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# LOCAL TIME FLOW RELATED TO SKEW BROWNIAN MOTION

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We define a local time flow of skew Brownian motions, that is, a family of solutions to the stochastic differential equation defining the skew Brownian motion, starting from different points but driven by the same Brownian motion. We prove several results on distributional and path properties of the flow. Our main result is a version of the Ray–Knight theorem on local times. In our case, however, the local time process viewed as a function of the spatial variable is a pure jump Markov process rather than a diffusion.

**1. Introduction.** We will present some results on a family of local time processes, including a new Ray–Knight type theorem. The results and techniques are directly inspired by those in a paper of Barlow, Burdzy, Kaspi and Mandelbaum (2000) on coalescence of skew Brownian motions. They are also related to an article of Bass and Burdzy (1999) where a family of local times on different random curves has been analyzed.

Suppose $B_t$ is a Brownian motion with $B_0 = 0$ defined on a probability space $(\Omega, \mathscr{F}, P)$ and $\beta \in (-1, 1)$ is a fixed constant. Consider the equation

$$X_t^x = x + B_t + \beta \widetilde{L}_t^x, \qquad t \geq 0, \tag{1.1}$$

where $\widetilde{L}_t^x$ is the symmetric local time of $X_t^x$ at 0, that is,

$$\widetilde{L}_t^x = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{|X_s^x| \leq \varepsilon\}}\, ds.$$

Note that $X_0^x = x$. It is known [Harrison and Shepp (1981)] that for every $x \in \mathbf{R}$ (1.1) has a strong solution $\{X_t^x,\ t \geq 0\}$ and the solution is pathwise unique. It follows that pathwise unique solutions $X_t^x$ can be constructed on a common probability space $(\Omega, \mathscr{F}, P)$ for all rational $x$ simultaneously; see Revuz and Yor (1991) for other equivalent characterizations for skew Brownian motion.

From now on we will assume that $0 < \beta < 1$. The analysis of the other cases requires no more than an application of symmetry. Let $L_t^x = x + \beta \widetilde{L}_t^x$ and for $x < y$ let

$$U(x, y) = \inf\{z \geq y\colon L_t^x = L_t^y = z \text{ for some } t\}.$$

It has been proved in Barlow, Burdzy, Kaspi and Mandelbaum (2000) that for fixed $x$ and $y$, the local time processes $L_t^x$ and $L_t^y$ meet with probability 1, that is, $U(x, y) < \infty$. The pathwise uniqueness of the strong solutions to

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(1.1) implies that the processes $L_t^x$ and $L_t^y$ are equal to each other after the first time they meet. This, time reversal and Proposition 1.7 below indicate, although we do not prove it here, that it is impossible to construct strong solution $X^x$ on a *common probability space* $(\Omega, \mathscr{F}, P)$ and still have pathwise uniqueness for all real $x$ *simultaneously*.

THEOREM 1.1. (i) *For every* $0 < x < y$, *the distributions of* $U(x, y) - x$ *and* $U(0, y - x)$ *are identical*.

(ii) *For every* $y > 0$, *the cumulative distribution function of* $U(0, y)/y$ *is given by*

$$F(u) = \left(\frac{u-1}{u}\right)^{(1-\beta)/2\beta}, \qquad u \geq 1. \tag{1.2}$$

Let $\mathbf{Q}$ and $\mathbf{R}$ denote the sets of all rational and real numbers; we will write $\mathbf{Q}^+$ and $\mathbf{R}^+$ to denote their subsets consisting of nonnegative elements. Let $T = \inf\{t \geq 0\colon L_t^0 = 1\}$. Note that the function $\mathbf{Q} \ni x \mapsto L_T^x$ is monotone with probability 1. Therefore, it can be extended in a unique way to a right-continuous function of real $x$. This definition will be in force whenever we refer to $L_T^x$ with real $x$.

THEOREM 1.2. (i) *The family* $\{L_T^x,\ x \in \mathbf{R}\}$ *constitutes a Markov process. The process* $\{L_T^x - x, x \in \mathbf{R}^+\}$ *is homogeneous and strong Markov*.

(ii) *For* $0 \leq x < a$ *and* $0 < \Delta x < a - x$,

$$P(L_T^{x+\Delta x} = L_T^x \mid L_T^x = a) = \left(1 - \frac{\Delta x}{a-x}\right)^{(1-\beta)/2\beta}.$$

*For fixed* $\beta$, $x$ *and* $a$, *and* $\Delta x \to 0$, *the probability of the complementary event is asymptotic to*

$$\frac{\Delta x}{a-x} \cdot \frac{1-\beta}{2\beta}.$$

(iii) *The process* $x \to L_T^x - x$ *has a constant negative drift of unit magnitude and isolated positive jumps. Let* $\mathbf{x}$ *be the "time" of the first jump for* $x \to L_T^x - x$ *with* $x \in \mathbf{R}^+$. *Then*

$$P(L_T^{\mathbf{x}} - L_T^{\mathbf{x}-} \in dz \mid L_T^{\mathbf{x}-} - \mathbf{x} = a) = \frac{1+\beta}{2\beta} a^{(1+\beta)/2\beta} (z+a)^{-(1+3\beta)/2\beta}\, dz,$$

*for* $a \geq 0$ *and* $z > 0$.

The term "homogeneous" in part (i) of Theorem 1.2 refers to a property usually called "time-homogeneity." Our statement means that the distributions of $\{L_T^{x+y} - (x+y), x > 0 \mid L_T^y - y = a\}$ and $\{L_T^{x+z} - (x+z), x > 0 \mid L_T^z - z = a\}$ are identical for all $a, y, z \in \mathbf{R}$. Using the word "time" may lead to a confusion since our "time" variable is $x$, the original space variable. Our result has the same flavor as the celebrated Ray–Knight theorem for Brownian local times.

By abuse of the notation, the superscript on $X$ and $L$ will indicate the value of the parameter $\beta$ in the following two theorems and a corollary.

For $\beta \in (0,1)$, let $L_t^\beta = \beta \widetilde{L}_t^\beta$, where $\widetilde{L}_t^\beta$ is the local time process corresponding to the solution $X_t^\beta$ of (1.1) with the parameter $\beta$ and $x = 0$, that is, $X_t^\beta = B_t + L_t^\beta$. Unique strong solutions to (1.1) exist for all rational $\beta \in (0,1)$ simultaneously.

THEOREM 1.3. (i) *For every fixed $t > 0$, the process $L_t^\beta$ is a nondecreasing function of $\beta \in (0,1) \cap \mathbf{Q}$, a.s.*

(ii) *Let $T = \inf\{t \geq 0\colon L_t^0 = 1\}$. The function $\beta \to L_T^\beta$ is a Markov process on $(0,1) \cap \mathbf{Q}$.*

Thus the process $\beta \to L_T^\beta$ can be extended in a unique way to a right continuous Markov process with the *real* parameter $\beta \in (0,\ 1)$.

THEOREM 1.4. *Assume that $0 < \beta_1 < \beta_2 < 1$.*

(i) *If $\beta_1 < \beta_2/(1+2\beta_2)$ then $L_t^{\beta_2} > L_t^{\beta_1}$ for all $t > 0$, a.s., and*

$$\lim_{t\to\infty} \left(L_t^{\beta_2} - L_t^{\beta_1}\right) = \infty.$$

(ii) *If $\beta_1 = \beta_2/(1+2\beta_2)$ then $L_t^{\beta_2} > L_t^{\beta_1}$ for all $t > 0$, a.s., and*

$$0 = \liminf_{t\to\infty} \left(L_t^{\beta_2} - L_t^{\beta_1}\right) < \limsup_{t\to\infty} \left(L_t^{\beta_2} - L_t^{\beta_1}\right) = \infty. \tag{1.3}$$

(iii) *If $\beta_1 > \beta_2/(1+2\beta_2)$ then a.s. for every $t_0 < \infty$ there exists $t > t_0$ such that $L_t^{\beta_2} = L_t^{\beta_1}$. Moreover, (1.3) holds.*

Let $M_t = \min_{s\leq t} B_s$ be the running minimum process of Brownian motion. It is a classical fact that for $x = 0$ and $\beta = 1$, equation (1.1) has a pathwise unique solution with $L_t^1 = -M_t$.

COROLLARY 1.5. *Suppose that $\beta > 1/3$. Then $L_t^\beta = -M_t$ for infinitely many $t \to \infty$, a.s.*

Before we state our next result, we warn the reader that the meaning of the superscripts is about to change for the second time in this section. We believe that changing superscript conventions will be less confusing to the reader than using multiple superscripts.

We will present some results on approximations of the solutions to the stochastic differential equation (1.1) which has a singular drift by solutions to stochastic differential equations with smooth drifts. The results are key ingredients in the proof of Theorem 1.3(i). As a by-product, we obtain Proposition 1.7 below.

Fix some $\beta \in (-1, 1)$ and $x_0$ and let $X_t^{x_0}$ be a solution to (1.1) with this $\beta$ and $X_0^{x_0} = x_0$. Let $f$ be a nonnegative smooth and symmetric function on $\mathbf{R}$, compactly supported on $[-1/2,\ 1/2]$ with $\int_{\mathbf{R}} f(x)\,dx = 1$. Denote $\frac{1}{2}\log((1+\beta)/(1-\beta))$ by $\gamma$ and let $f_n(x) = n\gamma f(nx)$ for $x \in \mathbf{R}$ and $n \geq 1$. Clearly, $f_n$ converges weakly to $\gamma\,\delta_0$, where $\delta_0$ is Dirac's delta function. Let $X^n$ be the unique strong solution to the following stochastic differential equation:

$$X_t^n = x_0 + B_t + \int_0^t f_n(X_s^n)\,ds \quad \text{for } t \geq 0. \tag{1.4}$$

THEOREM 1.6. *The sequence $\{X_t^n\}$ converges in probability to $X_t^{x_0}$ as $n \to \infty$ in $C([0,\infty), \mathbf{R})$, the space of continuous functions on $[0,\infty)$ equipped with the topology of uniform convergence on compact intervals. Moreover, a subsequence of $\{X_t^n\}$ converges to $X_t^{x_0}$ uniformly on compact intervals with probability* 1.

Extend the time domain of Brownian motion $B$ to the whole real line; that is, consider $\{B_t, t \in \mathbf{R}\}$ such that $B_0 = 0$, and the processes $\{B_t, t \geq 0\}$ and $\{B_{-t}, t \geq 0\}$ are independent Brownian motions. We can consider the following extension of (1.1):

$$X_t^x = x + B_t + \beta\widetilde{L}_t^x, \qquad t \in \mathbf{R}. \tag{1.5}$$

By reversing time, we see that there exists a unique strong solution $X_t^x$ of (1.5) for $t \leq 0$, with the self-evident new measurability property, namely, that for any fixed $t \leq 0$, the random variable $X_t^x$ is measurable with respect to $\sigma\{B_s,\ t \leq s \leq 0\}$. Having this measurability property in mind, we can combine solutions to (1.5) for $t \leq 0$ and $t \geq 0$ into one strong solution $\{X_t^x,\ t \in \mathbf{R}\}$. For $t \leq 0$, the value of $\widetilde{L}_t^x$ may be defined by saying that $\widetilde{L}_0^x - \widetilde{L}_t^x$ is the amount of local time accumulated by $X_s^x$ at the 0 level on the interval $(t, 0)$.

We have strong existence and pathwise uniqueness of solutions to (1.5) for all rational $x$ simultaneously. In view of the fact that $U(x, y) < \infty$, a.s., the topological structure of the processes $L_t^x$ for various values of the initial condition $X_0^x = x$ is given in Figure 1. In view of (1.5), the topological structure of $X_t^x$ is similar. This is completely different from the corresponding structure for the solutions to (1.4) because nonsingular SDEs have flows that are one-to-one. Note that each process $\{L_t^x,\ t \in \mathbf{R}\}$ is increasing (unlike a typical graph in Figure 1.

Let us further generalize (1.5) by moving the time origin to an arbitrary $s$; that is, consider the equation

$$X_t^{s,x} = x + (B_t - B_s) + \beta\widetilde{L}_t^{s,x}, \qquad t \in \mathbf{R}. \tag{1.6}$$

It is natural to ask whether solutions to various initial value problems (1.6) are consistent. The following proposition explains the meaning of "consistency" and gives a positive answer to the question.

PROPOSITION 1.7. *With probability* 1, *for all quadruples $(s_1, s_2, x_1, x_2) \in \mathbf{Q}^4$ simultaneously, either $X_t^{s_1,x_1} \leq X_t^{s_2,x_2}$ for all $t \in \mathbf{R}$ or $X_t^{s_1,x_1} \geq X_t^{s_2,x_2}$ for all $t \in \mathbf{R}$.*

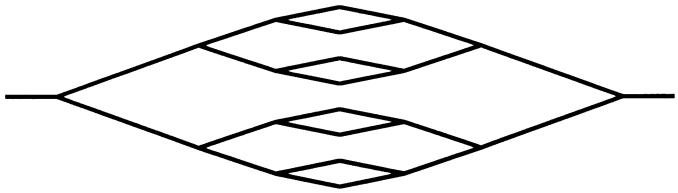

FIG. 1. *The typical topological structure of the processes* $t \mapsto x + \beta \tilde{L}_t^x$ *for various values of* $x$.

The following question arises naturally from Theorems 1.2 and 1.3. The methods used to prove Theorem 1.2(ii) and (iii) do not seem to yield an answer.

OPEN PROBLEM 1.8. *What is the distribution of the process* $\beta \to L_T^\beta$ *in Theorem* 1.3(ii)?

The next three sections contain proofs of our theorems, Section 2 for constant $\beta$ and Section 4 for variable $\beta$. These sections are separated by Section 3 with approximation arguments.

**2. Constant skewness parameter.** This section contains the proofs of Theorems 1.1 and 1.2. The proofs will be given in order reflecting the flow of logic in the proofs.

PROOF OF THEOREM 1.2(i). First we will show that the process $x \to L_T^x$ is Markov, for $x \in \mathbf{Q}$. The Markov property automatically extends to the process indexed by $x \in \mathbf{R}$, by right continuity.

Fix some rational $0 < x < y$ and $z < x$. The process $X_t^x$ is a skew Brownian motion. Let $\mathscr{E}^+ = \{(s, e_s^+)\}_{\{s \in S\}}$ be the Poisson point process of positive excursions of $X_t^x$; that is, the set $\{(s, e_s^+)\}_{\{s \in S\}}$ is the random collection of all excursions of $X_t^x$ above 0. Here $S$ is the set of all $s \geq x$ such that for some $0 < g_s < d_s < \infty$ we have $s = L_{g_s}^x$, $X_{g_s}^x = 0 = X_{d_s}^x$,

$$X_v^x > 0 \quad \text{for } v \in (g_s, d_s)$$

and

$$e_s^+ = e_s^+(u) = X_{u+g_s}^x, \qquad u \in [0, d_s - g_s).$$

We define in an analogous way the Poisson point process $\mathscr{E}^- = \{(s, e_s^-)\}_{\{s \in S\}}$ of negative excursions of $X_t^x$. The processes $\mathscr{E}^+$ and $\mathscr{E}^-$ are independent.

Let $\mathscr{L}_s^y = \mathscr{L}_s^{y,x} = \inf\{L_t^y \colon L_t^x > s\}$. We define $\mathscr{L}_s^0$ and $\mathscr{L}_s^z$ in a similar way. Note that $t \to L_t^x$ increases only when $X_t^x = 0$, that is, when $-B_t = L_t^x$. The inequalities $L_t^x \geq L_t^0$, $L_t^x \geq L_t^z$ hold for all $t \geq 0$ a.s. On all intervals where $-B_t$ stays strictly above $L_t^x$, that is, when $X_t^x < 0$, we also have $-B_t > L_t^0$ and $-B_t > L_t^z$, and so the processes $L_t^x, L_t^0$ and $L_t^z$ do not change. This and the strong existence and pathwise uniqueness of solutions to (1.1) imply that the processes $\mathscr{L}_u^z$ and $\mathscr{L}_u^0$ are measurable with respect to the filtration

$\mathscr{F}_u^+ = \mathscr{F}_u^{+,x} = \sigma\{(s, e_s^+)\}_{\{s\in S,\, s\le u\}}$. By analogy, the process $\mathscr{L}_u^y$ is adapted to $\mathscr{F}_u^- = \sigma\{(s, e_s^-)\}_{\{s\in S, s\le u\}}$. The random time $\mathscr{T} = \inf\{s\colon \mathscr{L}_s^0 \ge 1\} = L_T^x$ is a stopping time relative to $\mathscr{F}_u^+$. By the independence of $\mathscr{E}^+$ and $\mathscr{E}^-$, the random elements $\mathscr{L}_{\mathscr{T}}^z$ and $e_{\mathscr{T}}^+$ are independent of $\mathscr{L}_{\mathscr{T}}^y$ given the value of $\mathscr{T} = L_T^x$. However, this can be restated as the independence of $L_T^y$ and $L_T^z$ given $L_T^x$, since $\mathscr{L}_{\mathscr{T}}^y = L_T^y$ and $L_T^z$ is a function of $\mathscr{L}_{\mathscr{T}}^z$ and $e_{\mathscr{T}}^+$. This proves the Markov property for positive $x$. The proof is analogous for negative $x$.

Next we prove the homogeneity of the process $x \to L_T^x - x$ for $x > 0$. Suppose that $x, y \in \mathbf{Q}^+$. The strong Markov property of $B_t$ applied at the time $T_x = \inf\{t\colon -B_t = L_t^x\} = \inf\{t\colon B_t = -x\}$ and translation invariance of Brownian motion show that the distribution of the Poisson point process $\{(s, e_{s+x}^-)\}_{\{s\in S-x\}}$ does not depend on $x > 0$. Recall that $\mathscr{L}_u^{x+y,x}$ is pathwise determined by $\mathscr{E}^-$, in view of the pathwise uniqueness of solutions to (1.1). From these two observations, it is rather easy to see that the distribution of $\{\mathscr{L}_{u+x}^{x+y,x} - x, u \ge 0\}$ does not depend on $x$. Since $\mathscr{T} - x = L_T^x - x$ is a stopping time relative to $\{\mathscr{F}_{u+x}^{+,x}\}_{\{u\ge 0\}}$ and the point processes $\mathscr{E}^+$ and $\mathscr{E}^-$ are independent, the distribution of $\mathscr{L}_{(\mathscr{T}-x)+x}^{x+y,x} - x$ given any value of $L_T^x - x$ is independent of $x$. The number $y$ is fixed so the distribution of $\mathscr{L}_{(\mathscr{T}-x)+x}^{x+y,x} - (x+y)$ given $L_T^x - x$ is also independent of $x$. This can be rephrased by saying that the distribution of $L_T^{x+y} - (x+y)$ given $L_T^x - x = a$ is independent of $x$. We conclude that the process $x \to L_T^x - x$ is homogeneous.

See the proof of Theorem 1.2(iii) for an argument showing that $x \to L_T^x - x$ is strong Markov. □

REMARK 2.1. We are going to review several results presented in Barlow, Burdzy, Kaspi and Mandelbaum (2000) and needed for the proof of Theorem 1.1.

Let

$$
\begin{aligned}
&T_0 = 0,\\
&S_k = \inf\{t > T_k\colon -B_t = L_t^x\}, \qquad k \ge 0,\\
&T_k = \inf\{t > S_{k-1}\colon -B_t = L_t^0\}, \qquad k \ge 1,\\
&W_k = \frac{L_{S_{k-1}}^x - L_{S_{k-1}}^0}{L_{T_{k-1}}^x - L_{T_{k-1}}^0}, \qquad k \ge 1,\\
&V_k = \frac{L_{T_k}^x - L_{T_k}^0}{L_{S_{k-1}}^x - L_{S_{k-1}}^0}, \qquad k \ge 1,\\
&M_k = L_{T_k}^x - L_{T_k}^0, \qquad k \ge 0.
\end{aligned}
\tag{2.1}
$$

The random variables $W_k$ and $V_k$, $k \ge 1$, are jointly independent, due to the strong Markov property of Brownian motion and the fact that the skew Brownian motion $X_t^x$ is pathwise determined by $B_t$. Note that $M_0 = x > 0$ and $M_k = M_{k-1} V_k W_k$ for $k \ge 1$.

Let $\gamma_1 = (1-\beta)/(2\beta)$ and $\gamma_2 = (1+\beta)/(2\beta)$. It has been proved in Barlow, Burdzy, Kaspi and Mandelbaum (2000) that $P(V_k > v) = v^{-\gamma_2}$ for $v \geq 1$, and $P(W_k < w) = w^{\gamma_1}$ for $w \in (0,1)$. It follows that $V_k$, $k \geq 1$, are i.i.d. and the same is true of $W_k$'s. It was proved in the same paper that $\{M_k, k \geq 0\}$ is a martingale.

It follows from above that $L_t^x$ and $L_t^0$ cannot coalesce before any time $T_k$. However, an argument in the proof of Theorem 1.4 shows that $T_\infty = \lim_{k\to\infty} T_k$ is finite and that the two local time processes coalesce at time $T_\infty$.

PROOF OF THEOREM 1.1. (i) Note that $L_t^x = x$ for $t < \tau_{-x} = \inf\{s\colon B_s = -x\}$. Part (i) of the theorem follows easily by applying the strong Markov property to $X_t^x$ at the stopping time $\tau_{-x}$.

(ii) We will show that in order to prove Theorem 1.1(ii), it is enough to verify that the distribution in (1.2) satisfies a certain identity. Although this is sufficient for a rigorous proof of our assertion, such an argument provides no clue as to how one can derive the formula in (1.2). For this reason, we start with a derivation of (1.2) which gives the formula but fails to prove it for some values of $\beta$.

Consider $x$, $y$ and $z$ such that $0 < x < x+1 = y = z-\delta$, for some $\delta > 0$. Let $\mathscr{E}_y^+$ and $\mathscr{E}_y^-$ be the Poisson point processes of positive and negative excursions of $X_t^y$, defined in the same way as $\mathscr{E}^+$ and $\mathscr{E}^-$ in the proof of Theorem 1.2(i) above. Let $\mathscr{L}_s^z = \inf\{L_t^z\colon L_t^y > s\}$ and define $\mathscr{L}_s^x$ in the same way. Note that the coalescence time has probability zero of coinciding with the start or end of an excursion. We have

$$U(y,z) = \inf\{s\colon \mathscr{L}_s^z = s\}, \qquad U(x,y) = \inf\{s\colon \mathscr{L}_s^x = s\}.$$

Hence, the random variables $U(x,y)$ and $U(y,z)$ are measurable with respect to the $\sigma$-fields generated by $\mathscr{E}_y^+$ and $\mathscr{E}_y^-$, respectively. We conclude that $U(x,y)$ and $U(y,z)$ are independent.

Suppose that $0 < v < w$. Note that by Brownian scaling, $U(\delta v, \delta w)$ has the same distribution as $\delta U(v,w)$.

We have

$$\max(U(x,y), U(y,z)) = U(x,z).$$

Having in mind the scaling property of $U$, we can represent the last formula as follows. Suppose that $Y_1$ and $Y_2$ are independent and have the same distributions as $U(x,y)-x$. Then $\max(U(x,y), U(y,z))$ has the same distribution as

$$\max(Y_1 + x, \delta Y_2 + y) = \max(Y_1 + x, \delta Y_2 + x + 1),$$

while $U(x,z)$ has the same distribution as $(1+\delta)Y_1 + x$. Hence, the following random variables have identical distributions:

$$\max(Y_1 + x, \delta Y_2 + x + 1) \quad \text{and} \quad (1+\delta)Y_1 + x;$$

the same is true for

$$\max(Y_1, \delta Y_2 + 1) \quad \text{and} \quad (1+\delta)Y_1.$$

Let $F(u) = P(Y_1 \le u)$. Then

$$P(\max(Y_1, \delta Y_2 + 1) \le u) = P(Y_1 \le u)P(\delta Y_2 + 1 \le u) = P((1+\delta)Y_1 \le u),$$

and so,

$$F(u)F((u-1)/\delta) = F(u/(1+\delta)). \tag{2.2}$$

Let $a$ and $b$ be defined by $u = 1/(1-a)$ and $(u-1)/\delta = 1/(1-b)$. Then $u/(1+\delta) = 1/(1-ab)$ and so

$$F\Big(\frac{1}{1-a}\Big)F\Big(\frac{1}{1-b}\Big) = F\Big(\frac{1}{1-ab}\Big).$$

We now let $G(a) = F(1/(1-a))$ to obtain a functional equation

$$G(a)G(b) = G(ab).$$

Further substitutions $\tilde{a} = -\log a$, $\tilde{b} = -\log b$ and $\widetilde{G}(\tilde{a}) = G(a)$ yield

$$\widetilde{G}(\tilde{a})\widetilde{G}(\tilde{b}) = \widetilde{G}(\tilde{a} + \tilde{b}). \tag{2.3}$$

If we restrict the values of $u$ to $(1, \infty)$ and $\delta$ to $(0, u-1)$ then the argument of the function $F$ takes values in $(1, \infty)$ in all three instances in (2.2). For this range of values of $u$ and $\delta$, the variables $a$ and $b$ can be any pair of reals in $(0,1)^2$, and so $\tilde{a}$ and $\tilde{b}$ can be arbitrary positive numbers. Hence, a bounded function $\widetilde{G}$ satisfies (2.3) for any $\tilde{a}, \tilde{b} \in (0, \infty)$. This implies that [see Billingsley (1986), Corollary in Appendix A20]

$$\widetilde{G}(\tilde{a}) = e^{-\tilde{a}c_1},$$

for some $c_1$. Going back to $F$, we obtain,

$$F(u) = \Big(\frac{u-1}{u}\Big)^{c_1}, \tag{2.4}$$

for $u > 1$.

We will now determine the value of the constant $c_1$. It will turn out that $c_1 = \gamma_1 = \frac{1-\beta}{2\beta}$ but this part of the argument works only for $\beta > 1/3$. Recall the results reviewed in Remark 2.1. Suppose that $Y$ and $W$ are independent with distributions given by $P(V > v) = v^{-\gamma_2}$, $v \ge 1$ and $P(W < w) = w^{\gamma_1}$ for $w \in (0,1)$. Let $T_1$ be as in (2.1) with $x = 1$. The distribution of $L^1_{T_1} - 1$ is the same as that of $VW + (1-W) - 1 = W(V-1)$. Note that $U(0,1) \ge L^1_{T_1}$ in view of Remark 2.1. Let $Z = V - 1$ and $Q = W(V-1) = WZ$. Then $P(Z > z) = (z+1)^{-\gamma_2}$ for $z \ge 0$, and

$$\begin{aligned} P(U(0,1) < 1+q) &\le P(L^1_{T_1} < 1+q) = P(Q < q) \\ &= \int_0^1 \int_0^{q/w} f_W(w) f_Z(z)\, dz\, dw \\ &= \int_0^1 \int_0^{q/w} \gamma_1 w^{\gamma_1 - 1} \gamma_2 (z+1)^{-\gamma_2 - 1}\, dz\, dw \end{aligned} \tag{2.5}$$

$$= \int_0^1 \gamma_1 w^{\gamma_1 - 1}(-(z+1)^{-\gamma_2})|_0^{q/w}\, dw$$
$$= \int_0^1 \gamma_1 w^{\gamma_1 - 1}(1 - (q/w+1)^{-\gamma_2})\, dw.$$

After making the substitution $w/q = t$, we see that the last integral in (2.5) is equal to

$$\int_0^{1/q} \gamma_1 (tq)^{\gamma_1 - 1}(1 - (1/t+1)^{-\gamma_2})\, q\, dt$$
$$= q^{\gamma_1} \int_0^{1/q} \gamma_1 t^{\gamma_1 - 1}(1 - (1/t+1)^{-\gamma_2})\, dt \tag{2.6}$$
$$\le q^{\gamma_1}\left( c_2 \int_0^1 \gamma_1 t^{\gamma_1 - 1}\, dt + \int_1^{1/q} \gamma_1 t^{\gamma_1 - 1}(1 - (1/t+1)^{-\gamma_2})\, dt \right).$$

For large $t$, we have $(1 - (1/t+1)^{-\gamma_2}) \sim \gamma_2 t^{-1}$. If $\gamma_1 < 1$ then

$$\int_1^{1/q} \gamma_1 t^{\gamma_1 - 1}(1 - (1/t+1)^{-\gamma_2})\, dt \le c_3 \int_1^{\infty} \gamma_1 t^{\gamma_1 - 1}\gamma_2 t^{-1}\, dt < \infty,$$

and so, in view of (2.6),

$$P(U(0,1) < 1+q) \le P(Q < q) \le c_4 q^{\gamma_1}. \tag{2.7}$$

Recall that $U(0,1) \ge L^1_{T_1}$. By applying the strong Markov property at time $T_1$ to the skew Brownian motion and using the scaling property of $U(\cdot,\cdot)$, we see that $U(0,1)$ has the same distribution as $U(0,1)VW + (1-W)$, assuming $U(0,1)$, $V$ and $W$ are independent. Choose large $a \in (1,\infty)$ such that $P(U(0,1) < a) > 1/2$ and $P(V < a) > 1/2$. We have

$$\begin{aligned} P(U(0,1) < 1 + q(a^2-1)) &= P(U(0,1)VW + (1-W) < 1 + q(a^2-1)) \\ &\ge P(W < q, U(0,1) < a, V < a) \ge q^{\gamma_1}/4. \end{aligned} \tag{2.8}$$

For small $\delta > 0$, according to (2.4),

$$P(U \le 1+\delta) = \left(\frac{\delta}{1+\delta}\right)^{c_1} \sim \delta^{c_1}.$$

In view of (2.7) and (2.8), we must have $c_1 = \gamma_1 = (1-\beta)/(2\beta)$ in the case $\gamma_1 < 1$.

This proves Theorem 1.1(ii) in the case $(1-\beta)/(2\beta) < 1$, that is, when $\beta > 1/3$. The same argument does not seem to extend to other values of $\beta$ so we will have to proceed along different lines.

We have already noticed that the distribution of $U \stackrel{\text{def}}{=} U(0,1)$ is the same as that of $1 - W + VWU(0,1)$. Hence, $U - 1$ and $W(VU - 1)$ have identical distributions. We will first verify that the distributions are identical if we assume that $c_1 = \gamma_1$. Then we will argue that for other values of $c_1$, the

distributions of $U-1$ and $W(VU-1)$ must be different. For typographical convenience, denote $c_1$ by $\lambda$. We have the following formulas for densities:

$$f_V(v)=\gamma_2 v^{-\gamma_2-1},\quad f_W(w)=\gamma_1 w^{\gamma_1-1},$$
$$f_U(u)=\lambda\frac{(u-1)^{\lambda-1}}{u^{\lambda+1}}.$$

Let $Z=VU$. Then

$$\begin{aligned}
P(Z<z)&=\int_1^z\int_1^{z/u} f_U(u)f_V(v)\,dv\,du\\
&=\int_1^z\int_1^{z/u}\lambda\frac{(u-1)^{\lambda-1}}{u^{\lambda+1}}\gamma_2 v^{-\gamma_2-1}\,dv\,du\\
&=\int_1^z\lambda\frac{(u-1)^{\lambda-1}}{u^{\lambda+1}}(-v^{-\gamma_2})\Big|_1^{z/u}\,du\\
&=\int_1^z\lambda\frac{(u-1)^{\lambda-1}}{u^{\lambda+1}}\Big[1-\Big(\frac{z}{u}\Big)^{-\gamma_2}\Big]du\\
&=\int_1^z\lambda\frac{(u-1)^{\lambda-1}}{u^{\lambda+1}}\,du-z^{-\gamma_2}\int_1^z\lambda\frac{(u-1)^{\lambda-1}}{u^{\lambda+1-\gamma_2}}\,du\\
&=\Big(\frac{u-1}{u}\Big)^{\lambda}\Big|_1^z-z^{-\gamma_2}\int_1^z\lambda\frac{(u-1)^{\lambda-1}}{u^{\lambda+1-\gamma_2}}\,du\\
&=\Big(\frac{z-1}{z}\Big)^{\lambda}-z^{-\gamma_2}\int_1^z\lambda\frac{(u-1)^{\lambda-1}}{u^{\lambda+1-\gamma_2}}\,du.
\end{aligned}$$

This implies that

$$\begin{aligned}
f_Z(z)&=\lambda\frac{(z-1)^{\lambda-1}}{z^{\lambda+1}}+\gamma_2 z^{-\gamma_2-1}\int_1^z\lambda\frac{(u-1)^{\lambda-1}}{u^{\lambda+1-\gamma_2}}\,du-z^{-\gamma_2}\lambda\frac{(z-1)^{\lambda-1}}{z^{\lambda+1-\gamma_2}}\\
&=\gamma_2 z^{-\gamma_2-1}\int_1^z\lambda\frac{(u-1)^{\lambda-1}}{u^{\lambda+1-\gamma_2}}\,du.
\end{aligned}$$

Hence, the density of $VU-1=Z-1$ is

$$f_{VU-1}(z)=\gamma_2(z+1)^{-\gamma_2-1}\int_1^{z+1}\lambda\frac{(u-1)^{\lambda-1}}{u^{\lambda+1-\gamma_2}}\,du.$$

We see that

$$\begin{aligned}
&P(W(VU-1)<y)\\
&\quad=\int_0^1\int_0^{y/w} f_W(w)f_{VU-1}(z)\,dz\,dw\\
&\quad=\int_0^1\int_0^{y/w}\gamma_1 w^{\gamma_1-1}\gamma_2(z+1)^{-\gamma_2-1}\int_1^{z+1}\lambda\frac{(u-1)^{\lambda-1}}{u^{\lambda+1-\gamma_2}}\,du\,dz\,dw\\
&\quad=\int_0^1\int_0^{y/w}\gamma_1 w^{\gamma_1-1}\gamma_2(z+1)^{-\gamma_2-1}\int_0^{z}\lambda\frac{u^{\lambda-1}}{(u+1)^{\lambda+1-\gamma_2}}\,du\,dz\,dw.
\end{aligned}$$

We make the substitution $z = ty/w$ to obtain

$$P(W(VU-1) < y) = \int_0^1 \int_0^1 \gamma_1 w^{\gamma_1 - 1} \gamma_2 (ty/w + 1)^{-\gamma_2 - 1} \int_0^{ty/w} \lambda \frac{u^{\lambda-1}}{(u+1)^{\lambda+1-\gamma_2}} \, du \, dt \frac{y}{w} \, dw.$$

Another substitution $u = sty/w$ yields

$$\begin{aligned} P(W(VU-1) < y) &= \int_0^1 \int_0^1 \gamma_1 w^{\gamma_1 - 1} \gamma_2 (ty/w + 1)^{-\gamma_2 - 1} \int_0^1 \lambda \frac{(sty/w)^{\lambda-1}}{((sty/w)+1)^{\lambda+1-\gamma_2}} \, ds \frac{ty}{w} \, dt \frac{y}{w} \, dw \\ &= \int_0^1 \int_0^1 \int_0^1 \gamma_1 \gamma_2 w^{\gamma_1 - 1} \frac{ty^2}{w^2} (ty/w + 1)^{-\gamma_2 - 1} \lambda \frac{(sty/w)^{\lambda-1}}{((sty/w)+1)^{\lambda+1-\gamma_2}} \, ds \, dt \, dw. \end{aligned}$$

We continue our calculations with $\gamma_1$ substituted in place of $\lambda$. Then

$$\begin{aligned} P(W(VU-1) < y) &= \int_0^1 \int_0^1 \int_0^1 \gamma_1 \gamma_2 w^{\gamma_1 - 1} \frac{ty^2}{w^2} (ty/w + 1)^{-\gamma_2 - 1} \gamma_1 \frac{(sty/w)^{\gamma_1 - 1}}{((sty/w)+1)^{\gamma_1 + 1 - \gamma_2}} \, ds \, dt \, dw \\ &= \int_0^1 \int_0^1 \int_0^1 \gamma_1 \gamma_2 \frac{ty^2}{w^2} (ty/w + 1)^{-\gamma_2 - 1} \gamma_1 \frac{(sty)^{\gamma_1 - 1}}{((sty/w)+1)^{\gamma_1 + 1 - \gamma_2}} \, ds \, dt \, dw. \end{aligned}$$

We have $\gamma_1 + 1 - \gamma_2 = 0$ because $\gamma_1 = (1-\beta)/(2\beta)$ and $\gamma_2 = (1+\beta)/(2\beta)$. Hence

$$\begin{aligned} P(W(VU-1) < y) &= \int_0^1 \int_0^1 \int_0^1 \gamma_1 \gamma_2 \frac{ty^2}{w^2} (ty/w + 1)^{-\gamma_2 - 1} \gamma_1 (sty)^{\gamma_1 - 1} \, ds \, dt \, dw \\ &= \int_0^1 \int_0^1 \gamma_1 \gamma_2 \frac{ty^2}{w^2} (ty/w + 1)^{-\gamma_2 - 1} (ty)^{\gamma_1 - 1} \big(s^{\gamma_1}|_0^1\big) \, dt \, dw \\ &= \int_0^1 \int_0^1 \gamma_1 \gamma_2 \frac{ty^2}{w^2} (ty/w + 1)^{-\gamma_2 - 1} (ty)^{\gamma_1 - 1} \, dt \, dw \\ &= \int_0^1 \gamma_1 t^{\gamma_1 - 1} y^{\gamma_1} \int_0^1 \left[ \gamma_2 \frac{ty}{w^2} (ty/w + 1)^{-\gamma_2 - 1} \right] dw \, dt \\ &= \int_0^1 \gamma_1 t^{\gamma_1 - 1} y^{\gamma_1} \big[ (ty/w + 1)^{-\gamma_2}|_0^1 \big] \, dt \\ &= \int_0^1 \gamma_1 t^{\gamma_1 - 1} y^{\gamma_1} (ty + 1)^{-\gamma_2} \, dt \\ &= y^{\gamma_1} \int_0^1 \gamma_1 t^{\gamma_1 - 1} (ty + 1)^{-\gamma_2} \, dt. \end{aligned}$$

Recall that we have taken $\lambda$ to be $\gamma_1$ and note that $\gamma_1 = \gamma_2 - 1$. We proceed to obtain

$$\begin{aligned} P(W(VU-1) < y) &= y^{\gamma_1} \int_0^1 (\gamma_2 - 1) t^{\gamma_2 - 2} (ty+1)^{-\gamma_2}\, dt \\ &= y^{\gamma_1} \int_0^1 (\gamma_2 - 1) t^{-2} (y + 1/t)^{-\gamma_2}\, dt \\ &= y^{\gamma_1} (y + 1/t)^{-\gamma_2 + 1} |_0^1 \\ &= y^{\gamma_1} (y+1)^{-\gamma_2 + 1} \\ &= \left( \frac{y}{y+1} \right)^{\gamma_1} \\ &= \left( \frac{y}{y+1} \right)^{\lambda}. \end{aligned}$$

Since

$$P(U - 1 < y) = \left( \frac{y}{y+1} \right)^{\lambda},$$

we see that taking $\lambda = \gamma_1$ gives us the distributional identity for $U - 1$ and $W(VU - 1)$, as desired. We will show that no other $\lambda$ gives the same identity.

Consider a Markov chain $N_1, N_2, \ldots$ whose transition mechanism is described as follows. Given $N_1, N_2, \ldots, N_k$, take random variables $V_k$ and $W_k$, with the distributions $P(V_k > v) = v^{-\gamma_2}, v \geq 1$, and $P(W_k < w) = w^{\gamma_1}, w \in (0,1)$. Construct $V_j$'s and $W_j$'s so that they are all jointly independent and so that for every $k$, the random variables $V_k$ and $W_k$ are independent of $N_1, N_2, \ldots, N_k$. Let $N_{k+1} = 1 + W_k(V_k N_k - 1)$. Our calculations above showed that this Markov chain has a stationary distribution, namely the distribution in (2.4) with $c_1 = \gamma_1$. By Theorem 7.16 of Breiman (1968), there is only one stationary distribution for $\{N_k\}$, and so other values of $c_1$ do not give the distributional equality we have to have. □

PROOF OF THEOREM 1.2(ii). Recall the processes $\mathscr{E}^+$ and $\mathscr{E}^-$ from the proof of Theorem 1.2(i). The event $\{L_T^x = a\}$ is measurable with respect to the $\sigma$-field generated by $\mathscr{E}^+$ because $L_T^x = \mathscr{L}_{\mathscr{T}}^x$. Also the event $\{U(x, x+\Delta x) \leq a\}$ is measurable with respect to $\mathscr{E}^-$. By the independence of $\mathscr{E}^+$ and $\mathscr{E}^-$,

$$\begin{aligned} P(L_T^{x+\Delta x} = L_T^x \mid L_T^x = a) &= P(U(x, x+\Delta x) \leq a \mid L_T^x = a) \\ &= P(U(x, x+\Delta x) \leq a). \end{aligned}$$

Using Theorem 1.1(ii) we obtain

$$\begin{aligned} P(U(x,x+\Delta x)\le a) &= P(U(0,\Delta x)\le a-x)\\ &= P\biggl(\frac{U(0,\Delta x)}{\Delta x}\le\frac{a-x}{\Delta x}\biggr)\\ &= F\biggl(\frac{a-x}{\Delta x}\biggr)\\ &= \biggl(1-\frac{\Delta x}{a-x}\biggr)^{(1-\beta)/2\beta}. \end{aligned}$$

For small $\Delta x$,

$$1-\biggl(1-\frac{\Delta x}{a-x}\biggr)^{(1-\beta)/2\beta}\sim\frac{\Delta x}{a-x}\cdot\frac{1-\beta}{2\beta}. \qquad \square$$

PROOF OF THEOREM 1.2(iii). Fix some $a>0$ and consider a sequence of strictly positive numbers $a_n$ converging to $a$. Let us first find the density of $U(x,x+\frac{1}{n})$ given $\{L_T^{x+1/n}>L_T^x=x+a_n\}$, for $x\ge 0$. Fix some $v>x+a$ and consider $\Delta a>0$ such that $v>x+a_n+\Delta a$. As in the proof of Theorem 1.2(ii), we observe that the events $\{U(x,x+\frac{1}{n})>v\}$ and $\{L_T^x\in[x+a_n,x+a_n+\Delta a]\}$ are independent. The same is true if $v$ is replaced in the first event by $x+a_n$ or $x+a_n+\Delta a$. Hence,

$$\begin{aligned} &P\bigl(U(x,x+1/n)>v\mid L_T^{x+1/n}>L_T^x,\ L_T^x\in[x+a_n,x+a_n+\Delta a]\bigr)\\ &\quad=\frac{P(U(x,x+1/n)>v,L_T^{x+1/n}>L_T^x,L_T^x\in[x+a_n,x+a_n+\Delta a])}{P(L_T^{x+1/n}>L_T^x,\ L_T^x\in[x+a_n,x+a_n+\Delta a])}\\ &\quad=\frac{P(U(x,\ x+1/n)>v,L_T^x\in[x+a_n,x+a_n+\Delta a])}{P(L_T^{x+1/n}>L_T^x,L_T^x\in[x+a_n,x+a_n+\Delta a])}\\ &\quad\ge\frac{P(U(x,\ x+1/n)>v,L_T^x\in[x+a_n,x+a_n+\Delta a])}{P(U(x,\ x+1/n)>x+a_n,L_T^x\in[x+a_n,x+a_n+\Delta a])}\\ &\quad=\frac{P(U(x,\ x+1/n)>v)P(L_T^x\in[x+a_n,x+a_n+\Delta a])}{P(U(x,\ x+1/n)>x+a_n)P(L_T^x\in[x+a_n,x+a_n+\Delta a])}\\ &\quad=\frac{P(U(x,\ x+1/n)>v)}{P(U(x,\ x+1/n)>x+a_n)}. \end{aligned}$$

A similar calculation shows that

$$\begin{aligned} &P\bigl(U(x,x+1/n)>v\mid L_T^{x+1/n}>L_T^x,L_T^x\in[x+a_n,x+a_n+\Delta a]\bigr)\\ &\quad\le\frac{P(U(x,x+1/n)>v)}{P(U(x,x+1/n)>x+a_n+\Delta a)}. \end{aligned}$$

Since $a_n > 0$ and $\Delta a > 0$ are arbitrary numbers satisfying $v > x + a_n + \Delta a$, we conclude that, using Theorem 1.1,

$$\begin{aligned}
&P\big(U(x, x+1/n) > v \mid L_T^{x+1/n} > L_T^x = x + a_n\big) \\
&\quad = \frac{P(U(x, x+1/n) > v)}{P(U(x, x+1/n) > x + a_n)} \\
&\quad = \frac{P(U(0, \frac{1}{n}) > v - x)}{P(U(0, \frac{1}{n}) > a_n)} \\
&\quad = \frac{P(nU(0, \frac{1}{n}) > n(v-x))}{P(nU(0, \frac{1}{n}) > na_n)} \\
&\quad = \left[1 - \left(1 - \frac{1}{n(v-x)}\right)^{(1-\beta)/2\beta}\right] \cdot \left[1 - \left(1 - \frac{1}{na_n}\right)^{(1-\beta)/2\beta}\right]^{-1} \\
&\quad \to a/(v-x) \quad \text{as } n \to \infty.
\end{aligned} \tag{2.9}$$

After making the substitution $w = v - x - a_n$, we obtain,

$$\begin{aligned}
&\lim_{n\to\infty} P\big(U(x, x+1/n) - x - a_n \in dw \mid L_T^{x+1/n} > L_T^x = x + a_n\big) \\
&\quad = \frac{a}{(w+a)^2}\, dw,
\end{aligned} \tag{2.10}$$

for $w > 0$, uniformly on compact sets.

Define $G_n(dy) = P(L_T^{x+1/n} - L_T^x \in dy \mid L_T^{x+1/n} > L_T^x = x + a_n)$. Note that $G_n(dy)$ is independent of $x \geq 0$ since the Markov process $x \to L_T^x - x$ is homogeneous by Theorem 1.2(i). By the strong Markov property applied at the time $T$ to skew Brownian motion,

$$\begin{aligned}
&P\big(U(x, x+1/n) - x - a_n \in dw \mid L_T^{x+1/n} > L_T^x = x + a_n\big) \\
&\quad = \int_0^w P(U(x + a_n, x + a_n + y) - x - a_n \in dw)\, G_n(dy) \\
&\quad = \int_0^w P(U(0, y) \in dw)\, G_n(dy) \\
&\quad = \left(\int_0^w \frac{1-\beta}{2\beta}\left(1 - \frac{y}{w}\right)^{(1-3\beta)/2\beta} \frac{y}{w^2}\, G_n(dy)\right) dw.
\end{aligned} \tag{2.11}$$

This and (2.10) yield for $w > 0$,

$$\lim_{n\to\infty} \int_0^w (w-y)^r\, y\, G_n\,(dy) = \frac{a\, w^{r+2}}{(r+1)(w+a)^2}, \tag{2.12}$$

where $r = (1 - 3\beta)/2\beta$. Let

$$\widehat{G}_n(\theta) = \int_0^\infty e^{-y\theta} y\, G_n\,(dy).$$

Then, after multiplying (2.12) by $e^{-w\theta}$ and integrating, we get by the bounded convergence theorem,

$$\lim_{n\to\infty} \widehat{G}_n(\theta)\frac{\Gamma(r+1)}{\theta^{r+1}} = \frac{a}{r+1}\int_0^\infty e^{-w\theta}\frac{w^{r+2}}{(w+a)^2}\,dw$$
$$= \frac{a}{(r+1)\theta^{r+1}}\int_0^\infty e^{-u}\frac{u^{r+2}}{(u+a\theta)^2}\,du,$$

where $u = \theta w$. Since

$$\frac{1}{(u+a\theta)^2} = \int_0^\infty y e^{-y(a\theta+u)}\,dy,$$

we obtain

$$\begin{aligned}\lim_{n\to\infty}\widehat{G}_n(\theta) &= \frac{a}{(r+1)\Gamma(r+1)}\int_0^\infty y e^{-ya\theta}\int_0^\infty u^{r+2}e^{-u(y+1)}du\,dy\\ &= \frac{a\Gamma(r+3)}{\Gamma(r+2)}\int_0^\infty e^{-ya\theta}\frac{y}{(y+1)^{r+3}}\,dy\\ &= a(r+2)\int_0^\infty e^{-ya\theta}\frac{y}{(y+1)^{r+3}}dy\\ &= (r+2)\int_0^\infty e^{-\theta z}\frac{a^{r+2}z}{(z+a)^{r+3}}\,dz.\end{aligned} \tag{2.13}$$

For $b > 0$, by (2.9),

$$\begin{aligned}G_n(b,\infty) &= P(L_T^x > b + (x+a_n) \mid L_T^x > L_T^{x-1/n} = x + a_n)\\ &\le P(U(x-1/n,\ x) > x + b + a_n \mid L_T^x > L_T^{x-1/n} = x + a_n)\\ &\to \frac{a}{b+a} \quad \text{as } n \to \infty.\end{aligned}$$

Hence, the family $\{G_n(dy)\}_{n\ge1}$ of probability measures is tight. Let $\mu$ be any limiting measure for $\{G_n(dy)\}_{n\ge1}$. By passing to a subsequence, if necessary, we can assume that the sequence $G_n$ converges weakly to $\mu$. Weak convergence and (2.13) imply for $\theta > 0$,

$$\int_0^\infty e^{-\theta y}\,y\,\mu(dy) = \lim_{n\to\infty}\int_0^\infty e^{-\theta y}\,y\,G_n(dy)$$
$$= (r+2)\int_0^\infty e^{-\theta y}\frac{a^{r+2}y}{(y+a)^{r+3}}\,dy.$$

By uniqueness of Laplace transforms,

$$y\,\mu(dy) = \frac{(r+2)a^{r+2}y}{(y+a)^{r+3}}\,dy,$$

that is,

$$\mu(dy) = \frac{(r+2)a^{r+2}}{(y+a)^{r+3}}\,dy.$$

Since the weak limit of every subsequence of $\{G_n(dy)\}_{n\geq 1}$ is $((r+2)a^{r+2})/((y+a)^{r+3})\,dy$, we see that $\{G_n(dy)\}_{n\geq 1}$ converges weakly to $((r+2)a^{r+2})/((y+a)^{r+3})\,dy$ as $n\to\infty$. In particular, for any $z>0$,

$$\lim_{n\to\infty} G_n(z,\infty)=\mu(z,\infty)=\int_z^\infty \frac{(r+2)a^{r+2}}{(y+a)^{r+3}}\,dy. \tag{2.14}$$

We will prove that the convergence of $G_n(z,\infty)$ is uniform in an appropriate sense. In order to state this claim in a rigorous way we change the notation from $G_n$ to $G_n^{a_n}$ so that the dependence on $a_n$ becomes explicit (recall that $G_n^{a_n}$ does not depend on $x$). Let $G^a$ denote the weak limit of $G_n^{a_n}$, that is, the measure previously called $\mu$. Fix any interval $[\hat a_1,\hat a_2]$ with $\hat a_2>\hat a_1>0$. We will show that for any $\varepsilon>0$ there exists $n_0<\infty$ such that for all $n\geq n_0, z>y\geq 0, x\geq 0, a\in[\hat a_1,\hat a_2]$ and $a_n$ such that $|a-a_n|<1/n$, we have

$$|G_n^{a_n}(y,z)-G^a(y,z)|\leq\varepsilon. \tag{2.15}$$

Suppose that this is not true. Then there exist $a_k, b_k, n_k, y_k$ and $z_k$ such that $n_k\to\infty, z_k>y_k\geq 0, b_k\in[\hat a_1,\hat a_2]$, $|a_k-b_k|<1/n_k$ and

$$|G_{n_k}^{a_k}(y_k,z_k)-G^{b_k}(y_k,z_k)|>\varepsilon. \tag{2.16}$$

Note that $G^a(y,\infty)\to 0$ as $y\to\infty$, uniformly in $a\in[\hat a_1,\hat a_2]$ [see (2.14)]. Hence, the sequence $y_k$ is bounded. By compactness we can find a subsequence of $k$ along which $b_k$, $y_k$ and $z_k$ converge, with the possibility that the subsequence of $z_k$ goes to infinity. In order to simplify the notation we will assume that the original sequences converge. We will call the limit points $b_\infty, y_\infty$ and $z_\infty$. Note that $b_\infty\in[\hat a_1,\hat a_2]$, $y_\infty\in[0,\infty)$ and $z_\infty\in[0,\infty]$.

We see directly from (2.14) that $G^{b_k}(y_k,z_k)$ converge to $G^{b_\infty}(y_\infty,z_\infty)$. Since $a_k$ converge to $b_\infty$, the measures $G_{n_k}^{a_k}$ converge weakly to $G^{b_\infty}$. The limiting measure has a continuous density so $G_{n_k}^{a_k}(y_k,z_k)\to G^{b_\infty}(y_\infty,z_\infty)$. This contradicts (2.16).

We will now show that there exists a process with the distribution described in Theorem 1.2(ii) and (iii). Fix an arbitrary $\lambda_0$ and let $\xi_t$ be the process $t\mapsto\lambda_0-t$ killed according to a killing measure with intensity $(1-\beta)/(2\beta)y$. Let $\zeta$ be the lifetime of the strong Markov process $\xi_t$. It follows easily from the Feynman–Kac formula that $\xi_{\zeta-}>0$ a.s., and for $x>0$ and $0\leq t<x$,

$$P(\xi>t\mid\xi_0=x)=\left(1-\frac{t}{x}\right)^{(1-\beta)/2\beta}.$$

The process $\xi_t$ can be extended beyond $\zeta$ by a procedure described in Ikeda, Nagasawa and Watanabe (1966). Let $\Lambda_t(\omega)=\xi_t(\omega)$ for $t<\zeta(\omega)$. If $\xi_{\zeta-}=a$, let $\Lambda_\zeta(\omega)$ be a point $y$ distributed according to the density function $(1+\beta)/(2\beta)a^{(1+\beta)/2\beta}y^{-(1+3\beta)/2\beta}$ for $y>a$. Then glue an independent copy of $\xi$ starting from $\Lambda_\zeta(\omega)$. Iterating this procedure, we obtain a strong Markov process $\Lambda$ on $\mathbf{R}$ with right continuous sample paths and initial value $\lambda_0$ [see Ikeda, Nagasawa and Watanabe (1966)]. We now change the notation from $\Lambda_t$ to $\Lambda^x$

to be consistent with the rest of the proof. In other words, the "time" for $\Lambda$ will be denoted by a superscript $x$.

To complete the proof, it will suffice to show that the distribution of $L_T^x - x$ is the same as that of $\Lambda^x$. All we have to show is that the finite dimensional distributions of both processes are identical. Fix some $x_1 > x$, $a > x_1$, $a_2 > a_1 > 0$, and consider small $\Delta x, \Delta a > 0$. We will estimate the following probability, for small $\Delta x$,

$$
\begin{aligned}
P\Bigg( \bigcup_{j=1}^{(x_1-x)/\Delta x} \bigcup_{m=0}^{(a_2-a_1)/\Delta a} \Big[ &\{L_T^{x+j\Delta x} = L_T^x\} \cap \{L_T^{x+(j+1)\Delta x} \neq L_T^{x+j\Delta x}\} \\
&\cap \{L_T^{x+(j+1)\Delta x} - L_T^{x+j\Delta x} \in [a_1 + m\Delta a, a_1 + (m+1)\Delta a]\} \\
&\cap \{L_T^{x_1} = L_T^{x+(j+1)\Delta x}\} \Big] \Big| L_T^x = a \Bigg).
\end{aligned} \tag{2.17}
$$

By Theorem 1.2(ii),

$$
P(L_T^{x+j\Delta x} = L_T^x \mid L_T^x = a) = \left(1 - \frac{j\Delta x}{a - x}\right)^{(1-\beta)/2\beta},
$$

and for a number $b \in [a_1 - a, a_2 - a]$,

$$
\begin{aligned}
&P(L_T^{x_1} = L_T^{x+(j+1)\Delta x} \mid L_T^{x+(j+1)\Delta x} = a + b) \\
&\quad = \left(1 - \frac{x_1 - (x + (j+1)\Delta x)}{(a+b) - (x + (j+1)\Delta x)}\right)^{(1-\beta)/2\beta}.
\end{aligned}
$$

We have

$$
\begin{aligned}
&P\big(L_T^{x+(j+1)\Delta x} \neq L_T^{x+j\Delta x} \mid L_T^{x+j\Delta x} = a\big) \\
&\quad = 1 - \left(1 - \frac{\Delta x}{a - (x + j\Delta x)}\right)^{(1-\beta)/2\beta}.
\end{aligned}
$$

Finally by (2.15), for any given $\varepsilon > 0$, there is a $\delta > 0$ so that when $|\Delta x| < \delta$,

$$
\begin{aligned}
\big| P(L_T^{x+(j+1)\Delta x} - L_T^{x+j\Delta x} \in [y,\, z) \,|\, L_T^{x+(j+1)\Delta x} \neq L_T^{x+j\Delta x} = a) \\
- G^{a-(x+j\Delta x)}[y,\, z) \big| < \varepsilon
\end{aligned}
$$

for any $z > y \geq 0$. Using the Markov property of $x \to L_T^x$, we can multiply the probabilities of individual events in (2.17), except for the event in the second line, which needs a conditional probability. The result is a Riemann sum approximation to

$$
\begin{aligned}
&\int_0^{x_1 - x} \frac{1-\beta}{2\beta} \frac{1}{a-x} \left(1 - \frac{y}{a-x}\right)^{(1-3\beta)/2\beta} \\
&\quad \times \int_{a_1}^{a_2} \left(1 - \frac{x_1 - (x+y)}{(a+b) - (x+y)}\right)^{(1-\beta)/2\beta} G^{a-(x+y)}(db)\, dy.
\end{aligned} \tag{2.18}
$$

Note that $y$ plays the role of $j\Delta x$. The expression between the integrals in (2.18) is the derivative of the probability in Theorem 1.2(ii) with respect to $\Delta x$ (but $y$ plays here the role of $\Delta x$).

The quantity in (2.18) is equal to the probability that $z \to \Lambda^z + z$ makes only one jump in $(x, x_1)$ and $\Lambda^{x_1} + x_1 \in [a + a_1, a + a_2]$, assuming $\Lambda^x + x = a$. The case when $a \le x_1$ requires only minor modifications.

One can find an event disjoint from that in (2.17), depending on $\Delta x$, whose probability converges, as $\Delta x \to 0$, to the probability that $\Lambda^z + z$ makes exactly two jumps in $(x, x_1)$ and $\Lambda^{x_1} + x_1 \in [a + a_1, a + a_2]$, given $\Lambda^x + x = a$.

Proceeding in this way and summing over all possible numbers of jumps, we conclude that

$$P(L_T^{x_1} \in [a + a_1, a + a_2] \mid L_T^x = a) \\ \ge P_\Lambda(\Lambda^{x_1} + x_1 \in [a + a_1, a + a_2] \mid \Lambda^x + x = a).$$

Since a similar inequality can be proved for arbitrary intervals in the complement of $[a + a_1, a + a_2]$, the inequality is in fact an equality. Hence, the transition probabilities for $L_T^x$ are the same as for $\Lambda^x + x$. Recall that $L_T^0 = 1$. So if we assume that $\Lambda^0 = 1$, then by the Markov property of $L_T^x$ and $\Lambda^x + x$, we conclude that the finite dimensional distributions of $L_T^x$ and $\Lambda^x + x$ are identical. Since both processes are right continuous, they are identical in distribution. This implies that $L_t^x - x$ is a homogeneous strong Markov process and the formula in Theorem 1.2(iii) holds. □

## 3. Convergence of approximations.

PROOF OF THEOREM 1.6. We will first prove weak convergence of $X_t^n$ to $X_t^{x_0}$.

Define the scale function [cf. Bass (1998)]

$$s_n(x) = \int_0^x \exp\left(-2\int_0^y f_n(t)\,dt\right) dy \quad \text{for } x \in \mathbf{R}.$$

Clearly,

$$\lim_{n\to\infty} s_n(x) = s(x) \equiv \begin{cases} e^{\gamma}x, & \text{if } x < 0, \\ e^{-\gamma}x, & \text{if } x \ge 0. \end{cases}$$

By Itô's formula,

$$s_n(X_t^n) = s_n(x_0) + \int_0^t s_n'(X_s^n)\,dB_s, \qquad t \ge 0.$$

Since $s_n$ is a strictly increasing function, its inverse $s_n^{-1}$ exists. If let $Y_t^n = s_n(X_t^n)$ and $\sigma_n(x) = s_n' \circ s_n^{-1}(x)$, then

$$Y_t^n = y_n + \int_0^t \sigma_n(Y_s^n)\,dB_s, \qquad t \ge 0,$$

where $y_n = s_n(x_0)$. Note that $\lim_{n\to\infty} y_n = y \equiv s(x_0)$ and

$$s_n'(x) = \exp\left(-2\int_0^x f_n(t)\,dt\right)$$

is a monotone function bounded by $e^{-\gamma}$ and $e^{\gamma}$. Thus $\{s_n\}_{n\geq 1}$ are equicontinuous and so are their inverse functions $\{s_n^{-1}\}_{n\geq 1}$. Therefore $s_n$ and $s_n^{-1}$ converge uniformly on compact intervals in $\mathbf{R}$ to $s$ and $s^{-1}$, respectively. Define

$$\sigma(y) = \begin{cases} e^{\gamma}, & \text{for } y < 0, \\ e^{-\gamma}, & \text{for } y > 0. \end{cases}$$

Then $\lim_{n\to\infty} \sigma_n(y) = \sigma(y)$ for $y \neq 0$. Let $W$ be a Brownian motion on $\mathbf{R}$ with $W_0 = 0$. It is proved in Harrison and Shepp (1981) that $X^{x_0}$ is a strong solution to (1.1) if and only if $Y = s(X^x)$ satisfies

$$Y_t = s(x_0) + \int_0^t \sigma(Y_s)\,dW_s. \tag{3.1}$$

Since the function $\sigma$ has finite variation, pathwise uniqueness holds for solutions to (3.1), by Nakao (1972). Thus by proof of Theorem B of Kaneko and Nakao (1988),

$$\lim_{n\to\infty} E\left[\max_{0\leq t\leq T} |Y_t^n - Y_t|^2\right] = 0, \tag{3.2}$$

for every $T > 0$. This implies that $Y^n$ converge weakly to $Y$ on $C([0, \infty, \mathbf{R})$. Since $s_n^{-1}$ converge uniformly on compact intervals in $\mathbf{R}$ to $s^{-1}$, we see that $X^n$ converge weakly on $C([0, \infty, \mathbf{R})$ to $X^{x_0}$. It follows from (3.2) that there is a subsequence $n_k$ such that

$$\lim_{k\to\infty} \max_{0\leq t\leq T} |Y_t^{n_k} - Y_t| = 0 \quad \text{a.s.}$$

Hence for each $T > 0$,

$$\lim_{k\to\infty} \max_{0\leq t\leq T} |X_t^{n_k} - X_t| = 0 \quad \text{a.s.} \qquad \square$$

PROOF OF THEOREM 1.3. (i) Consider $0 < \beta < \beta_1 < 1$ and recall the notation from Theorem 1.6. We will apply that theorem to solutions of (1.4) with $x_0 = 0$. Let $\gamma_1 = \frac{1}{2}\log((1+\beta_1)(1-\beta_1))$. Clearly $\gamma_1 > \gamma > 0$. Define $\tilde{f}_n = c(\gamma_1/\gamma)f_n$. Let $\widetilde{X}^n$ be the diffusion determined by (1.4) with $\tilde{f}_n$ in place of $f_n$. By Theorem 1.6 and the uniqueness of strong solutions to (1.1), passing to a subsequence, if necessary, we see that $X_t^n$ and $\widetilde{X}_t^n$ converge uniformly on compact intervals to $X_t^{\beta}$ and $X_t^{\beta_1}$, respectively, almost surely (recall that $X_t^{\beta}$ and $X_t^{\beta_1}$ are solutions to (1.1) starting from 0). Since $\tilde{f}_n \geq f_n$, the stochastic comparison theorem [see Proposition 5.2.18 of Karatzas and Shreve (1994)] yields

$$P(\widetilde{X}_t^n \geq X_t^n \text{ for } t \geq 0) = 1.$$

This implies that $P(X_t^{\beta_1} \geq X_t^{\beta} \text{ for } t \geq 0) = 1$. The result follows from this and (1.1).

(ii) The proof is analogous to the proof of Theorem 1.2(i) and so it is omitted. □

PROOF OF PROPOSITION 1.7. It is enough to prove the result for a single quadruple since the set $\mathbf{Q}^4$ is countable. Hence, fix a quadruple $(s_1, s_2, x_1, x_2) \in \mathbf{Q}^4$, and without loss of generality, assume that $0 \leq s_1 \leq s_2$. It suffices to show the proposition for $t \in [s_1, s_2]$ because outside this interval the arrow of time points in the same direction for both solutions and the result follows from the uniqueness of the strong solution to (1.1). Recall functions $f_n$ defined in the Introduction. By Theorem 4.5.1 and the last paragraph on page 115 of Kunita (1997), for every fixed $n$, the following stochastic differential equation:

$$dX_t^n = dB_t + f_n(X_t^n)\,dt$$

yields a stochastic flow of homeomorphism $\phi_{st}^n(x, \omega)$ defined on $\mathbf{R} \times \Omega$. That is, there is a $P$-null set $N$ such that for all $\omega \in N^c$:

(i) $\phi_{st}(\cdot, \omega)$ is a continuous homeomorphism from $\mathbf{R}$ to $\mathbf{R}$ for all $s, t \in \mathbf{R}_+$,
(ii) $\phi_{sr}(\omega) = \phi_{st}(\omega) \circ \phi_{tr}(\omega)$ for all $s, t, r \in \mathbf{R}_+$,
(iii) $\phi_{ss}(\omega)$ is the identity map of $\mathbf{R}$,
(iv) $\phi_{st}(\omega)^{-1} = \phi_{ts}(\omega)$ and $X_t^{n,s,x} \equiv \phi_{st}(x, \cdot)$ solves

$$X_t^{n,s,x} = x + (B_t - B_s) + \int_s^t f_n(X_u^{n,s,x})\,du, \qquad t, s \in \mathbf{R}_+ .$$

By property (i) and the continuity of $t \to X_t^{n,s,x}$, we have

$$P\big(\text{either } X_t^{n,s_1,x_1} \leq X_t^{n,s_2,x_2} \text{ for } t \in [s_1, s_2] \text{ or } X_t^{n,s_1,x_1} \geq X_t^{n,s_2,x_2} \text{ for } t \in [s_1, s_2]\big) = 1.$$

In view of Theorem 1.6, after letting $n \to \infty$, we get

$$P\big(\text{either } X_t^{s_1,x_1} \leq X_t^{s_2,x_2} \text{ for } t \in [s_1, s_2] \text{ or } X_t^{s_1,x_1} \geq X_t^{s_2,x_2} \text{ for } t \in [s_1, s_2]\big) = 1.$$ □

## 4. Variable skewness parameter.

PROOF OF THEOREM 1.4. We start by noting that for $\beta_1 \neq \beta_2$, the processes $L_t^{\beta_1}$ and $L_t^{\beta_2}$ cannot agree on any interval on which one of them (and so both of them) increase. This is because in such a case, the equality would hold for the skew Brownian motions $X_t^{\beta_1}$ and $X_t^{\beta_2}$ on an interval where the local time at 0 would increase. On any such interval, the ratio of numbers of small excursions of skew Brownian motion on both sides of the real axis determines the skewness parameter $\beta$ with probability 1. Since $\beta_1 \neq \beta_2$, this leads to a contradiction. Hence, we can fix a small $m_0 > 0$ and wait until $L_t^{\beta_1}$ and $L_t^{\beta_2}$ are $m_0$ units apart. By letting $m_0 \to 0$, we will take account of all trajectories.

We start with a few definitions very similar to those in Remark 2.1. Note the crucial difference that in the present case we have two parameters $\beta_1$ and $\beta_2$ rather than just $\beta$. Let

$$
\begin{aligned}
T_0 &= \inf\{t > 0\colon |L_t^{\beta_2} - L_t^{\beta_1}| = m_0\},\\
S_k &= \inf\{t > T_k\colon B_t = -L_t^{\beta_2}\}, \qquad k \geq 0,\\
T_k &= \inf\{t > S_{k-1}\colon B_t = -L_t^{\beta_1}\}, \qquad k \geq 1,\\
W_k &= \frac{L_{S_{k-1}}^{\beta_2} - L_{S_{k-1}}^{\beta_1}}{L_{T_{k-1}}^{\beta_2} - L_{T_{k-1}}^{\beta_1}}, \qquad k \geq 1,\\
V_k &= \frac{L_{T_k}^{\beta_2} - L_{T_k}^{\beta_1}}{L_{S_{k-1}}^{\beta_2} - L_{S_{k-1}}^{\beta_1}}, \qquad k \geq 1,\\
M_k &= L_{T_k}^{\beta_2} - L_{T_k}^{\beta_1}, \qquad k \geq 0.
\end{aligned}
$$

As in Remark 2.1, we note that the random variables $W_k$ and $V_k$, $k \geq 1$, are jointly independent and $M_k = M_{k-1} V_k W_k$ for $k \geq 1$.

Let $\gamma_1 = (1 - \beta_1)/(2\beta_1)$ and $\gamma_2 = (1 + \beta_2)/(2\beta_2)$. The derivation of the cumulative distribution function for $W_k$ given in Barlow, Burdzy Kaspi and Mandelbaum (2000) does not depend on the fact that the same parameter $\beta$ is used in the definition of $V_k$. Hence, those distributional results apply in the present case and we see that $P(V_k > v) = v^{-\gamma_2}$ for $v \geq 1$, and $P(W_k < w) = w^{\gamma_1}$ for $w \in (0, 1)$. It follows that $V_k$, $k \geq 1$, are i.i.d. and the same can be said about $W_k$'s.

We have for $k \geq 1$,

$$M_k = M_0 \prod_{j=1}^{k} V_j W_j,$$

and so

$$\log M_k = \log M_0 + \sum_{j=1}^{k} (\log V_j + \log W_j). \tag{4.1}$$

The distribution of $\log V_j$ is exponential with mean $1/\gamma_2$ and that of $-\log W_j$ is exponential with mean $1/\gamma_1$. It follows that $\log V_j + \log W_j$ is a random variable with mean $1/\gamma_2 - 1/\gamma_1$ and finite variance.

If $1/\gamma_2 - 1/\gamma_1 > 0$, that is; when $\beta_1 < \beta_2/(1 + 2\beta_2)$, then the sum in (4.1) goes to infinity a.s. and so $M_k$ goes to infinity. An identical argument shows that $M_k W_{k+1}$ goes to infinity a.s. Hence, $L_{S_k}^{\beta_2} - L_{S_k}^{\beta_1} = M_k W_{k+1}$ goes to infinity. Note that $L_t^{\beta_2} - L_t^{\beta_1}$ is nondecreasing on intervals of the form $[S_k, T_{k+1}]$ and nonincreasing on intervals of the form $[T_k, S_k]$. This and the unbounded growth of $L_{S_k}^{\beta_2} - L_{S_k}^{\beta_1}$ imply that $L_t^{\beta_2} - L_t^{\beta_1}$ goes to infinity without ever taking the value 0. Part (i) of the theorem has been proved.

If $1/\gamma_2 - 1/\gamma_1 = 0$, that is, when $\beta_1 = \beta_2/(1+2\beta_2)$, the process $\log M_k$ is a mean zero random walk so it will oscillate between $-\infty$ and $+\infty$. This implies that $M_k$ will oscillate strictly between 0 and $\infty$. It follows that $L_t^{\beta_2} - L_t^{\beta_1}$ has to oscillate between 0 and $\infty$, a.s., which implies part (ii) of the theorem.

Next assume that $1/\gamma_2 - 1/\gamma_1 < 0$, that is, $\beta_1 > \beta_2/(1+2\beta_2)$. Let $T_\infty = \lim_{k\to\infty} T_k$. We have using Theorem 1.3(i),

$$L_{T_{k+1}}^{\beta_1} = L_{S_k}^{\beta_1} \le L_{S_k}^{\beta_2} = L_{T_k}^{\beta_2} = M_k + L_{T_k}^{\beta_1},$$

so

$$L_{T_\infty}^{\beta_1} \le L_{T_0}^{\beta_1} + \sum_{k=0}^{\infty} M_k.$$

If we can prove that $\sum_{k=0}^{\infty} M_k < \infty$, then $L_{T_\infty}^{\beta_1} < \infty$ and hence $T_\infty < \infty$.

Since $1/\gamma_2 - 1/\gamma_1 < 0$, the expectation of $\log V_j + \log W_j$ is negative and we see that a.s., there exists $c_1 > 0$ such that for large $k$ we will have

$$\sum_{j=1}^{k} (\log V_j + \log W_j) < -c_1 k.$$

From this inequality and (4.1) we deduce that $M_k \le c_2 e^{-c_1 k}$ for large $k$, and so $\sum_{k=0}^{\infty} M_k < \infty$. We conclude that $T_\infty < \infty$ a.s. Combining this with the observation that $M_k \to 0$, we see that there is some $t < \infty$ (namely, $t = T_\infty$) such that $L_t^{\beta_2} = L_t^{\beta_1}$.

It remains to show that $\limsup_{t\to\infty}(L_t^{\beta_2} - L_t^{\beta_1}) = \infty$. This follows from the excursion theory. Excursions of the process $(L_t^{\beta_1}, L_t^{\beta_2})$ from the diagonal can have arbitrarily large size and sooner or later some of them will exceed any given level. This completes the proof of the theorem. □

PROOF OF COROLLARY 1.5. We have dealt only with $\beta < 1$ but it is not hard to see that equation (1.1) defines reflected Brownian motion when we set $\beta = 1$ and then $L_t^{\beta}$ is the running minimum of $B_t$. The arguments given in the last proof easily extend to $\beta_2 = 1$ and then the corollary can be obtained from Theorem 1.4(iii). □

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