian motion, which is α-Hölder continuous for any $\alpha < 1/2$. For a general X^x, we see that the martingale part of the diffusion process X^x is of α-Hölder continuous for any $\alpha < 1/2$, and the nonmartingale part is differentiable, so it is even smoother, so (A.1) is true here as well. Now, note that the paths which do not satisfy

$$\sup_{a\le s\le t\le a+b} |X^{x,1}(t) - X^{x,1}(s)|^p \le c_{1,p} b^{c_{2,p} p}$$

have exponentially small probability, so they can be thrown away when $t-s$ is small.

THEOREM 5.1. *There is a positive constant c such that, for each $p > 0$, there is a positive constant $C(p)$ satisfying*

$$\text{(A.2)} \qquad \mathbb{E}_{\mathbb{P}}\big|\mathbb{X}^{x,k}_{B,c}(s) - \mathbb{X}^{x,k}_{B,c}(t)\big|^p \le C(p)|s-t|^{cp} \quad \forall\, 0 \le qs \le t < \infty,\ \forall\, k \in \mathbb{N},$$

and this is enough to conclude that there is a subsequence of $\{\mathbb{X}^{x,k}_{B,c}\}$ converging weakly to $\mathbb{X}^x_{B,c}$, as $k \to \infty$.

PROOF. Let $A_{i,s} \triangleq [\mathbb{X}^{x,k}_{B,c}(s) = X^{x,i}(|B(s)|)]$, for $1 \le i \le k$ and $0 \le s < \infty$. We then have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\big|\mathbb{X}^{x,k}_{B,c}(s) - \mathbb{X}^{x,k}_{B,c}(t)\big|^p &= \sum_{i,j=1}^{k} \mathbb{E}_{\mathbb{P}}\big\{1_{A_{i,s}} 1_{A_{j,t}} \big|\mathbb{X}^{x,k}_{B,c}(s) - \mathbb{X}^{x,k}_{B,c}(t)\big|^p\big\} \\
&= \sum_{\substack{i,j=1\\ i\ne j}}^{k} \mathbb{E}_{\mathbb{P}}\big\{1_{A_{i,s}} 1_{A_{j,t}} \big|X^{x,i}(|B(s)|) - X^{x,j}(|B(t)|)\big|^p\big\} \\
&\quad + \sum_{i=1}^{k} \mathbb{E}_{\mathbb{P}}\big\{1_{A_{i,s}} 1_{A_{i,t}} \big|X^{x,i}(|B(s)|) - X^{x,i}(|B(t)|)\big|^p\big\} \\
&= k(k-1)\mathbb{E}_{\mathbb{P}}\big\{1_{A_{s,1}} 1_{A_{t,2}} \big|X^{x,1}(|B(s)|) - X^{x,2}(|B(t)|)\big|^p\big\} \\
&\quad + k\mathbb{E}_{\mathbb{P}}\{1_{A_{1,s}} 1_{A_{1,t}} |X^{x,1}(|B(s)|) - X^{x,1}(|B(t)|)|^p\},
\end{aligned}$$

where the last equality follows from symmetry. From the definition of $\mathbb{X}^{x,k}_{B,c}(\cdot)$, it is easy to see that the following inclusion of events is true when $i \ne j$:

$$\big[\mathbb{X}^{x,k}_{B,c}(s) = X^{x,i}(|B(s)|)\big] \cap \big[\mathbb{X}^{x,k}_{B,c}(t) = X^{x,j}(|B(t)|)\big] \subset \big[\inf_{s\le u\le t} |B(u)| = 0\big] \triangleq S_{s,t}.$$

Thus, by symmetry,

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\big|\mathbb{X}^{x,k}_{B,c}(s)-\mathbb{X}^{x,k}_{B,c}(t)\big|^p \\
&\qquad \le \mathbb{E}_{\mathbb{P}}\big\{1_{S_{s,t}}\big|X^{x,1}(|B(s)|)-X^{x,2}(|B(t)|)\big|^p\big\} \\
&\qquad\quad + k\mathbb{E}_{\mathbb{P}}\big\{1_{A_{1,s}}1_{A_{1,t}}\big|X^{x,1}(|B(s)|)-X^{x,1}(|B(t)|)\big|^p\big\} \\
&\qquad \le C_p\mathbb{E}_{\mathbb{P}}\big\{1_{S_{s,t}}\big[\big|X^{x,1}(|B(s)|)-x\big|^p+\big|x-X^{x,2}(|B(t)|)\big|^p\big]\big\} \\
&\qquad\quad + k\mathbb{E}_{\mathbb{P}}\big\{\big|X^{x,1}(|B(s)|)-X^{x,1}(|B(t)|)\big|^p\big\}.
\end{aligned}
\tag{A.3}
$$

As $x = X^{x,i}(0)$, then by (A.1) and the remarks following it and (A.3), we obtain

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\big|\mathbb{X}^{x,k}_{B,c}(s)-\mathbb{X}^{x,k}_{B,c}(t)\big|^p \\
&\qquad \le Cc_{1,p}\mathbb{E}_{\mathbb{P}}\big\{1_{S_{s,t}}\big[|B(s)|^{c_{2,p}p}+|B(t)|^{c_{2,p}p}\big]\big\} \\
&\qquad\quad + c_{1,p}\mathbb{E}_{\mathbb{P}}\big\{\big||B(s)|-|B(t)|\big|^{c_{2,p}p}\big\} \\
&\qquad \le Cc_{1,p}\mathbb{E}_{\mathbb{P}}\big\{1_{S_{s,t}}[|B(s)|^{c_{2,p}p}+|B(t)|^{c_{2,p}p}]\big\} + c_{4,p}\mathbb{E}_{\mathbb{P}}\big\{|t-s|^{c_{5,p}p}\big\},
\end{aligned}
\tag{A.4}
$$

where C is a generic constant whose value may vary from line to line and $c_{4,p}$ and $c_{5,p}$ are new constants obtained by the well-known property of Brownian motion: there is a constant $\overline{C}_p$ such that

$$
\mathbb{E}_{\mathbb{P}}\left\{\sup_{s_0\le s\le t\le t_0}\big\{|B(t)-B(s)|^p\big\}\right\} \le \overline{C}_p|t_0-s_0|^{p/2} \qquad \forall\ 0\le s\le t<\infty. \tag{A.5}
$$

On the other hand, it is easy to see that

$$
1_{S_{s,t}}\left\{|B(s)|^{c_{2,p}p}+|B(t)|^{c_{2,p}p}\right\} \le 2\sup_{s\le u\le v\le t}\{|B(v)-B(u)|^p\}. \tag{A.6}
$$

Thus, (A.2) can be easily deduced from (A.4), (A.6) and (A.5).

It is well known (see, e.g., [7] and [12]) that Kolmogorov's criterion implies that the sequence of processes $\{\mathbb{X}^{x,k}_{B,c}(t);\ 0\le t<\infty\}_k$ is tight in law under the uniform convergence topology. It is easy to check that any limit of the convergent subsequence of $\{\mathbb{X}^{x,k}_{B,c}\}$ gives the law of $\mathbb{X}^{x}_{B,c}$. Thus we proved statement 1 in Section 0. □

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