QUASI-STATIONARY MEASURES FOR CONSERVATIVE DYNAMICS IN THE INFINITE LATTICE

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We study quasi-stationary measures for conservative particle systems in the infinite lattice. Existence of quasi-stationary measures is established for a fairly general class of reversible systems. For the special cases of a system of independent random walks and the symmetric simple exclusion process, it is shown that qualitative features of quasi-stationary measures change drastically with dimension.

1. Introduction. In this article we establish existence and qualitative properties of quasi-stationary measures for stochastic systems of particles moving in the infinite lattice $\mathbb{Z}^d$. To illustrate the notion of quasi-stationarity, consider particles moving in a large but finite subset Λ of $\mathbb{Z}^d$, that are subject to exclusion (at most one particle can occupy a site $i \in \mathbb{Z}^d$), to a sufficiently strong attractive potential and some stochastic exchange dynamics, in such a way that typical configurations in the stationary measure consist of a single cluster of particles. Suppose that at time $t = 0$ particles are uniformly distributed in Λ with a very low density. The density will remain nearly homogeneous until the formation of a sufficiently large droplet (the *critical droplet*), which in a relatively short time grows to the single cluster.

A relevant problem in this context is to describe how typical configurations look like before the critical droplet forms, that is, in the *metastable regime*.

More generally, let $(\eta_t)_{t \geq 0}$ be a Markov process taking values in some measurable space $(E, \mathscr{E})$, and $A \subset E$ be such that

$$\tau_A = \inf\{t \geq 0 \colon \eta_t \in A\}$$

is a stopping time. Also, for a given probability measure μ on E, let P_μ be the law of η_t with initial measure μ, and E_μ the corresponding expectation.

The phenomenon described above motivates the following definition.

DEFINITION 1. A probability measure μ on $(E, \mathscr{E})$ is a quasi-stationary measure if $P_\mu(\tau_A > 0) > 0$ and if for every $t \geq 0$ and every $f \colon E \to \mathbb{R}$ measurable and bounded

(1.1) $$E_\mu(f(\eta_t)|\tau_A > t) = \int f \, d\mu.$$

We denote the set of quasi-stationary measures by $\mathscr{D}$.

We remark that, in the context of Markov chains with countable state space, quasi-stationary measures are usually defined for absorbing sets A; no such assumption on A is made here.

The aim of this paper is to establish existence and various properties of quasi-stationary measures for a class of conservative particle systems with configuration space $\mathscr{C} \subset S^{\mathbb{Z}^d}$, with $S = \{0, 1\}$ or $S = \mathbb{N}$.

After showing that, for $\mu \in \mathscr{D}$, there exists a nonnegative $\lambda(\mu)$ such that

$$P_\mu(\tau_A > t) = e^{-\lambda(\mu)t},$$

we prove for a fairly general class of reversible dynamics that $\mathscr{D}$ is nonempty, and that $\lambda(\mu) = \lambda_A$, where λ_A is the bottom of the spectrum of the generator of the process "stopped" when entering A.

The identification of $\lambda(\mu)$ with λ_A is particularly relevant: we will see that for some examples of conservative particle systems, $\lambda_A = 0$ is equivalent to $\mathscr{D} = \{\delta_{\mathbf{0}}\}$, where $\delta_{\mathbf{0}}$ is the Dirac measure on $\mathbf{0}$ the configuration with no particles.

There are, at least, two natural approaches. Let L be the generator of the process and ν be a reversible measure for the dynamics. Consider the eigenvalue problem

$$Lf + \lambda f = 0$$

in the Hilbert space $H_A = \{f \in L^2(\nu): f(\eta) = 0 \text{ for } \nu\text{-almost every } \eta \in A\}$. It is not hard to show that if we can find an eigenvalue λ with a nonnegative eigenvector then the probability measure $d\mu = f d\nu / \int f \, d\nu$ is quasi-stationary. This approach is certainly suitable in the context of finite Markov chains (see, e.g., [7]), where one can rely on the Perron–Frobenius theorem. In more general cases, however, it seems hard to obtain existence of quasi-stationary measures from purely functional analytic considerations.

Another approach is based on the remark that any $\mu \in \mathscr{D}$ is a fixed point for the maps T_t, $t > 0$, acting on probability measures on E, defined by

$$\int f d(T_t \mu) = E_\mu(f(\eta_t)|\tau_A > t).$$

For a given probability ν, define $\nu_t = T_t \nu$. If the limit $\mu = \lim_{t \to +\infty} \nu_t$ exists, then it is called the *Yaglom limit*. Since T_t has the semigroup property, μ is a natural candidate for being in $\mathscr{D}$. In general, the existence of the Yaglom limit is a nontrivial matter. Our approach in this paper consists in showing that *any* limit point of the Cesaro's means of the ν_t belongs to $\mathscr{D}$, provided $\nu_0 = \nu$ is a reversible measure for the system. For a class of nonreversible systems with countable state space and discrete time, the existence of quasi-stationary measures has been established in [6].

The paper is organized as follows. In Section 2, we introduce the class of conservative dynamics which we consider. Section 3 contains existence results and characterization of the parameter of the exponential time τ_A, when initial configurations are drawn from $\mu \in \mathscr{D}$; these results are very general. Indeed, it is only in the second part of Theorem 2 that the actual form of the generator

appears, and it is clear that even that part can be modified to cover many other dynamics, including Glauber dynamics and various classes of infinite dimensional diffusions. However, for simplicity, we restrict ourselves to a class of conservative particle dynamics in the infinite lattice.

Systems of independent random walks are studied in Section 4. We show that, for a suitable class of sets A, there is a trichotomy. In dimensions 1 and 2, we show that starting from ν, a reversible measure for the dynamics, the Yaglom limit exists and is equal to the trivial measure $\delta_{\mathbf{0}}$. In dimensions $d \geq 3$, for every ν extremal and reversible, we obtain different nontrivial limit points of the Cesaro means. For $d \geq 5$, we show that the Yaglom limit exists and that it is absolutely continuous with respect to the initial measure ν. In dimensions 3 and 4, we give an example in which the Yaglom limit exists but is singular with respect to ν. Some open problems are listed in Section 6.

2. Models and notations. Let $S = \{0, 1\}$ or $S = \mathbb{N}$. We denote by $\mathscr{C}$ the configuration space; for $S = \{0, 1\}$, we choose $\mathscr{C} = S^{\mathbb{Z}^d}$ provided with the product topology, while, for $S = \mathbb{N}$, we choose

$$\mathscr{C} = \left\{ \eta \in S^{\mathbb{Z}^d} : \|\eta\| \equiv \sum_{i \in \mathbb{Z}^d} \eta(i) e^{-|i|} < +\infty \right\}$$

with the topology induced by the norm $\|\cdot\|$. For $\eta \in \mathscr{C}$ and $\Lambda \subset \mathbb{Z}^d$, we denote by $\eta(\Lambda)$ the element of S^Λ obtained by restricting η to the components in Λ. For every finite subset Λ of $\mathbb{Z}^d$ (we write $\Lambda \Subset \mathbb{Z}^d$), let $\Phi_\Lambda \colon S^\Lambda \to \mathbb{R}$ be bounded maps satisfying $\Phi_{\Lambda+i} = \Phi_\Lambda \circ \theta_i$ for all $\Lambda \Subset \mathbb{Z}^d$, $i \in \mathbb{Z}^d$, where θ_i is the shift map on $S^{\mathbb{Z}^d} \colon (\theta_i \eta)(j) = \eta(i + j)$. We assume

$$(2.1) \qquad \sum_{\Lambda \ni 0} |\Lambda| \|\Phi_\Lambda\|_\infty < +\infty.$$

After letting, for $X \Subset \mathbb{Z}^d$, $\eta \in \mathscr{C}$,

$$H_X(\eta) = \sum_{\Lambda \cap X \neq \varnothing} \Phi_\Lambda(\eta),$$

we denote by $\mathscr{G}_\rho(\Phi)$ the set of probability measures (*Gibbs measures*) μ on $\mathscr{C}$ whose conditional measures $\mu(\eta(X)|\eta(X^c))$, $X \Subset \mathbb{Z}^d$, are given by

$$\mu(\eta(X)|\eta(X^c)) = \frac{e^{-H_X(\eta)}}{Z_X(\eta(X^c))} \prod_{i \in X} \nu_\rho(\eta(i)),$$

where ν_ρ is the Bernoulli measure of density $\rho \in [0, 1)$ in the case $S = \{0, 1\}$, and the Poisson measure of density $\rho \in [0, +\infty)$ in the case $S = \mathbb{N}$, and Z_X is the normalization factor.

For $i, j \in \mathbb{Z}^d$, we write $i \sim j$ if $|i - j| = 1$. When $S = \{0, 1\}$, we define $T^{i,j}\eta(j) = \eta(i)$, $T^{i,j}\eta(i) = \eta(j)$ and for $k \neq i, j$ $T^{i,j}\eta(k) = \eta(k)$. Also, when $S = \mathbb{N}$, we define $T^{i,j}\eta(j) = \eta(j) + 1$, $T^{i,j}\eta(i) = \eta(i) - 1$ and for $k \neq i, j$ $T^{i,j}\eta(k) = \eta(k)$. For every pair $i \sim j$, we introduce the rates of jump

$$(2.2) \qquad c_{i,j}(\eta) = \exp\bigl[-(H_{i,j}(T^{i,j}\eta) - H_{i,j}(\eta))/2\bigr].$$

For each $i \sim j$, $c_{i,j}: \mathscr{C} \to \,]0, +\infty)$ is a bounded continuous function which satisfies $c_{i,j} = c_{j,i} = c_{0,j-i} \circ \theta_i$, and the *detailed balance* property

$$c_{i,j}(\eta)e^{-H_{\{i,j\}}(\eta)} = c_{i,j}(T^{i,j}\eta)e^{-H_{\{i,j\}}(T^{i,j}\eta)}.$$

Finally, let L be the operator acting on functions from $\mathscr{C}$ to $\mathbb{R}$ formally defined by

$$(2.3) \qquad Lf(\eta) = \sum_{i \in \mathbb{Z}^d} \sum_{j \sim i} \eta(i)c_{i,j}(\eta)\big[f(T^{i,j}(\eta)) - f(\eta)\big].$$

For the case $S = \{0, 1\}$, it is an easy computation, which we omit, to check that the rates satisfy Liggett's condition [formula (3.8) of [10], page 26] (with the notation $\eta^k(i) = \eta(i)$ for $i \neq k$ and $\eta^k(k) = 1 - \eta(k)$),

$$(2.4) \qquad \sup_i \sum_{k \in \mathbb{Z}^d} \sum_{j \sim i} \sup_\eta |c_{i,j}(\eta^k) - c_{i,j}(\eta)| < \infty.$$

Condition (2.4) guarantees that the restriction of L to local functions (functions depending on a finite number of components) has a graph closure in the space of continuous functions on $\mathscr{C}$ which is the infinitesimal generator of a Feller process on the Skorokhod space of right continuous functions with left limits (Theorem 3.9 of [10]).

For the case $S = \mathbb{N}$, the process can be constructed following step by step Liggett's construction in [9]. For a large class of such processes, the zero range processes, the associated semigroups map local functions into continuous ones in the norm topology on $\mathscr{C}$ (see proof of Lemma 2.2 of [1]).

Both for $S = \{0, 1\}$ and $S = \mathbb{N}$, given $\mu \in \mathscr{I}_\rho(\Phi)$, the restriction of L to bounded local functions has a graph closure in $L^2(\mu)$ that is a self-adjoint operator (see [11], Chapter 1). It follows that $P_\mu \equiv \int P_\eta \mu(d\eta)$ is the law of a stationary reversible process.

We finally remark that the special case $\Phi_\Lambda \equiv 0$, $c_{i,j} \equiv 1$ corresponds to the symmetric simple exclusion process (SSEP) when $S = \{0, 1\}$ and to a system of independent random walks (IRW) when $S = \mathbb{N}$.

3. General results. Let ν be a fixed element of $\mathscr{I}_\rho(\Phi)$. For a given $\Lambda \Subset \mathbb{Z}^d$, let $\mathscr{F}_\Lambda$ be the σ-algebra generated by $\{\eta(i): i \in \Lambda\}$. The elements of $\cup_{\Lambda \Subset \mathbb{Z}^d} \mathscr{F}_\Lambda$ are called local. In what follows, we fix $A \in \mathscr{F}_\Lambda$ and we write τ for τ_A. We recall

$$\int f d\nu_t = E_\nu(f(\eta_t)|\tau > t)$$

and define

$$\bar{\nu}_t = \frac{1}{t} \int_0^t \nu_s \, ds.$$

We remark that the conditional expectation $E_\nu(f(\eta_t)|\tau > t)$ makes sense since $P_\nu(\tau > t) > 0$ for every $t > 0$. Indeed, by locality of A and the fact that, for $S = \{0, 1\}$, we assume $\rho < 1$, it is easy to see that $P_\nu(\tau > s) > 0$ for s small

enough. Moreover, by reversibility, $P_\nu(\tau > 2s) = \int \nu(d\eta)P_\eta^2(\tau > s) > 0$, and by iteration, we get $P_\nu(\tau > t) > 0$ for all $t > 0$.

A relevant object is the stopped process $\eta_{t\wedge\tau}$. By $\widetilde{S}_t f(\eta) = E_\eta(f(\eta_{t\wedge\tau}))$ we define a sub-Markovian semigroup on the Hilbert space $H_A = \{f \in L^2(\nu): f \equiv 0 \text{ on } A\}$, with the scalar product induced by $L^2(\nu)$. Let $\mathscr{D}$ be the domain obtained by closing, on $L^2(\nu) \times L^2(\nu)$, the graph of L restricted to bounded local functions; we still denote by L the extended self-adjoint operator in $\mathscr{D}$. Let $\mathscr{D}_A = \mathscr{D} \cap H_A$. The following rather elementary fact will be repeatedly used in the paper.

LEMMA 1. *The operator $\widetilde{L}$ defined on $\mathscr{D}_A$ by*

$$\widetilde{L}f(\eta) = 1_{A^c}(\eta)Lf(\eta)$$

is self-adjoint on H_A, and $\widetilde{S}_t = e^{t\widetilde{L}}$.

PROOF. Self-adjointness of $\widetilde{L}$ on $\mathscr{D}_A$ is elementary. In order to show that $\widetilde{S}_t = e^{t\widetilde{L}}$, it is enough to show that, for every local bounded function f, such that $f \equiv 0$ on A, we have

(3.1) $$E_\eta(f(\eta_{t\wedge\tau})) - f(\eta) = E_\eta \int_0^t \widetilde{L}f(\eta_{s\wedge\tau})\,ds,$$

where E_η denotes expectation with respect to P_η. To see (3.1), observe that the process $f(\eta_t) - f(\eta) - \int_0^t Lf(\eta_s)\,ds$ is a P_η-martingale. Thus, by the optional sampling theorem, the process $f(\eta_{t\wedge\tau}) - f(\eta) - \int_0^{t\wedge\tau} Lf(\eta_s)\,ds$ is also a P_η-martingale. This, and the fact that $\eta_\tau \in A$ P_η-a.s. (since $t \mapsto \eta_t$ is a right continuous path), yields

$$E_\eta(f(\eta_{t\wedge\tau})) - f(\eta) = E_\eta \int_0^{t\wedge\tau} Lf(\eta_s)\,ds = E_\eta \int_0^t 1_{A^c}(\eta_{s\wedge\tau})Lf(\eta_{s\wedge\tau})\,ds$$

$$= E_\eta \int_0^t \widetilde{L}f(\eta_{s\wedge\tau})\,ds. \qquad \square$$

In Theorem 1 below, the Feller property of the stopped semigroup $\widetilde{S}_t$ is crucial. For $S = \{0, 1\}$, the arguments of [10], Theorem 3.9, yield that $\widetilde{L}$ is a sub-Markovian generator on the Banach space of continuous functions so that $\widetilde{S}_t$ is Feller.

For $S = \mathbb{N}$, this property is more delicate. We will in general assume that for each bounded local function f, $\widetilde{S}_t f$ is continuous with respect to the norm topology. In Section 4, we show that it holds for independent random walks.

3.1. *Existence.* Henceforth, A will be a local event. Also, when $S = \mathbb{N}$, we assume here that for each bounded local function f, $\widetilde{S}_t f$ is continuous with respect to the norm topology.

As in the case of stationary measures (see, e.g., [10], Proposition I.1.8), limit points of the Cesaro's mean are more convenient than limit points of ν_t.

THEOREM 1. *Suppose that the Cesaro's means $(\bar{\nu}_t)_{t \geq 0}$ have a limit point μ in the weak topology. Then μ is a quasi-stationary measure.*

For $S = \{0, 1\}$, Theorem 1 implies that the weak limit points of $\bar{\nu}_{\rho, t}$ belong to $\mathscr{D}$. For $S = \mathbb{N}$, tightness of Cesaro's means has to be shown.

The key step consists in determining the long-time behavior of the probability of the conditioning event $\{\tau > t\}$.

LEMMA 2. *There exists $\lambda \in [0, +\infty]$ such that for every $s > 0$,*

$$\lim_{t \to +\infty} \frac{P_\nu(\tau > t + s)}{P_\nu(\tau > t)} = e^{-\lambda s}.$$

PROOF. First note that

$$\frac{P_\nu(\tau > t + s)}{P_\nu(\tau > t)} = \frac{E_\nu[e^{(t+s)\widetilde{L}}1_{A^c}]}{E_\nu[e^{t\widetilde{L}}1_{A^c}]} = \frac{\int_0^{+\infty} e^{-(t+s)x} P(dx)}{\int_0^{+\infty} e^{-tx} P(dx)},$$

where P is the spectral measure associated to $-\widetilde{L}$ and to the function $f = \frac{1_{A^c}}{[\nu(A^c)]^{1/2}}$. Since P is a probability measure, we may consider a positive random variable X with law P. We have

(3.2)
$$\frac{P_\nu(\tau > t + s)}{P_\nu(\tau > t)} = \frac{E(e^{-sX}e^{-tX})}{E(e^{-tX})}.$$

Convexity of the logarithmic moment generating function $\log Ee^{-tX}$ immediately implies that the expressions in (3.2) are nondecreasing in t, and therefore they have a limit in $[0, 1]$ that we denote by $a(s)$. It is easily seen that $a(t + s) = a(t)a(s)$ for all $s, t > 0$, and this completes the proof. $\square$

PROOF OF THEOREM 1. Suppose

$$\mu = \lim_n \frac{1}{t_n} \int_0^{t_n} \nu_s \, ds.$$

We first note that $\nu_t(A^c) = 1$, and, as 1_{A^c} is a bounded continuous function, $\mu(A^c) = 1$. This, together with locality of A, implies $P_\mu(\tau > 0) > 0$.

We first show that, for f bounded and local,

(3.3)
$$E_\mu[f(\eta_t)1_{\tau>t}] = e^{-\lambda t} \int f \, d\mu,$$

where λ is the constant introduced in Lemma 2.

Indeed, using Lemma 2 and the assumed Feller property of $\widetilde{S}_T$

$$E_\mu[f(\eta_t)1_{\tau>t}] = \lim_n \frac{1}{t_n} \int_0^{t_n} \frac{E_\nu[f(\eta_{s+t})1_{\tau>s+t}]}{P_\nu(\tau > s)} \, ds$$

$$= \lim_n \frac{1}{t_n} \int_0^{t_n} \frac{E_\nu[f(\eta_{s+t})1_{\tau>s+t}]}{P_\nu(\tau > s + t)} \frac{P_\nu(\tau > s + t)}{P_\nu(\tau > s)} \, ds$$

$$= e^{-\lambda t} \lim_n \frac{1}{t_n} \int_t^{t+t_n} \frac{E_\nu[f(\eta_s)1_{\tau > s}]}{P_\nu(\tau > s)} \, ds$$

$$= e^{-\lambda t} \lim_n \frac{1}{t_n} \int_0^{t_n} \frac{E_\nu[f(\eta_s)1_{\tau > s}]}{P_\nu(\tau > s)} \, ds = e^{-\lambda t} \int f \, d\mu,$$

where we have used the fact that

$$\left| \int_t^{t+t_n} \frac{E_\nu[f(\eta_s)1_{\tau > s}]}{P_\nu(\tau > s)} \, ds - \int_0^{t_n} \frac{E_\nu[f(\eta_s)1_{\tau > s}]}{P_\nu(\tau > s)} \, ds \right| \le 2t\|f\|_\infty.$$

In particular, if we set $f \equiv 1$ in (3.3) we obtain

(3.4)
$$P_\mu(\tau > t) = e^{-\lambda t}$$

(which actually implies $\lambda < +\infty$ since, as shown above, $P_\mu(\tau > 0) > 0$) so that

$$E_\mu(f(\eta_t)|\tau > t) = \int f \, d\mu.$$

The extension of the last equality to all bounded measurable function follows from Dynkin class theorem (see, e.g., [4], Theorem 3, page 16). □

REMARK 1. Property (3.4) is shared also by all limit points of (ν_t). Indeed, if $\mu = \lim_n \nu_{t_n}$, then

$$P_\mu(\tau > t) = \lim_n \frac{P_\nu(\tau > t + t_n)}{P_\nu(\tau > t_n)} = e^{-\lambda t}$$

by Lemma 2.

3.2. *Characterization of λ.* In this section we characterize the constant $\lambda \in [0, +\infty)$ as the bottom of the spectrum of $-\widetilde{L}$ in H_A, given by

(3.5)
$$\tilde{\lambda}(\nu, A) = \inf_{f \in \mathscr{D}_A, f \neq 0} \frac{(f, -\widetilde{L}f)_\nu}{(f, f)_\nu} = \inf_{f \in \mathscr{D}_A, f \neq 0} \frac{(f, -Lf)_\nu}{(f, f)_\nu},$$

where $(\cdot, \cdot)_\nu$ is the scalar product in (H_A, ν).

We first note that Lemma 2 yields

$$\lim_{t \to +\infty} [\log P_\nu(\tau > t + 1) - \log P_\nu(\tau > t)] = -\lambda$$

and therefore

$$-\lambda = \lim_{t \to \infty} \frac{1}{t} \log P_\nu(\tau > t).$$

THEOREM 2.

$$\lambda = \tilde{\lambda}(\nu, A).$$

PROOF. Upper bound. Let $f = 1_{A^c}/(\nu(A^c))$. We have

$$P_\nu(\tau > t) = E_\nu[e^{t\widetilde{L}}f]\nu(A^c) = \int_{\tilde\lambda(\nu,A)}^{+\infty} e^{-tx} P(dx)\nu(A^c) \le e^{-t\tilde\lambda(\nu,A)},$$

where P is the spectral measure associated to $-\widetilde{L}$ and f.

Lower bound. Let $g: \mathscr{C} \to [0, +\infty]$ be a local function such that $g(\eta) = +\infty$ for $\eta \in A$, and g is bounded on A^c. In other words, e^{-g} is any bounded local function with value 0 on A and strictly positive on A^c. Consider the generator

$$L^g f(\eta) = \sum_{i\in\mathbb{Z}^d}\sum_{i\sim j} \eta(i)c_{i,\,j}(\eta)e^{g(\eta)-g(T^{i,j}\eta)}[f(T^{i,j}\eta) - f(\eta)].$$

Since g is local, the Markov family P_η^g associated to L^g may be obtained by the Girsanov transformation

$$\frac{dP_\eta^g}{dP_\eta}\bigg|_{\mathscr{F}_t} = \exp\bigg[g(\eta_t) - g(\eta_0) - \sum_i\sum_{j\sim i}\int_0^t \eta_s(i)c_{i,\,j}(\eta_s)[e^{g(\eta_s)-g(T^{i,j}\eta_s)} - 1]\,ds\bigg],$$

where $\mathscr{F}_t = \sigma\{\eta_s: s \le t\}$. L^g has been built to satisfy detailed balance for the probability measure ν_g, supported on A^c, given by

$$\frac{d\nu_g}{d\nu}(\eta) = \frac{e^{-2g(\eta)}}{\int e^{-2g(\eta)}\nu(d\eta)}.$$

In particular, $P^g = \int P_\xi^g \nu_g(d\xi)$ is the law of a stationary, reversible Markov process.

Now, since the event $\{\tau > t\}$ is $\mathscr{F}_t$-measurable, using Jensen's inequality,

$$\log P_\nu(\tau > t) = \log \int 1_{\tau>t}\bigg(\frac{dP^g}{dP_\nu}\bigg|_{\mathscr{F}_t}\bigg)^{-1} dP^g$$

$$(3.6) \qquad \ge \int \log\bigg(\frac{d\nu_g}{d\nu}\bigg)d\nu_g + t\sum_i\sum_{j\sim i}\int \eta(i)c_{i,\,j}(\eta)[e^{g(\eta)-g(T^{i,j}\eta)} - 1]\nu_g(d\eta)$$

$$= \int \log\bigg(\frac{d\nu_g}{d\nu}\bigg)d\nu_g - t\frac{(e^{-g}, -Le^{-g})_\nu}{(e^{-g}, e^{-g})_\nu}.$$

It follows that

$$(3.7) \qquad \lim_{t\to\infty}\frac{1}{t}\log P_\nu(\tau > t) \ge -\frac{(f, -Lf)_\nu}{(f, f)_\nu}$$

for every function f vanishing on A and strictly positive on A^c. Inequality (3.7) is easily extended to all nonnegative bounded local functions f vanishing on A and with $(f, f)_\nu > 0$, by applying (3.7) to $f_\varepsilon = f + \varepsilon 1_{A^c}$, and observing that $(f_\varepsilon, -Lf_\varepsilon)_\nu \to (f, -Lf)_\nu$ and $(f_\varepsilon, f_\varepsilon)_\nu \to (f, f)_\nu$ as $\varepsilon \downarrow 0$. Moreover, one checks directly that if f is a bounded and local function, then

$$(3.8) \qquad (f, -Lf)_\nu \ge (|f|, -L|f|)_\nu.$$

By (3.7) and (3.8), we have that

$$\lim_{t \to \infty} \frac{1}{t} \log P_\nu(\tau > t) \geq -\frac{(f, -Lf)_\nu}{(f, f)_\nu}$$

for every bounded local f vanishing on A. Since bounded local functions are a core for the Dirichlet form $(f, -Lf)$, the conclusion follows from [11], Lemma I.2.12. $\square$

4. Results for independent random walks. In the language of Section 2, we are considering the system with $S = \mathbb{N}$, $\Phi_\Lambda \equiv 0$ and $c_{i, j} \equiv 1$. In this case $\mathscr{I}_\rho(\Phi)$ has a unique element ν_ρ, that is the infinite product of Poisson measures of density ρ. For simplicity, we write $\nu_{\rho, t}$ (resp. $\bar{\nu}_{\rho, t}$) instead of $(\nu_\rho)_t$ [resp. $\overline{(\nu_\rho)}_t$].

In what follows the natural partial order on $\mathscr{C}$ will be useful. For $\eta, \xi \in \mathscr{C}$, we say that $\eta \leq \xi$ if $\eta(i) \leq \xi(i)$ for all $i \in \mathbb{Z}^d$. Monotonicity of functions from $\mathscr{C}$ to $\mathbb{R}$ will be meant with this partial order; in particular, we will say that $A \subset \mathscr{C}$ is increasing if its indicator function is increasing. Finally, for given probability measures ν, μ on $\mathscr{C}$, we say that $\nu \leq \mu$ if $\int f \, d\nu \leq \int f \, d\mu$ for every increasing function f.

THEOREM 3. *Suppose that $A \neq \mathscr{C}$ is a local, increasing subset of $\mathscr{C}$. Then, for each $\rho > 0$ we have*:

(a) *When f is bounded and local, $\widetilde{S}_t f$ is continuous. Moreover, the Cesaro means $(\bar{\nu}_{\rho, t})_{t \geq 0}$ as well as the measures $(\nu_{\rho, t})_{t \geq 0}$ form tight families of probability measures.*

(b) *For $d = 1$ and $d = 2$ the Yaglom limit exists and equals δ_0, the Dirac measure concentrated on the configuration with no particles.*

(c) *For $d \geq 5$ the Yaglom limit μ_ρ exists. Moreover, $\mu_\rho \ll \nu_\rho$ and $\frac{d\mu_\rho}{d\nu_\rho} \in L^p(\nu_\rho)$ for all $1 \leq p < \infty$.*

(d) *For $d \geq 3$, limit points of the Cesaro means corresponding to distinct values of $\rho \geq 0$ are distinct.*

REMARK 2. We conjecture that the Yaglom limit exists also for $d = 3, 4$.

4.1. *Proof of Theorem* 3(a). Let C be the support of f, Λ_A the support of 1_A and $\{\eta_t, \xi_t, t \geq 0\}$ two systems of independent random walks coupled as in [3]. In this coupling, η and ξ-particles at given site occupy different levels: one ladder for the η particles and one for the ξ-particles. When an η-particle and a ξ-particle happen to be at the same level of the same site, they are matched and evolve from that point on as one random walk independent of all other η and ξ particles. Unmatched particles evolve as random walks independent of all other η and ξ particles. If $P_{\eta, \xi}$ denotes the law of these coupled systems

with initial configurations η and ξ, then we have

$$|E_\eta(f(\eta_t)1_{\tau>t}) - E_\xi(f(\xi_t)1_{\tau>t})|$$
$$\leq 2\|f\|_\infty P_{\eta,\xi}(\exists i \in C \cup \Lambda_A, \exists s \leq t\colon \eta_s(i) \neq \xi_s(i)).$$

It is easy to see that if there is $i \in C \cup \Lambda_A$ and $s \leq t$ such that $\eta_s(i) \neq \xi_s(i)$, then one unmatched particle has entered $C \cup \Lambda_A$ within time t. On the other hand, unmatched particles at time 0 are located on $V := \{i \in \mathbb{Z}^d\colon \eta(i) \neq \xi(i)\}$. Since the probability that a simple random walk travels a distance d within time t is bounded above by $t^d/d!$ ([5], page. 12), we have if $d(i) = \text{dist}(i, C \cup \Lambda_A)$,

$$P_{\eta,\xi}(\exists i \in C \cup \Lambda_A, \exists s \leq t\colon \eta_s(i) \neq \xi_s(i)) \leq \sum_{i \in V}[\eta(i) \vee \xi(i)]\frac{t^{d(i)}}{d(i)!}$$

$$\leq [\|\eta\| \vee \|\xi\|]\sum_{i \in V} e^{|i|}\frac{t^{d(i)}}{d(i)!}.$$

Now, as Λ_A and C are fixed, the right-hand side converges to 0 when $\|\eta - \xi\| \to 0$. The continuity of $E_\eta(f(\eta_t)1_{\tau>t})$ follows.

We show now that $\{\nu_{\rho,t}, t > 0\}$ is a tight family (the proof for $\{\bar\nu_{\rho,t}, t > 0\}$ follows then easily). Note that, for a bounded measurable g,

$$\int g \, d\nu_{\rho,t} = E_{\nu_\rho}[g(\eta_t)1_{\tau>t}]/P_{\nu_\rho}(\tau > t).$$

By reversibility,

$$E_{\nu_\rho}[g(\eta_t)1_{\tau>t}] = E_{\nu_\rho}[g(\eta_0)1_{\tau>t}] = \int g(\eta)P_\eta(\tau > t)\nu_\rho(d\eta).$$

Thus,

$$(4.1) \qquad\qquad f_t(\eta) \equiv \frac{d\nu_{\rho,t}}{d\nu_\rho}(\eta) = \frac{P_\eta(\tau > t)}{P_{\nu_\rho}(\tau > t)}.$$

Using increasingness of A, it easily seen from (4.1) that f_t is a decreasing function of η. By using FKG inequalities for the product measure ν_ρ, we have that for every increasing function g

$$\int g \, d\nu_{\rho,t} = \int g f_t \, d\nu_\rho \leq \int g \, d\nu_\rho$$

and, therefore, $\nu_{\rho,t} \leq \nu_\rho$. In particular, for every $i \in \mathbb{Z}^d$, $\nu_{\rho,t}(\eta(i) > k_i) \leq \nu_\rho(\eta(i) > k_i)$. Now, we choose $k_i = L(|i| + 1)$. Given $\varepsilon > 0$, we can find $L > 0$ such that

$$\sup_{t>0}\nu_{\rho,t}\left(\bigcup_{i \in \mathbb{Z}^d}\{\eta_i \geq k_i\}\right) \leq \nu_\rho\left(\bigcup_{i \in \mathbb{Z}^d}\{\eta_i \geq k_i\}\right) \leq \sum_{i \in \mathbb{Z}^d}\frac{\rho^{L(|i|+1)}}{(L(|i|+1))!} \leq \varepsilon.$$

The set $K = \bigcap_{i \in \mathbb{Z}^d}\{\eta_i \leq k_i\}$ is compact in the norm topology. Indeed, let $\{\eta_n\}$ be a sequence in K. As K is obviously compact in the product topology, let

$\{\eta_{n_k}\}$ be a further subsequence converging to $\eta \in K$ in the product topology. For any $\varepsilon > 0$, there is n such that

$$\sum_{|i| \geq n} |\eta(i) - \eta_{n_k}(i)|e^{-|i|} \leq \sum_{|i| \geq n} 2k_i e^{-|i|} \leq \varepsilon.$$

On the other hand, for k large enough, η and η_{n_k} coincide on $\{i: |i| \leq n\}$. Thus, for k large enough $||\eta_{n_k} - \eta|| \leq \varepsilon$. Tightness follows then from Prohorov's theorem. $\square$

4.2. *Proof of Theorem* 3(b).   Let Λ be a finite subset of $\mathbb{Z}^d$ such that 1_A depends only on $\{\eta(i): i \in \Lambda\}$. Since $A \neq \mathscr{C}$ and A is increasing, it follows that

$$A \subset B = \left\{\eta: \sum_{i \in \Lambda} \eta(i) > 0\right\}$$

and, therefore, $\tau_B \leq \tau$. Define ν_t^B by

$$\int f \, d\nu_t^B = E_{\nu_\rho}[f(\eta_t)|\tau_B > t].$$

Consider now a finite set $U \subset \mathbb{Z}^d \backslash \Lambda$ and, for $i \in U$, let $f_i: \mathbf{N} \to \mathbb{R}$ be bounded. We have

$$\int \prod_{i \in U} f_i(\eta(i))\nu_t^B(d\eta) = \frac{E_{\nu_\rho}[\prod_{i \in U} f_i(\eta_0(i))1_{\tau_B > t}]}{P_{\nu_\rho}(\tau_B > t)}$$

$$= \prod_{i \in U} \frac{\sum_{k=0}^\infty f_i(k)[P(X(i, s) \notin \Lambda \,\forall\, s \in [0, t])]^k e^{-\rho}\frac{\rho^k}{k!}}{\sum_{k=0}^\infty [P(X(i, s) \notin \Lambda \,\forall\, s \in [0, t])]^k e^{-\rho}\frac{\rho^k}{k!}},$$

where $X(i, \cdot)$ is a simple random walk starting at i. It follows that

$$\nu_t^B(d\eta) = \left[\bigotimes_{i \in \Lambda} \delta_0(d\eta(i))\right]\bigotimes\left[\bigotimes_{i \in \Lambda^c} \nu_{\alpha(i, t)}(d\eta(i))\right],$$

where $\nu_{\alpha(i, t)}$ is the Poisson measure of density

$$\alpha(i, t) \equiv P(X(i, s) \notin \Lambda \,\forall\, s \in [0, t]),$$

and the operator "$\otimes$" denotes product of probability measures. For $d = 1$ and $d = 2$, recurrence of the simple random walk implies that $\lim_{t \to \infty} \alpha(i, t) = 0$ for every $i \in \Lambda^c$, so we get the Yaglom limit

$$\lim_{t \to \infty} \nu_t^B = \delta_0.$$

By the results in Section 3, this implies that

$$\lim_{t \to \infty} \frac{1}{t} \log P_{\nu_\rho}(\tau_B > t) = 0.$$

Since $\tau \geq \tau_B$, we have

(4.2) $$\lim_{t \to \infty} \frac{1}{t} \log P_{\nu_\rho}(\tau > t) = 0.$$

Thus, using Remark 1, if μ is a limit point of $(\nu_{\rho,t})$, (4.2) implies

$$(4.3) \qquad\qquad P_\mu(\tau > t) = 1.$$

Define now $l \in \mathbb{N}$ to be the minimum number of particles of an element of A. The dynamics of independent random walks induce a natural dynamics on the family of unordered sequences of length l with elements in $\mathbb{Z}^d$: if the sequence X_0 denotes the positions of l particles in $\mathbb{Z}^d$, then X_t denotes the positions at time t of the l particles that have evolved as independent simple random walks. The resulting Markov process of the positions of the given l particles is irreducible, so that every state is reachable within a given time t with positive probability.

Let now $\eta \in \mathscr{C}$ be a configuration with at least l particles and $t > 0$ be fixed. We label l particles of η and disregard the others. By the argument above, there is a positive probability that, within time t, the configuration consisting of those l particles, evolving as independent random walks, belongs to A. Since A is increasing, the same holds if we consider the evolution of the whole configuration η. In other words,

$$P_\eta(\tau > t) < 1$$

for every η such that $\sum_i \eta(i) \geq l$. Comparing this fact with (4.3), we deduce that μ is concentrated on configurations with less than l particles. Note, also, that (4.3) and the fact that $\mu \in \mathscr{D}$ imply that μ is actually a *stationary* measure for IRW. Now, if U is a finite subset of $\mathbb{Z}^d$ and η is a configuration with finitely many particles, then let $X_1(i_1, \cdot), \ldots, X_k(i_k, \cdot)$ be independent simple random walks starting from i_j, $j = 1, \ldots, k$, and such that $\eta(i) = |\{ j : i_j = i \}|$ for every $i \in \mathbb{Z}^d$. Thus

$$P_\eta\left(\sum_{i \in U} \eta_t(i) > 0 \right) = P_\eta(\exists \ j = 1, \ldots, k \colon X_j(i_j, t) \in U)$$

$$\leq \sum_{j=1}^k P(X_j(i_j, t) \in U),$$

which, by standard Gaussian estimates for random walks, yield

$$(4.4) \qquad \lim_{t \to \infty} P_\eta\left(\sum_{i \in U} \eta_t(i) > 0 \right) = \sum_{j=1}^k \lim_{t \to \infty} P(X_j(i_j, t) \in U) = 0.$$

It follows that

$$\mu\left(\sum_{i \in U} \eta(i) > 0 \right) = \lim_{t \to \infty} \int P_\eta\left(\sum_{i \in U} \eta_t(i) > 0 \right) d\mu = 0.$$

Thus, $\mu = \delta_0$. □

4.3. *Proof of Theorem* 3(c). Let, as above,

$$f_t = \frac{d\nu_{\rho,t}}{d\nu_\rho} = \frac{P_\eta(\tau > t)}{P_{\nu_\rho}(\tau > t)}.$$

We begin by showing that

(4.5) $$\sup_{t \geq 0} \int f_t^p \, d\nu_\rho < +\infty$$

for every $1 \leq p < +\infty$. First, define

$$\alpha_i = \rho P(X(i,s) \notin \Lambda \; \forall s \geq 0)$$

and let ν_{α_i} be the Poisson measure of density α_i. Consider the product measure

$$\nu_{\alpha(\cdot)}(d\eta) = \bigotimes_{i \in \mathbb{Z}^d} \nu_{\alpha_i}(d\eta(i)).$$

Let $\mathscr{G}_n$ be the σ-field in $\mathscr{C}$ generated by the projection $\eta \to \eta(i)$ for $|i| \leq n$, and denote by $\frac{d\nu_{\alpha(\cdot)}}{d\nu_\rho}\big|_{\mathscr{G}_n}$ the Radon–Nykodim derivative between the restrictions of $\nu_{\alpha(\cdot)}$ and ν_ρ to $\mathscr{G}_n$. A simple explicit computation shows that, for each $p \in [1, +\infty)$ there is a $C > 0$ such that

(4.6) $$\max\left[\int\left(\frac{d\nu_{\alpha(\cdot)}}{d\nu_\rho}\Big|_{\mathscr{G}_n}\right)^p d\nu_\rho, \int\left(\frac{d\nu_\rho}{d\nu_{\alpha(\cdot)}}\Big|_{\mathscr{G}_n}\right)^p d\nu_{\alpha(\cdot)}\right] \leq \exp\left[C \sum_{i \in \Lambda_n} \left(1 - \frac{\alpha_i}{\rho}\right)^2\right].$$

Standard estimates on random walks show that [8]

(4.7) $$\left(1 - \frac{\alpha_i}{\rho}\right) \leq \frac{B}{|i|^{d-2}}.$$

Thus, the series

$$\sum_{i \in \mathbb{Z}^d} \left(1 - \frac{\alpha_i}{\rho}\right)^2$$

converges for $d \geq 5$. By (4.6), for $d \geq 5$,

$$\left(\frac{d\nu_{\alpha(\cdot)}}{d\nu_\rho}\Big|_{\mathscr{G}_n}\right)^p, \left(\frac{d\nu_\rho}{d\nu_{\alpha(\cdot)}}\Big|_{\mathscr{G}_n}\right)^p$$

are uniformly integrable submartingales under ν_ρ and $\nu_{\alpha(\cdot)}$, respectively. Then, by the martingale convergence theorem, for $d \geq 5$, $\nu_{\alpha(\cdot)}$ and ν_ρ are equivalent measures, and $\frac{d\nu_{\alpha(\cdot)}}{d\nu_\rho} \in L^p(\nu_\rho)$ for every $p \in [1, +\infty)$.

Now, for $\eta \in \mathscr{C}$, let $B_i \eta$ be defined by $B_i \eta(j) = \eta(j) + \delta_{ij}$, and, for $f \colon \mathscr{C} \to \mathbb{R}$, let $B_i f(\eta) \equiv f(B_i \eta)$. Increasingness of A and a simple coupling argument yield

$$0 \leq P_\eta(\tau > t) - P_{B_i \eta}(\tau > t) \leq P_\eta(\tau > t) P(\forall t, \; X(i,t) \cap \Lambda \neq \varnothing),$$

so that

$$(4.8) \qquad\qquad B_i f_t(\eta) \geq \frac{\alpha_i}{\rho} f_t(\eta).$$

Direct inspection shows that $B_i \frac{d\nu_{\alpha(\cdot)}}{d\nu_\rho} = \frac{\alpha_i}{\rho} \frac{d\nu_{\alpha(\cdot)}}{d\nu_\rho}$ (in particular $\frac{d\nu_{\alpha(\cdot)}}{d\nu_\rho}$ is decreasing). Thus, (4.8) implies that $B_i \frac{d\nu_{\rho,t}}{d\nu_{\alpha(\cdot)}} \geq \frac{d\nu_{\rho,t}}{d\nu_{\alpha(\cdot)}}$ or, equivalently, that $\frac{d\nu_{\rho,t}}{d\nu_{\alpha(\cdot)}}$ is increasing. Since $\nu_{\alpha(\cdot)}$ as any product measure, satisfies FKG inequalities, we deduce that $\nu_{\rho,t} \geq \nu_{\alpha(\cdot)}$. In particular, for $i \geq 1$ and $j \geq 0$ (recalling that f_t and $d\nu_{\alpha(\cdot)}/d\nu_\rho$ are decreasing),

$$\int f_t^i \left(\frac{d\nu_{\alpha(\cdot)}}{d\nu_\rho}\right)^j d\nu_\rho = \int f_t^{i-1}\left(\frac{d\nu_{\alpha(\cdot)}}{d\nu_\rho}\right)^j d\nu_t \leq \int f_t^{i-1}\left(\frac{d\nu_{\alpha(\cdot)}}{d\nu_\rho}\right)^j d\nu_{\alpha(\cdot)}$$

$$= \int f_t^{i-1}\left(\frac{d\nu_{\alpha(\cdot)}}{d\nu_\rho}\right)^{j+1} d\nu_\rho.$$

So, by induction, for each $n \geq 1$,

$$(4.9) \qquad\qquad \int f_t^n \, d\nu_\rho \leq \int \left(\frac{d\nu_{\alpha(\cdot)}}{d\nu_\rho}\right)^n d\nu_\rho.$$

Since the r.h.s. of (4.9) is bounded for $d \geq 5$, (4.5) follows.

The uniform bound (4.5) implies that any limit point of the family $(\nu_{\rho,t})_{t \geq 0}$, as well as $(\bar{\nu}_{\rho,t})_{t \geq 0}$, has a density with respect to ν_ρ that belongs to $L^p(\nu_\rho)$ for all $p \in [1, +\infty)$. It remains to show that there is a unique limit point.

Let μ be a limit point of $(\bar{\nu}_{\rho,t})_{t \geq 0}$. As shown in Section 3, $\mu \in \mathscr{D}$, and

$$P_\mu(\tau > s) = \lim_{t \to \infty} \frac{P_{\nu_{\rho,t}}(\tau > t + s)}{P_{\nu_\rho}(\tau > t)}.$$

Suppose that $\bar{\nu}_{\rho,t_n} \to \mu$ weakly as $n \to +\infty$. Note that

$$\frac{d\bar{\nu}_{\rho,t_n}}{d\nu_\rho} = \frac{1}{t_n}\int_0^{t_n} f_s \, ds.$$

Since $\int \left(\frac{d\bar{\nu}_{\rho,t_n}}{d\nu_\rho}\right)^2 d\nu_\rho$ is uniformly bounded, it is not restrictive to assume that $\frac{d\bar{\nu}_{\rho,t_n}}{d\nu_\rho} \to \frac{d\mu}{d\nu_\rho}$ weakly in $L^2(\nu_\rho)$. In particular,

$$\int \left(\frac{d\mu}{d\nu_\rho}\right)^2 d\nu_\rho = \lim_{n \to +\infty} \frac{1}{t_n}\int_0^{t_n} f_s \frac{d\mu}{d\nu_\rho} \, d\nu_\rho \, ds = \lim_{n \to +\infty} \frac{1}{t_n}\int_0^{t_n} \frac{P_\mu(\tau > s)}{P_{\nu_\rho}(\tau > s)} \, ds.$$

Since

$$\frac{P_\mu(\tau > s)}{P_{\nu_\rho}(\tau > s)} = \lim_{t \to +\infty} \frac{P_{\nu_\rho}(\tau > t + s)}{P_{\nu_\rho}(\tau > s) P_{\nu_\rho}(\tau > t)}$$

and, as seen in Lemma 2, $\frac{P_{\nu_\rho}(\tau > t+s)}{P_{\nu_\rho}(\tau > s)}$ is increasing in s, it follows that $\frac{P_\mu(\tau > s)}{P_{\nu_\rho}(\tau > s)}$ is increasing in s. Thus

$$(4.10) \qquad \int \left(\frac{d\mu}{d\nu_\rho} \right)^2 dv_\rho = \lim_{s \to +\infty} \frac{P_\mu(\tau > s)}{P_{\nu_\rho}(\tau > s)} \equiv a.$$

Let now $\tilde{\mu}$ be a limit point of $(\nu_{\rho,t})_{t \geq 0}$. As before, we find a sequence t_n such that $f_{t_n} \to \frac{d\tilde{\mu}}{d\nu_\rho}$ weakly in $L^2(\nu_\rho)$. Thus

$$(4.11) \qquad \int \frac{d\mu}{d\nu_\rho} \frac{d\tilde{\mu}}{d\nu_\rho} \, dv_\rho = \lim_{n \to +\infty} \int \frac{d\mu}{d\nu_\rho} f_{t_n} \, dv_\rho = a.$$

Finally,

$$
\begin{aligned}
\int \left(\frac{d\tilde{\mu}}{d\nu_\rho} \right)^2 dv_\rho &= \lim_{m \to +\infty} \lim_{n \to +\infty} \int f_{t_n} f_{t_m} \, dv_\rho \\
(4.12) \qquad &= \lim_{m \to +\infty} \lim_{n \to +\infty} \int \frac{P_\eta(\tau > t_n) P_\eta(\tau > t_m)}{P_{\nu_\rho}(\tau > t_n) P_{\nu_\rho}(\tau > t_m)} \, dv_\rho \\
&= \lim_{m \to +\infty} \lim_{n \to +\infty} \frac{P_{\nu_\rho}(\tau > t_n + t_m)}{P_{\nu_\rho}(\tau > t_n) P_{\nu_\rho}(\tau > t_m)} = a,
\end{aligned}
$$

where in the one but last equality we used reversibility. By (4.10), (4.11) and (4.12), we obtain

$$\int \left(\frac{d\mu}{d\nu_\rho} - \frac{d\tilde{\mu}}{d\nu_\rho} \right)^2 dv_\rho = 0$$

and so $\mu = \tilde{\mu}$.   □

REMARK 3.   The last part of the previous proof does not use the fact that we are considering a system of IRW's. Thus, with the assumptions of Section 2, if

$$\sup_{t>0} \int \left(\frac{d\nu_{\rho,t}}{d\nu} \right)^2 dv < +\infty,$$

then the Yaglom limit exists and belongs to $\mathscr{D}$.

4.4. *Proof of Theorem* 3(d).   Let μ be a weak limit point of the Cesaro means. We have seen that

$$(4.13) \qquad P_\mu(\tau > t) = e^{-\lambda(\rho)t}.$$

We prove that for $\rho_1, \rho_2 \geq 0$ and $\rho = \rho_1 + \rho_2$,

$$(4.14) \qquad \lambda(\rho) \geq \lambda(\rho_1) + \lambda(\rho_2).$$

Inequality (4.14) implies that $\lambda(\cdot)$ is strictly increasing, and therefore proves Theorem 3(d), once we recall that for $d \geq 3$, $\lambda(\rho) > 0$ (see [3] where the proof

was given for a special choice of A, but only increasingness and locality were used).

The initial condition η drawn from ν_ρ can be decomposed as $\eta = \eta^1 + \eta^2$ where η^1 and η^2 are independently drawn from ν_{ρ_1} and ν_{ρ_2}, respectively. If we denote by η_t^i, $i = 1, 2$, the configuration at time t of a system of IRW starting from η^i, we define

$$\tau_i = \inf\{t \geq 0 \colon \eta_t^i \in A\}.$$

Noting that, by increasingness of A,

$$\{\eta_t \notin A\} \subset \{\eta_t^1 \notin A\} \cap \{\eta_t^2 \notin A\},$$

we have

$$(4.15) \quad P_{\nu_\rho}(\tau > t) \leq P_{\nu_\rho}(\{\tau_1 > t\} \cap \{\tau_2 > t\}) = P_{\nu_{\rho_1}}(\tau > t) P_{\nu_{\rho_2}}(\tau > t),$$

which, together with (4.13), implies (4.14). $\square$

4.5. *An example for $d = 3$ and $d = 4$.* In Theorem 3 no statement about regularity in $d = 3$ and $d = 4$ is made. We show here a case in which we can compute explicitly the Yaglom limit, and show that it is singular with respect to ν_ρ.

As we noticed in Section 4.3, the measure $\nu_{\rho,t}$ can be explicitly computed in the case

$$A = \left\{\eta \colon \sum_{i \in \Lambda} \eta(i) > 0\right\}.$$

For simplicity we take here $\Lambda = \{0\}$. Recall that

$$\nu_{\rho,t}(d\eta) = \bigotimes_{i \in \mathbb{Z}^d} \nu_{\alpha_i(t)}(d\eta(i)),$$

where $\alpha_0(t) \equiv 0$, $\alpha_i(t) = \rho P(X(i,s) \neq 0 \; \forall s \in [0,t])$. It follows that the Yaglom limit is

$$\mu(d\eta) = \bigotimes_{i \in \mathbb{Z}^d} \nu_{\alpha_i}(d\eta(i)),$$

where $\alpha_0 = 0$, $\alpha_i = \rho P(X(i,s) \neq 0 \; \forall s \geq 0)$.

Denote by

$$\varepsilon_i \equiv P(X(i,s) = 0 \text{ for some } s \geq 0).$$

Using the notations of Section 4.3, we get

$$\frac{d\mu}{d\nu_\rho}\bigg|_{\mathscr{I}_n} = \exp\left(\sum_{i \in \Lambda_n} Y_i\right) \quad \text{with } Y_i = \rho\varepsilon_i + \eta(i)\log(1 - \varepsilon_i),$$

where $\Lambda_n = \{i \in \mathbb{Z}^d \colon |i| \leq n\}$. Our aim is to show that μ-a.s.,

$$(4.16) \qquad\qquad\qquad \sum_{i \in \Lambda_n} Y_i \longrightarrow +\infty$$

as $n \to +\infty$. We first recall classical asymptotics estimates for $d \geq 2$ (see, e.g., [8]),

(4.17) $$\lim_{|i| \to \infty} \varepsilon_i \, |i|^{d-2} = c_d > 0.$$

Also, we define $X_i = Y_i - E_\mu[Y_i]$ and

$$Z_n = \sum_{|i|=n} X_i.$$

Note that due to our cubic lattice, there is $\gamma_d > 0$ such that

$$\lim_{n \to \infty} \frac{|\{i : |i| = n\}|}{n^{d-1}} = \gamma_d.$$

Therefore

(4.18) $$E_\mu[Z_n^2] \sim \rho^2 c_d \gamma_d \frac{n^{d-1}}{n^{2d-4}} = \frac{\rho^2 c_d \gamma_d}{n^{d-3}}.$$

If $d = 3$ we have that $E_\mu[Z_n^2] \sim \rho^2 c_3 \gamma_3$, whereas if $d = 4$, $E_\mu[Z_n^2] \sim \rho^2 c_4 \gamma_4 / n$.

We will recall now two classical results: Kolmogorov's theorem and Kronecker's lemma (see [4], Theorem 1 and Lemma 2, pages 110 and 111). Kolmogorov: if X_n are independent, $E[X_n] = 0$ and $\sum E[X_n^2] < \infty$, then $X_1 + \cdots + X_n$ converges almost surely; Kronecker: if a_n, $b_n > 0$ are real numbers, b_n increases to infinity and

$$\sum_{i=1}^{n} \frac{a_i}{b_i} \text{ converges, then } \frac{1}{b_n} \sum_{i=1}^{n} a_i \longrightarrow 0.$$

CASE $d = 3$. We choose $b_n = n$; then there is a constant C such that

(4.19) $$\sum_{i=1}^{n} E_\mu \left[\left(\frac{Z_i}{b_i} \right)^2 \right] \leq \sum_{i=1}^{n} \frac{C}{i^2} < C'.$$

Thus, by Kolmogorov's theorem and Kronecker's lemma,

$$\frac{1}{n} \sum_{i=1}^{n} Z_i \longrightarrow 0, \quad \mu\text{-a.s.}$$

Recalling that

$$\frac{1}{n} \sum_{i=1}^{n} Z_i = \frac{1}{n} \left(\sum_{i \in \Lambda_n} Y_i - \sum_{i \in \Lambda_n} E_\mu[Y_i] \right)$$

and after noticing that

$$E_\mu[Y_i] \geq C \varepsilon_i^2 \geq \frac{C'}{|i|^2}$$

for some $C, C' > 0$, we obtain

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{i \in \Lambda_n} E_\mu[Y_i] \geq \lim_{n \to \infty} \frac{1}{n} \sum_{i \in \Lambda_n} \frac{C'}{|i|^2} > 0.$$

We conclude that

(4.20)
$$\lim_{n \to +\infty} \sum_{i \in \Lambda_n} Y_i = \infty, \quad \mu\text{-a.s.}$$

CASE $d = 4$. We choose $b_n = \log(n)$. There is a constant C such that

$$\sum_{i=1}^{n} E_\mu \left[\left(\frac{Z_i}{b_i} \right)^2 \right] \sim \sum_{i=1}^{n} \frac{C}{i(\log(i))^2} < C'.$$

Thus,

$$\frac{1}{\log n} \sum_{i=1}^{n} Z_i \longrightarrow 0, \quad \mu\text{-a.s.}$$

On the other hand,

$$\liminf_{n \to \infty} \frac{\sum_{i \in \Lambda_n} E_\mu[Y_i]}{\log(n)} \geq C > 0$$

implies that

$$\lim_{n \to \infty} \sum_{i \in \Lambda_n} Y_i = \infty, \quad \mu\text{-a.s.}$$

Finally, in both cases we have

$$\lim_{n \to \infty} \frac{d\mu}{d\nu_\rho} \bigg|_{\Lambda_n} = +\infty, \quad \mu\text{-a.s.},$$

which implies that μ is singular with respect to ν_ρ. □

5. Results for the symmetric simple exclusion process. In this section, $S = \{0, 1\}$, and ν_ρ is the product of Bernoulli measures of density $\rho \in [0, 1)$.

THEOREM 4. *Let $A \subset \mathscr{C}$ be local and increasing, and $A \neq \mathscr{C}$. Then*:

(a) *For $d = 1$ and $d = 2$ the Yaglom limit exists, and equals δ_0.*
(b) *For $d \geq 3$ limit points of the Cesaro means are different from δ_0, and they are distinct for distinct values of $\rho \in (0, 1)$.*

As before, let

(5.1)
$$\lambda(\rho) = -\lim_{t \to +\infty} \frac{1}{t} \log P_{\nu_\rho}(\tau > t).$$

We show that:

1. For $d = 1$ and $d = 2$ we have $\lambda(\rho) = 0$.
2. For $d \geq 3$, $\lambda(\rho) > 0$ for $\rho > 0$ and, for $\rho_1, \rho_2 > 0$,

(5.2)
$$\lambda(\rho_1 + \rho_2) \geq \lambda(\rho_1) + \lambda(\rho_2).$$

The derivation from these results of (a) and (b) is identical to the IRW case, observing that (4.4) can be proved for SSEP by replacing independent random walks by the stirring process [10] and noting that independence of particles was not used in deriving (4.4).

We begin with $d = 1$ and $d = 2$. As for IRW we assume $1_A(\eta)$ depends on $\{\eta(i): i \in \Lambda\}$ only, and let

$$B = \left\{\eta: \sum_{i \in \Lambda} \eta(i) \geq 1\right\}.$$

By increasingness of A we have $A \subset B$, and thus

(5.3) $$\lambda(\rho) \leq -\lim_{t \to +\infty} \frac{1}{t} \log P_{\nu_\rho}(\tau_B > t).$$

Following Arratia [2], we decompose our SSEP $\{\eta(t),\ t \geq 0\}$ as a Stirring process $\{\xi(x, t),\ t \geq 0,\ x \in \mathbb{Z}^d\}$ and an initial condition η drawn from ν_ρ. We define H_t to be the set of sites whose "marks" end up touching Λ,

$$H_t = \{x \in \mathbb{Z}^d: \xi(x, s) \in \Lambda \text{ for some } s \leq t\}.$$

Thus, by reversibility,

$$\begin{aligned} E|H_t| &= \sum_{x \in \mathbb{Z}^d} P(\xi(x, s) \in \Lambda \text{ for some } s \leq t) \\ &= \sum_{x \in \mathbb{Z}^d} P(\xi(i, s) = x \text{ for some } i \in \Lambda \text{ for some } s \leq t) \\ &= ER_t, \end{aligned}$$

where R_t is the range of the Stirring particles which started on Λ,

$$R_t = \sum_y 1(\xi(i, s) = y \text{ for some } i \in \Lambda \text{ for some } s \leq t).$$

Now,

$$\int P^\eta(\tau_B > t) d\nu_\rho = \int P(\eta(i) = 0 \text{ for all } i \in H_t) d\nu_\rho = E[(1 - \rho)^{|H_t|}]$$

$$\geq (1 - \rho)^{E|H_t|} = (1 - \rho)^{ER_t}.$$

Now,

$$E[R_t] \leq \sum_{i \in \Lambda} \sum_y P(\xi(i, s) = y \text{ for some } s \leq t) = \sum_{i \in \Lambda} E[R_t^i],$$

where R_t^i is the range of the Stirring particle which started on i. We use now the classical estimates for large t, for $d = 1$,

(5.4) $$E[R_t^i] \sim 4\sqrt{\frac{t}{2\pi}},$$

whereas in $d = 2$,

$$(5.5) \qquad\qquad E[R_t^i] \sim \frac{\pi t}{\log(t)}$$

to conclude that in both cases

$$(5.6) \qquad\qquad \lim_{t\to\infty} \frac{1}{t} \log(P^\eta(\tau_B > t)) = 0,$$

which, by (5.3), concludes the case $d = 1, 2$.

We now consider $d \geq 3$. The fact that $\lambda(\rho) > 0$ follows from the same argument as in [3], using the original inequality of Varadhan ([12]).

The proof of (5.2) given for IRW does not apply here. Instead we rely on the variational representation for $\lambda(\rho)$,

$$(5.7) \qquad 2\lambda(\rho) = \inf\left\{ \sum_{i\in\mathbb{Z}^d} \sum_{j\sim i} \int \eta(i)\left(\sqrt{T^{ij}\varphi(\eta)} - \sqrt{\varphi(\eta)}\right)^2 d\nu_\rho(\eta) : \right.$$

$$\left. \varphi \text{ local}, \ \varphi \geq 0, \ \varphi = 0 \text{ in } A, \ \int \varphi d\nu_\rho = 1\right\}.$$

Identity (5.7) follows from (3.5) with $\sqrt{\varphi}$ in place of f, after having observed that (i) due to (3.8) the infimum in (3.5) may be taken over positive functions; (ii) local functions in $\mathscr{D}_A$ form a core of $\mathscr{D}_A$.

Now, we think of $d\nu_\rho$ as the law of $\eta = \zeta + \xi$, where the joint distribution of ζ and ξ is a product measure $\tilde\nu = \otimes_{i\in\mathbb{Z}^d}\nu_i$, and

$$(5.8) \qquad \begin{aligned} \nu_i(\zeta(i) = 1, \quad \xi(i) = 0) = \rho_1, \quad \nu_i(\zeta(i) = 0, \ \xi(i) = 0) = 1 - \rho, \\ \nu_i(\zeta(i) = 0, \quad \xi(i) = 1) = \rho_2, \end{aligned}$$

with $\rho_1 + \rho_2 = \rho$. So that $\tilde\nu$ has marginals ν_{ρ_1} and ν_{ρ_2}. We rewrite now (5.7), with the notation $\eta = \zeta + \xi$ as

$$(5.9) \qquad 2\lambda(\rho) = \inf\left\{ \sum_{i\in\mathbb{Z}^d} \sum_{j\sim i} \int (\zeta(i) + \xi(i))\left(\sqrt{T^{ij}\psi(\zeta, \xi)} - \sqrt{\psi(\zeta, \xi)}\right)^2 d\tilde\nu(\zeta, \xi) : \right.$$

$$\left. \psi \geq 0, \ \psi(\zeta, \xi) = 0 \text{ if } \zeta + \xi \in A, \ \int \psi \, d\tilde\nu = 1\right\}.$$

When acting on a function $\psi(\zeta, \xi)$ the operator T^{ij} acts on both variables ζ and ξ. We will use now three simple facts. 1. The convexity of the Dirichlet form, which we write for any nonnegative measurable functions f, g and σ-algebra Γ,

$$(5.10) \qquad E\left[\left(\sqrt{f} - \sqrt{g}\right)^2 |\Gamma\right] \geq \left(\sqrt{E[f|\Gamma]} - \sqrt{E[g|\Gamma]}\right)^2.$$

2. The identity $E[T^{ij}\psi|\sigma(\zeta)] = T^{ij}E[\psi|\sigma(\zeta)]$, which is shown by direct inspection.

3. As $\tilde{\nu}$ is a homogeneous product measure,

(5.11)
$$\frac{dT^{ij}\tilde{\nu}}{d\tilde{\nu}} \equiv 1.$$

Now, for a function $\psi(\zeta, \xi)$, combining (1), (2) and (3) we have

(5.12)
$$\sum_{i \in \mathbb{Z}^d} \sum_{j \sim i} \int (\zeta(i) + \xi(i))\left(\sqrt{T^{ij}\psi(\zeta, \xi)} - \sqrt{\psi(\zeta, \xi)}\right)^2 d\tilde{\nu}(\zeta, \xi)$$
$$\geq \sum_{i \in \mathbb{Z}^d} \sum_{j \sim i} \int \zeta(i)\left(\sqrt{T^{ij}E[\psi|\zeta]} - \sqrt{E[\psi|\zeta]}\right)^2 d\nu_{\rho_1}(\zeta)$$
$$+ \sum_{i \in \mathbb{Z}^d} \sum_{j \sim i} \int \xi(i)\left(\sqrt{T^{ij}E[\psi|\xi]} - \sqrt{E[\psi|\xi]}\right)^2 d\nu_{\rho_2}(\xi).$$

We recall the key constraint on ψ: $\psi(\zeta, \xi) = 0$ if $\zeta + \xi \in A$. As A is an increasing event, $\zeta \in A$ implies that $\zeta + \xi \in A$, for any ξ. Thus,

(5.13)
$$E[\psi|\zeta](\zeta) = 0 \quad \text{if } \zeta \in A.$$

To conclude the proof, it is enough to take the infimum in inequality (5.12) over the function ψ satisfying the constraints of (5.9). $\square$

6. Open problems. There are basically three directions where problems look interesting.

1. For SSEP can one establish regularity of some elements of $\mathcal{D}$, say μ, in high dimension ($d \geq 5$)? A first step would be to estimate the density of particles far away from the origin under μ.
2. What if we draw initial configurations from a nonstationary measure ν? For instance, if $d\nu/d\nu_\rho \in L^2(\nu_\rho)$, do the Cesaro means obtained starting from ν have the same limit points as the ones starting from ν_ρ?
3. When we know that $\lambda = 0$, what are the correct asymptotics for $P_{\nu_\rho} \times (\tau > t)$? This problem goes back to Arratia [2], which establishes that there exist constants $0 < c_1 < c_2$ such that for all $t > 0$:

$$\text{for } d = 1, \quad \frac{1}{\sqrt{t}}P_{\nu_\rho}(\tau > t) \in [c_1, c_2],$$

$$\text{for } d = 2, \quad \frac{1}{\sqrt{t}\log t}P_{\nu_\rho}(\tau > t) \in [c_1, c_2].$$

The problem is to show existence of the limit as $t \to +\infty$ of the expressions above and estimate it.

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