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ISOPERIMETRY FOR GIBBS MEASURES[1]

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We show that a strong mixing condition implies a Bakry–Bobkov–Ledoux inequality for a probability measure on infinite-dimensional space.

1. Introduction. In recent years we observe an interesting and intensive development in the area of isometry and its relation to coercive inequalities in infinite dimensions; see, for example, [2, 5, 17, 18] and references given there. In particular in [2] and [5] the following new functional inequality was introduced:

$$\mathscr{I}(\mu f) \le \mu(\mathscr{I}^2(f) + C|\nabla f|^2)^{1/2} \tag{1}$$

for any function $0 \le f \le 1$ for which the right-hand side is finite; here $\mathscr{I} \equiv \gamma \circ \Gamma^{-1}$ is the Lévy–Gromov isoperimetric function of a standard Gaussian distribution Γ and density γ, μF denotes the expectation of a function F with a probability measure μ and ∇f is a natural gradient of the function f. It was shown there that such inequality has a product property; that is, if it is true for two probability measures, it also holds for their product. Thus it is suitable for the infinite-dimensional setting. Additionally, in [5] (using the arguments similar to the Γ_2 criterion of Bakry–Emery [1]), the authors demonstrated that this inequality can also be proved for nontrivial measures when the underlying space is given as a product of smooth finite-dimensional Riemannian manifolds with strictly positive Ricci curvature.

In this paper we explore further the functional side of the isoperimetry in infinite dimensions. Refining the ideas introduced in the last decades in the course of studying the logarithmic Sobolev inequality (see, e.g., [9], [14]–[16], [20]–[23] and references therein), in Section 2 we show that more general inequalities propagate to infinite dimensions provided that the corresponding finite-dimensional conditional expectations satisfy them together with a strong mixing condition and some mild regularity condition (see Theorems 2.2 and 2.4). Section 3 is devoted to a closer discussion of the regularity and mixing in a comprehensive class of situations when the underlying space is defined as a infinite product associated to either a smooth connected finite-dimensional Riemannian manifolds or a finite set. In particular this allows including the results of [5]. Additionally, in Section 4 we prove that the inequalities of interest to us are true for finite-dimensional conditional expectations in the discrete

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setting and so the strategy developed in the previous sections applies (see Theorem 4.4). By this we generalize the results of [2] in an essential way.

As a by-product of our work (by general arguments of [5]), we get yet another proof of the logarithmic Sobolev inequality.

Summarizing, in our work we have introduced a family of functional inequalities which potentially allow the study of various classes probability measures and give a deeper insight into the corresponding isoperimetric problem. One may also hope that such inequalities could be of interest to the geometric measure theory.

2. Coercive inequalities for infinite mixing systems. Let $\mathscr{F}$ denote the family of all finite subsets in $\mathbb{Z}^d$. For $\Lambda \in \mathscr{F}$, its cardinality will be denoted by $|\Lambda|$. Let $\Omega \equiv \mathbf{M}^{\mathbb{Z}^d}$, where $\mathbf{M}$ is either a smooth connected finite-dimensional Riemannian manifold or simply a finite set. We will say that a function $f\colon \Omega \to \mathbb{R}$ is localized in a finite set $\Lambda \subset \mathbb{Z}^d$ iff it depends only on coordinates $\{\omega_i\colon i \in \Lambda\}$ and the smallest localization set for f will be denoted by Λ_f. Let $\nabla_j f$ denote the gradient and the discrete gradient (with respect to the ith coordinate), given by $\nabla_j f \equiv f - \nu_j f$, on a Riemannian manifold and a finite set, respectively, where ν_j denotes a copy of the uniform probability measure on $\mathbf{M}$ with the subscript indicating the integration over the coordinate ω_j of $\omega \equiv (\omega_k)_{k\in\mathbb{Z}^d}$. For a set $\Lambda \subset \mathbb{Z}^d$, we denote $\nabla_\Lambda f \equiv (\nabla_j f)_{j\in\Lambda}$ and we define

$$|\nabla_\Lambda f|_p^p \equiv \sum_{j\in\Lambda} |\nabla_j f|^p.$$

For $\Lambda \in \mathscr{F}$ and a configuration $\omega \in \Omega \equiv \mathbf{M}^{\mathbb{Z}^d}$, let μ_Λ^ω be a probability kernel on Ω given $\{\omega_j,\ j \in \mathbb{Z}^d\backslash\Lambda\}$. We will assume later on that the kernel μ_Λ^ω is of range $R \in (0,\infty)$; that is, for any bounded measurable function f localized in Λ, its expectation $\mu_\Lambda^\omega f$ depends only on ω_j, $j \in \mathbb{Z}^d\backslash\Lambda$, $\operatorname{dist}(j,\Lambda) < R$ (clearly, by definition, for any function the expectation $\mu_\Lambda^\omega f$ is independent of ω_i, with $i \in \Lambda$).

Let $\mathscr{U}$ be a nonnegative concave function on an interval $I \subset \mathbb{R}$. Let C_Λ be the best constant such that the following inequality $\mathbf{BBL}_p(\Lambda)$, $p \in [1,\infty)$, is true uniformly in external conditions $\omega \in \Omega$,

$$\mathscr{U}(\mu_\Lambda^\omega f) \le \mu_\Lambda^\omega \left(\mathscr{U}(f)^p + C_\Lambda |\nabla_\Lambda f|_p^p\right)^{1/p} \tag{2}$$

for all functions $f\colon \Omega \to I$ for which the right-hand side is finite.

Given a set $\Gamma \equiv \bigcup_k \Lambda_k$ consisting of disjoint finite cubes $\Lambda_k \in \mathscr{F}$, $k \in \mathbb{N}$, which are translates of a given cube Λ_0 and such that $\operatorname{dist}(\Lambda_k, \Lambda_l) \ge 2R$ for $k \ne l$, we define the following product measure

$$E_\Gamma \equiv \bigotimes_{\Lambda_k\subset\Gamma} \mu_{\Lambda_k}^\omega$$

Following [5] and [2], we have the following preliminary lemma.

LEMMA 2.1. *Suppose* $\mathbf{BBL}_p(\Lambda)$ *is true for some* $p \in [1,\infty)$. *Then we have*

$$\mathscr{U}(E_\Gamma f) \le E_\Gamma\big(\mathscr{U}(f)^p + C_\Gamma |\nabla_\Lambda f|_p^p\big)^{1/p} \tag{3}$$

with a constant $C_\Gamma = C_\Lambda$ for all functions $f\colon \Omega \to I$ for which the right-hand side is finite.

PROOF. The proof is similar to the one given in [5] and [2] and goes by the following inductive arguments. Let Λ_k, $k \in \mathbb{N}$, be a lexicographic ordering of Γ. For $n \in \mathbb{N}$ we define $\Gamma^{(n)} \equiv \cup_{k=1,\dots,n}\Lambda_k$. Suppose for some $n \in \mathbb{N}$, we have

$$\mathscr{U}(E_{\Gamma^{(n)}} f) \le E_{\Gamma^{(n)}}\big(\mathscr{U}(f)^p + C_\Lambda |\nabla_{\Gamma^{(n)}} f|_p^p\big)^{1/p} \tag{4}$$

for all functions f for which the right-hand side is well defined. Then for $E_{\Gamma^{(n+1)}} = E_{\Gamma^{(n)}} \otimes \mu_\Lambda^\omega$, $\Gamma^{(n)} \cap \Lambda = \varnothing$, we have

$$\begin{aligned} \mathscr{U}(E_{\Gamma^{(n+1)}} f) &= \mathscr{U}(E_{\Gamma^{(n)}} \otimes \mu_\Lambda^\omega f) \\ &\le E_{\Gamma^{(n)}}\big(\mathscr{U}(\mu_\Lambda^\omega f)^p + C_\Lambda |\nabla_{\Gamma^{(n)}} \mu_\Lambda^\omega f|_p^p\big)^{1/p} \end{aligned} \tag{5}$$

Now, using our assumption (2) we can bound the right-hand side as follows:

$$\begin{aligned} &E_{\Gamma^{(n)}}\big(\mathscr{U}(\mu_\Lambda^\omega f)^p + C_\Lambda |\nabla_{\Gamma^{(n)}} \mu_\Lambda^\omega f|_p^p\big)^{1/p} \\ &\quad\le E_{\Gamma^{(n)}}\Big(\big(\mu_\Lambda^\omega\big(\mathscr{U}(f)^p + C_\Lambda|\nabla_\Lambda f|_p^p\big)^{1/p}\big)^p + C_\Lambda |\mu_\Lambda^\omega \nabla_{\Gamma^{(n)}} f|_p^p\Big)^{1/p}. \end{aligned} \tag{6}$$

Applying the Minkowski inequality,

$$\left(|\mu_\Lambda^\omega F_0|^p + \sum_{l\in\mathbb{N}} |\mu_\Lambda^\omega F_l|^p\right)^{1/p} \le \mu_\Lambda^\omega\left(|F_0|^p + \sum_{l\in\mathbb{N}} |F_l|^p\right)^{1/p} \tag{7}$$

with $F_0 \equiv (\mathscr{U}(f)^p + C_\Lambda|\nabla_\Lambda f|_p^p)^{1/p}$ and $F_l \equiv C_\Lambda^{1/p}\nabla_{j_l} f$, $j_l \in \Gamma^{(n)}$, and rearranging the terms we arrive at the desired inequality for $E_{\Gamma^{(n+1)}}$. By induction this shows inequality (3) for any local function with finite gradient. Since such functions are dense in the set $\{f\colon \Omega \to I/E_\Gamma|\nabla_\Gamma f|_p < \infty\}$, this ends the proof. $\square$

Let Γ_l, $0 \le l \le K \equiv 2^d$, be the suitable translations of the Γ_0 considered above, so that their union covers the lattice $\mathbb{Z}^d$; see [16]. We will set $E_l \equiv E_{\Gamma_l}$ and define the following transfer matrix:

$$\mathscr{P} \equiv E_{K-1} \cdots E_0.$$

In the following result we formulate sufficient conditions on the transfer matrix $\mathscr{P}$, which allow us to prove the $\mathbf{BBL}_p$ inequality for a nonproduct measure on the infinite-dimensional space Ω.

THEOREM 2.2. *Suppose the following conditions are satisfied*:

(Ci) *There is a constant $\widetilde{C} \in (0, \infty)$ such that*

$$\mathscr{U}(\mathscr{P} f) \le \mathscr{P}\big(\mathscr{U}(f)^p + \widetilde{C}|\nabla f|_p^p\big)^{1/p} \tag{8}$$

for any function f for which the right-hand side is finite.

(Cii) *There is* $\lambda \in (0,1)$ *such that*

$$|\nabla \mathscr{P} f|_p^p \le \lambda \sum_j (\mathscr{P}|\nabla_j f|)^p. \tag{9}$$

Then for any $N \in \mathbb{N}$, *we have*

$$\big(\mathscr{U}(\mathscr{P}^N f)^p + \widetilde{C}|\nabla \mathscr{P}^N f|_p^p\big)^{1/p} \le \mathscr{P}^N\big(\mathscr{U}(f)^p + \widetilde{C}_N|\nabla f|_p^p\big)^{1/p} \tag{10}$$

for any function f *for which the right-hand side is finite with a constant*

$$\widetilde{C}_N \equiv \widetilde{C} \sum_{k=0}^{N-1} \lambda^k. \tag{11}$$

Hence the following $\mathbf{BBL}_p$ *inequality is true for the infinite volume measure* $\mu \equiv \lim_{N\to\infty} \mathscr{P}^N$:

$$\mathscr{U}(\mu f) \le \mu\big(\mathscr{U}(f)^p + C|\nabla f|_p^p\big)^{1/p} \tag{12}$$

with a constant

$$C \le \widetilde{C}/(1-\lambda). \tag{13}$$

PROOF. By condition (Ci), we have

$$\mathscr{U}(\mathscr{P}^N f) \le \mathscr{P}(\mathscr{U}(\mathscr{P}^{N-1} f)^p + \widetilde{C}|\nabla \mathscr{P}^{N-1} f|_p^p)^{1/p}. \tag{14}$$

Applying (Ci) to the first term in the bracket on the right-hand side, we have

$$\mathscr{U}(\mathscr{P}^{N-1} f)^p \le (\mathscr{P}(\mathscr{U}(\mathscr{P}^{N-2} f)^p + \widetilde{C}|\nabla \mathscr{P}^{N-2} f|_p^p)^{1/p})^p. \tag{15}$$

On the other hand, using (Cii), we get the following estimate for the second term in the bracket on the right-hand side of (14):

$$|\nabla \mathscr{P}^{N-1} f|_p^p \le \lambda \sum_j (\mathscr{P}|\nabla_j \mathscr{P}^{N-2} f|)^p. \tag{16}$$

Combining this and using the Minkowski inequality, we arrive at

$$\begin{aligned} \mathscr{U}(\mathscr{P}^N f) &\le \mathscr{P}(\mathscr{U}(\mathscr{P}^{N-1} f)^p + \widetilde{C}|\nabla \mathscr{P}^{N-1} f|_p^p)^{1/p} \\ &\le \mathscr{P}^2(\mathscr{U}(\mathscr{P}^{N-2} f)^p + \widetilde{C}(1+\lambda)|\nabla \mathscr{P}^{N-2} f|_p^p)^{1/p}. \end{aligned} \tag{17}$$

Hence, by induction we get

$$\mathscr{U}(\mathscr{P}^N f) \le \mathscr{P}^N(\mathscr{U}(f)^p + \widetilde{C}_N|\nabla f|_p^p)^{1/p} \tag{18}$$

with

$$\widetilde{C}_N \equiv \widetilde{C} \sum_{k=0}^{N-1} \lambda^k. \tag{19}$$

Since under the assumption (Cii) we have $\mathscr{P}^N f \to \mu f$ for any local $\mathscr{C}^1$ function f, this also implies (12). This ends the proof of Theorem 2.2. $\square$

Now we would like to formulate a local condition in terms of the conditional expectation μ_Λ^ω for a cube Λ of a finite size, which allows us to verify the desired properties (C) of Theorem 2.2.

DEFINITION 2.3. A local specification $\{\mu_\Lambda^\omega\}_{\Lambda\in\mathscr{F},\,\omega\in\Omega}$ is called *regular* iff there are constants $\alpha_{ij}^{(\Lambda)}$, $ij \in \mathbb{Z}^d$, such that

$$|\nabla_i \mu_\Lambda^\omega f| \le \sum_{j\in\Lambda\cup\{i\}} \alpha_{ij}^{(\Lambda)} \mu_\Lambda^\omega |\nabla_j f| \tag{20}$$

for any differentiable function f; because of the definition of conditional expectation we can and do assume that $\alpha_{ij}^{(\Lambda)} \equiv 0$ if $i \in \Lambda$.

A local specification $\{\mu_\Lambda^\omega\}_{\Lambda\in\mathscr{F},\,\omega\in\Omega}$ is called *mixing* iff additionally the constants $\alpha_{ij}^{(\Lambda)}$, $i, j \in \mathbb{Z}^d$, satisfy

$$\alpha_{ij}^{(\Lambda)} \le a e^{-M|i-j|} \tag{21}$$

with some constants $a, M \in (0,\infty)$ independent of i, j and the set $\Lambda \in \mathscr{F}$.

Using this definition we show the following result.

THEOREM 2.4. (i) *Suppose the local specification is regular and for a cube $\Lambda \in \mathscr{F}$ the corresponding conditional expectation satisfies the inequality* $\mathbf{BBL}_p(\Lambda)$*with some constant $C_\Lambda \equiv C(|\Lambda|)$. Then the transfer matrix $\mathscr{P}$ satisfies the condition* (Ci).

(ii) *If additionally, the local specification is mixing, then the condition* (Cii) *is satisfied provided the size of the cubes $\Lambda_l \subset \Gamma$ is sufficiently large.*

PROOF. We begin from the arguments which allow us to estimate the gradients. For a diferentiable local function f we define

$$f_k \equiv E_k f_{k-1} \equiv E_k E_{k-1} \cdots\ E_0 f \tag{22}$$

for $k \in \mathbb{N}$. Given a point $i \notin \Gamma_k$ there is at most one cube $\Lambda(i) \subset \Gamma_k$ such that $\operatorname{dist}(i, \Lambda(i)) \le R$ and $\operatorname{dist}(j, \Lambda_l) > R$ for any other cube $\Lambda_l \subset \Gamma_k$, $\Lambda_l \ne \Lambda(i)$. Using this together with the regularity of the local specification we get

$$\begin{aligned} |\nabla_i E_k f_{k-1}| &= |E_{\Gamma_k\setminus\Lambda(i)} \nabla_i \mu^{\cdot}_{\Lambda(i)} f_{k-1}| \le E_{\Gamma_k\setminus\Lambda(i)} |\nabla_i \mu^{\cdot}_{\Lambda(i)} f_{k-1}| \\ &\le \sum_{j\in\Lambda\cup\{i\}} \alpha_{ij}^{(\Lambda(i))} E_k |\nabla_j f_{k-1}|. \end{aligned} \tag{23}$$

Iterating this argument we arrive at the following bound:

$$|\nabla_i E_k f_{k-1}| \le \sum_j \eta_{ij}^{(k)} E_k E_{k-1} \cdots\ E_0 |\nabla_j f|, \tag{24}$$

where

$$\eta_{ij}^{(k)} \equiv \sum_{\mathscr{W}_{ij}=\{i,\, j_1,\, \ldots,\, j_k,\, j\}} \alpha_{ij_1}^{(\Lambda(i))} \alpha_{j_1 j_2}^{(\Lambda(j_1))} \cdots \alpha_{j_k j}^{(\Lambda(j_k))} \tag{25}$$

with the summation going over the paths $\mathscr{W}_{ij} \equiv \{i, j_1, \ldots, j_k, j\}$ such that $j_l \in \Lambda(j_{l-1})\backslash\Gamma_{k-l-1} \cup \{j_{l-1}\}$ for $l = 1, \ldots, k-1$.

Applying the Hölder inequality to (24), we obtain

$$|\nabla_i f_k|^p \leq \left[\sum_j \eta_{ij}^{(k)}\right]^{p-1} \sum_j \eta_{ij}^{(k)} \big(E_k E_{k-1} \cdots E_0 |\nabla_j f|\big)^p, \tag{26}$$

whence

$$|\nabla f_k|_p^p \leq A_k \sum_j (E_k E_{k-1} \cdots E_0 |\nabla_j f|)^p \tag{27}$$

with a constant A_k given by

$$A_k \equiv \max_{i \in \mathbb{Z}^d} \left[\sum_j \eta_{ij}^{(k)}\right]^{p-1} \max_{i \in \mathbb{Z}^d} \sum_j \eta_{ji}^{(k)}. \tag{28}$$

Hence in particular we obtain

$$|\nabla \mathscr{P} f|_p^p \leq \lambda \sum_j (\mathscr{P} |\nabla_j f|)^p \tag{29}$$

with $\lambda \equiv A_K$. At this point, following [16], we observe that if $k = K$ in any path $\mathscr{W}_{ij}$ there has to be a large step, say $\{j_l, j_{l+1}\}$, such that $\mathrm{dist}(j_l, j_{l+1}) \geq \frac{1}{4}\mathrm{diam}(\Lambda_0)$. In the case when the local specification is mixing, this implies that

$$\alpha_{j_l, j_{l+1}} \leq a \exp\{-M \tfrac{1}{2} \mathrm{diam}(\Lambda_0)\}.$$

Hence one can easily see that by choosing Λ_0 sufficiently large we can get $\lambda \in (0, 1)$. This ends the proof of (Cii).

Given (27) and (28) and using Lemma 2.1 we can complete the proof of (Ci) as follows. In the first step, by Lemma 2.1, we have

$$\mathscr{U}(\mathscr{P} f) = \mathscr{U}(E_K f_{K-1}) \leq E_K \big(\mathscr{U}(f_{K-1})^p + C_\Lambda |\nabla f_{K-1}|_p^p\big)^{1/p}. \tag{30}$$

Hence, applying (27) and (28), we get with $A \equiv max_{k=0,\ldots,K} A_k$,

$$\begin{aligned} \mathscr{U}(\mathscr{P} f) &= \mathscr{U}(E_K f_{K-1}) \\ &\leq E_K \bigg(\mathscr{U}(E_{K-1} f_{K-2})^p + C_\Lambda A \sum_j |E_{K-1} \nabla_j f_{K-2}|^p\bigg)^{1/p} \\ &\leq E_K \big((E_{K-1}(\mathscr{U}(f_{K-2})^p + C_\Lambda |\nabla f_{K-2}|_p^p)^{1/p})^p \\ &\quad + C_\Lambda A \sum_j |E_{K-1} \nabla_j f_{K-2}|^p\big)^{1/p}, \end{aligned} \tag{31}$$

where in the last step we have applied Lemma 2.1 again. From (31), using the Minkowski inequality, we obtain

$$\mathscr{U}(\mathscr{P}f) \le E_K E_{K-1}(\mathscr{U}(f_{K-2})^p + C_\Lambda(1+A)|\nabla f_{K-2}|^p)^{1/p}. \tag{32}$$

Applying these arguments inductively we arrive at the following bound:

$$\mathscr{U}(\mathscr{P}f) \le \mathscr{P}(\mathscr{U}(f)^p + \widetilde{C}|\nabla f|_p^p)^{1/p} \tag{33}$$

with a constant $\widetilde{C}$ given by

$$\widetilde{C} \equiv C_\Lambda \sum_{k=0}^{K} A^k. \tag{34}$$

This ends the proof of Theorem 2.4. □

REMARK 2.5. We note that, although we have given here a proof for conditional expectations of finite range, one can use a modification of a strategy (similar to the one considered in [22]) to prove the corresponding result for the case of infinite range.

3. Regularity and mixing conditions. For a set $\Lambda \in \mathscr{F}$, we define a finite volume Gibbs measure μ_Λ^ω with external condition $\omega \in \Omega$ as follows:

$$\mu_\Lambda^\omega(f) \equiv \delta_\omega(\nu_\Lambda(\rho_\Lambda f)) \tag{35}$$

with

$$\rho_\Lambda \equiv \frac{e^{-U_\Lambda}}{\nu_\Lambda e^{-U_\Lambda}},$$

where δ_ω is a point measure concentrated on a configuration ω, $\nu_\Lambda \equiv \otimes_{j\in\Lambda}\nu_j$ with $\nu_j = \nu_0(d\omega_j)$ being the uniform probability measure on $\mathbf{M}$ and

$$U_\Lambda \equiv \sum_{X\cap\Lambda\neq\varnothing} \Phi_X$$

where Φ_X is a continuous (respectively, $\mathscr{C}^1$ if $\mathbf{M}$ is a smooth compact connected Riemannian manifold) function localized in a finite set $X \in \mathscr{F}$; that is, it depends only on the coordinates $\omega_X \equiv \{\omega_j \in \mathbf{M}\colon j \in X\}$. The collection $\Phi \equiv \{\Phi_X\}_{X\in\mathscr{F}}$ is called an interaction and we assume that

$$\|\Phi\| \equiv \sup_{j\in\mathbb{Z}^d} \sum_{X\colon X\ni j} \|\Phi_X\|_u < \infty.$$

Under this condition we have $|U_\Lambda| \le \|\Phi\|\cdot|\Lambda|$, where $|\Lambda|$ denotes the cardinality of the finite set Λ, and therefore our finite volume measure μ_Λ^ω is well defined. To simplify the exposition, later on we restrict ourselves to interactions Φ of finite range $R > 0$, that is, such that $\Phi_X \equiv 0$ if $\operatorname{diam}(X) > R$.

The following strong mixing condition is well established in the literature (see [6], [13], etc.) and can be verified in a large number of interesting situations.

DEFINITION 3.1 The strong mixing condition. There is a constant $M \in (0, \infty)$ such that for any cube $\Lambda \subset \mathbb{Z}^d$ and any configuration $\omega \in \Omega$, we have

$$\left|\mu_\Lambda^\omega\left((F - \mu_\Lambda^\omega(F))(G - \mu_\Lambda^\omega(G))\right)\right| \leq e^{-M\,\mathrm{dist}(\Lambda_F,\Lambda_G)}|||F||| \cdot |||G||| \tag{36}$$

for all local functions F and G dependent on coordinates ω_i, $i \in \Lambda_F$ and ω_i, $i \in \Lambda_G$, respectively, for some subsets $\Lambda_F, \Lambda_G \subset \Lambda$; here

$$|||g||| \equiv \sum_i |||\nabla_i g|||_u.$$

We remark that the strong mixing condition, if it is satisfied for sufficiently large cubes, is also satisfied for all unions of such cubes (see [6], [12], [13], etc.).

In this section we show the following result.

THEOREM 3.2. *The local specification $\{\mu_\Lambda^\omega\}_{\Lambda\in\mathscr{F},\,\omega\in\Omega}$ is regular; that is, there are constants $\alpha_{ij}^{(\Lambda)}$, $i, j \in \mathbb{Z}^d$, such that*

$$|\nabla_i \mu_\Lambda^\omega f| \leq \sum_{j\in\Lambda\cup\{i\}} \alpha_{ij}^{(\Lambda)} \mu_\Lambda^\omega |\nabla_j f|. \tag{37}$$

for any differentiable function f; because of the definition of conditional expectation we can and do assume that $\alpha_{ij}^{(\Lambda)} \equiv 0$ if $i \in \Lambda$

Moreover, if the strong mixing condition is satisfied, then for any cube $\Lambda \in \mathscr{F}$ the constants $\alpha_{ij}^{(\Lambda)}$, $i, j \in \mathbb{Z}^d$, satisfy

$$\alpha_{ij}^{(\Lambda)} \leq a e^{-M|i-j|} \tag{38}$$

with some constants $a, M \in (0, \infty)$ independent of i, j and the cube $\Lambda \in \mathscr{F}$.

Before we begin the proof, let us notice that the following fact is true.

LEMMA 3.3. *For any set $\Lambda \subset \mathbb{Z}^d$ and any finite set $\Delta \subset \Lambda$ the following L_1-Poincaré inequality is true*:

$$m_\Delta \mu_\Lambda^\omega |F - \mu_\Lambda^\omega F| \leq \mu_\Lambda^\omega |\nabla_\Delta F|_1 \tag{39}$$

for any differentiable function F localized in Δ with a constant $m_\Delta \in (0, \infty)$ independent of Λ, ω and a function F.

PROOF. First we note that for the product measure we have [10]

$$m_\Delta^{(0)} \nu_\Delta |F - \nu_\Delta F| \leq \nu_\Delta |\nabla_\Delta F|_1 \tag{40}$$

with some constant $m_\Delta^{(0)}$ independent of a $\mathscr{C}^1$ function F localized in Δ. From this the desired inequality follows by the following sequence of inequalities for any $\mathscr{C}^1$ function F localized in Δ:

$$\begin{aligned}\mu_\Lambda^\omega|F-\mu_\Lambda^\omega F| &\le 2e^{2\|\Phi\|\cdot|\Delta|}\nu_\Delta|F-\nu_\Delta F| \le 2e^{2\|\Phi\|\cdot|\Delta|}\big[m_\Delta^{(0)}\big]^{-1}\nu_\Delta|\nabla_\Delta F|_1\\ &\le 2e^{4\|\Phi\|\cdot|\Delta|}\big[m_\Delta^{(0)}\big]^{-1}\mu_\Lambda^\omega|\nabla_\Delta F|_1.\end{aligned} \tag{41}$$

This ends the proof of Lemma 3.3. □

PROOF OF THEOREM 3.2 (The continuous case). We consider first the case when **M** is a Riemannian manifold and therefore differentiation satisfies the Leibnitz rule; similar arguments for the discrete case are given later. Let Δ be a cube contained in a larger cube Λ. For a differentiable function f, let $F_\Delta \equiv E_{\mu_\Lambda^\omega}(f|\omega_\Delta) = \mu_{\Lambda\setminus\Delta}^\omega f$; that is, F is a conditional expectation given ω_Δ associated to the measure μ_Λ^ω. We notice first that for any local $\mathscr{C}^1$ functions and any $j\in\Lambda^c$ such that $\operatorname{dist}(j,\Delta)>R$, we have

$$\begin{aligned}\nabla_j\mu_\Lambda^\omega(f) &= \nabla_j\mu_\Lambda^\omega(\mu_{\Lambda\setminus\Delta}^{\cdot}f)\equiv\nabla_j\mu_\Lambda^\omega(F_\Delta)\\ &=\mu_\Lambda^\omega(\nabla_jF_\Delta)-\mu_\Lambda^\omega(\nabla_jU_\Lambda;F_\Delta),\end{aligned} \tag{42}$$

where for two bounded measurable functions g and h we have set

$$\mu(g;h)\equiv\mu(gh)-\mu(g)\mu(h)$$

to denote the covariance with a probability measure μ. The first term on the right hand side has the right structure already, so we need to consider only the second one. Since F_Δ depends only on coordinates in the set Δ (the others in Λ^c are fixed), we can write that term as follows:

$$\mu_\Lambda^\omega(\nabla_jU_\Lambda;F_\Delta)=\mu_\Lambda^\omega(\mu_{\Lambda\setminus\Delta}^\omega(\nabla_jU_j);F_\Delta), \tag{43}$$

where we have also used the local structure of U_Λ to get $\nabla_jU_\Lambda=\nabla_jU_j$. Using (43) together with the L_1-Poincaré inequality of Lemma 3.3 we arrive at

$$|\mu_\Lambda^\omega(\nabla_jU_\Lambda;F_\Delta)|\le m_\Delta^{-1}\operatorname{Var}_\Delta(\mu_{\Lambda\setminus\Delta}^\omega(\nabla_jU_j))\mu_\Lambda^\omega|\nabla_\Delta F_\Delta|_1, \tag{44}$$

where

$$\operatorname{Var}_\Delta(g)\equiv\sup_{\omega,\tilde\omega:\omega_{\Delta^c}=\tilde\omega_{\Delta^c}}|g(\omega)-g(\tilde\omega)|.$$

Combining (44) and (42) we obtain the following inequality:

$$|\nabla_j\mu_\Lambda^\omega(f)|\le\sum_{l\in\Delta\cap\{j\}}\gamma_{jl}\mu_\Lambda^\omega|\nabla_l\mu_{\Lambda\setminus\Delta}^\omega(f)| \tag{45}$$

with

$$\gamma_{jl}\equiv m_\Delta^{-1}\operatorname{Var}_\Delta\big(\mu_{\Lambda\setminus\Delta}^\omega(\nabla_jU_j)\big) \tag{46}$$

for $l\in\Delta$ and $\gamma_{jj}\equiv1$. Since the terms on the right-hand side of (46) are of the same structure as that we started with, we can apply these arguments inductively and after a finite number of steps obtain the regularity statement.

To prove the second property of interest to us we observe that if the strong mixing condition is satisfied then by standard arguments we have

$$\mathrm{Var}_\Delta(\mu^\omega_{\Lambda\setminus\Delta}(\nabla_j U_j)) \le \tilde{a} e^{-M \operatorname{dist}(j,\Delta)}. \tag{47}$$

with a constant $\tilde{a} \in (0,\infty)$ dependent only on $\nabla_j U_j$ and $\|\Phi\|$. Using this observation we can optimize our inductive procedure (which led us to the regularity) so that at every consecutive step we choose the next cube to be lying as far as possible from the previous cube and in the way that the remaining new set has a shape similar to a cube (similarly as in [14]). After resuming the resulting expansion we arrive at the desired bound (38) on the corresponding coefficients α_{ji}. This ends the proof of Theorem 3.2 in the continuous case. □

PROOF OF THEOREM 3.2 (The discrete case). Now we consider the case when $\mathbf{M}$ is a finite set. In this case the differentiation does not satisfy the Leibnitz rule, but we have

$$\nabla_j(FG) \equiv FG - \nu_j(FG) = \nabla_j(F)G + \nu_j(F)\nabla_j(G) - \nu_j(F\nabla_j(G)).$$

Let Δ be a cube contained in a larger cube Λ. For a function f, let $F_\Delta \equiv E_{\mu^\omega_\Lambda}(f|\omega_\Delta) = \mu^\omega_{\Lambda\setminus\Delta} f$; that is, F is a conditional expectation given ω_Δ associated to the measure μ^ω_Λ. We notice first that for any functions and any $j \in \Lambda^c$ such that $\operatorname{dist}(j,\Delta) > R$, we have

$$\nabla_j \mu^\omega_\Lambda(f) = \nabla_j \mu^\omega_\Lambda(\mu^{\cdot}_{\Lambda\setminus\Delta} f) \equiv \nabla_j \mu^\omega_\Lambda(F_\Delta). \tag{48}$$

Since μ^ω_Λ has density $\rho_{\Lambda,\omega}$ with respect to the product measure ν_Λ, we have

$$\begin{aligned} \nabla_j \mu^\omega_\Lambda(F_\Delta) &= \nu_\Lambda(\nabla_j(\rho_{\Lambda,\omega} F_\Delta)) \\ &= \mu^\omega_\Lambda(\nabla_j(F_\Delta)) + \mu^\omega_\Lambda(\rho^{-1}_{\Lambda,\omega}\nabla_j(\rho_{\Lambda,\omega})\nu_j(F_\Delta)) \\ &\quad + \nu_\Lambda(\nu_j(\nabla_j \rho_{\Lambda,\omega})F_\Delta). \end{aligned} \tag{49}$$

The first term on the right-hand side has the right structure already, so we need to consider only the second and third. We note that because F_Δ depends only on coordinates in the set Δ, so does $\nu_j F_\Delta$ (the others in Λ^c are fixed). Therefore we can write the second term as follows:

$$\mu^\omega_\Lambda(\rho^{-1}_{\Lambda,\omega}\nabla_j(\rho_{\Lambda,\omega})\nu_j(F_\Delta)) = \mu^\omega_\Lambda(\mu^\omega_{\Lambda\setminus\Delta}(\rho^{-1}_{\Lambda,\omega}\nabla_j(\rho_{\Lambda,\omega}))\nu_j(F_\Delta - \mu^\omega_\Lambda(F_\Delta))), \tag{50}$$

where we have also used that

$$\mu^\omega_\Lambda(\rho^{-1}_{\Lambda,\omega}\nabla_j(\rho_{\Lambda,\omega}) = 0.$$

Now we can bound (50) as follows:

$$\begin{aligned} &|\mu^\omega_\Lambda(\rho^{-1}_{\Lambda,\omega}\nabla_j(\rho_{\Lambda,\omega})\nu_j(F_\Delta))| \\ &\quad \le \mathrm{Var}_\Delta(\mu^\omega_{\Lambda\setminus\Delta}(\rho^{-1}_{\Lambda,\omega}\nabla_j(\rho_{\Lambda,\omega}))\mu^\omega_\Lambda \nu_j|(F_\Delta - \mu^\omega_\Lambda(F_\Delta))| \\ &\quad \le m_\Delta^{-1} e^{4\|\Phi\|}\mathrm{Var}_\Delta\big(\mu^\omega_{\Lambda\setminus\Delta}(\rho^{-1}_{\Lambda,\omega}\nabla_j(\rho_{\Lambda,\omega})\big)\mu^\omega_{\{j\}}\mu^{\cdot}_\Lambda|\nabla_\Delta F_\Delta|_1, \end{aligned} \tag{51}$$

where in the last step we have used the L_1-Poincaré inequality of Lemma 3.3 (and some simple arguments to replace ν_j under the expectation with the measure μ_Λ^ω by the conditional expectation $\mu_{\{j\}}^\omega$). The third term from the right-hand side of (49) we treat as follows. We observe first that we can write it as

$$\nu_j(\mu_\Lambda^\omega(\rho_{\Lambda,\omega}^{-1}(\nabla_j\rho_{\Lambda,\omega})F_\Delta)) = \nu_j(\mu_\Lambda^\omega([\mu_{\Lambda\setminus\Delta}^\omega\rho_{\Lambda,\omega}^{-1}(\nabla_j\rho_{\Lambda,\omega})](F_\Delta - \mu_\Lambda^\omega F_\Delta))). \tag{52}$$

Hence, by similar arguments used to estimate the second term on the right hand side (49), we obtain the following bound:

$$|\nu_\Lambda(\nu_j(\nabla_j\rho_{\Lambda,\omega}F_\Delta)) \le m_\Delta^{-1}e^{2\|\Phi\|}\mathrm{Var}_\Delta\big(\mu_{\Lambda\setminus\Delta}^\omega(\rho_{\Lambda,\omega}^{-1}\nabla_j(\rho_{\Lambda,\omega}))\mu_{\{j\}}^\omega\mu_\Lambda^\cdot|\nabla_\Delta F_\Delta|_1. \tag{53}$$

Combining (48), (51) and (53) we obtain the following inequality:

$$|\nabla_j\mu_\Lambda^\omega(f)| = |\nabla_j\mu_\Lambda^\omega(F_\Delta)| \le \sum_{l\in\Delta\cup\{j\}}\gamma_{jl}(\delta_{jl}\delta_\omega + (1-\delta_{jl})\mu_{\{j\}}^\omega)\mu_\Lambda^\cdot|\nabla_l(F_\Delta)| \tag{54}$$

with

$$\gamma_{jl} \equiv 2m_\Delta^{-1}e^{4\|\Phi\|}\mathrm{Var}_\Delta\big(\mu_{\Lambda\setminus\Delta}^\omega(\rho_{\Lambda,\omega}^{-1}\nabla_j(\rho_{\Lambda,\omega})\big) \tag{55}$$

for $l \in \Delta$ and $\gamma_{jj} \equiv 1$. Since the terms on the right-hand side of (55) with $l \neq j$ are similar to the structure we started with, we can apply these arguments inductively and after a finite number of steps obtain the regularity statement.

To prove the second property of interest to us we observe that if the strong mixing condition is satisfied then by standard arguments we have

$$\mathrm{Var}_\Delta\big(\mu_{\Lambda\setminus\Delta}^\omega(\rho_{\Lambda,\omega}^{-1}\nabla_j(\rho_{\Lambda,\omega})\big) \le \tilde{a}e^{-M\,\mathrm{dist}(j,\Delta)} \tag{56}$$

with a constant $\tilde{a} \in (0,\infty)$ dependent only on $\|\Phi\|$ and $|\Delta|$. Using this observation we can optimize our inductive procedure (which led us to the regularity) so that at every consecutive step we choose the next cube to be lying as far as possible and in the way that the remaining new set has a shape similar to a cube (similarly as in [14]). After resuming the resulting expansion we arrive at the desired bound (38) on the corresponding coefficients α_{ji}. This ends the proof of Theorem 3.2 in the discrete case. □

4. Applications. In this section we consider in more detail the $\mathbf{BBL}_2$ inequality originally introduced in [2] and [5] with

$$\mathcal{U} = \mathcal{I} \equiv \gamma \circ \Gamma^{-1},$$

where Γ and γ denote the distribution and density of the Gaussian measure with mean zero and covariance 1. We have already mentioned that [5] includes a proof of such inequalities for the case when the configuration space is given by $\Omega \equiv \mathbb{M}^{\mathbb{Z}^d}$, with $\mathbb{M}$ being a smooth compact and connected Riemannian manifold with strictly positive Ricci curvature. Such an assumption makes it possible to use an excellent idea similar to the Γ_2 criterion of Bakry and Emery [1], invented for the case of logarithmic Sobolev inequality. If one would like to apply this idea directly in the infinite-dimensional setting, it would require some delicate smoothness justification. One way to overcome them is via the use of the method of [5] to get $\mathbf{BBL}_2(\Lambda)$ for large cubes and then follow the strategy described in the previous sections. In fact we believe that following our route it is possible to extend the domain where $\mathbf{BBL}_2$ is true to the cases when Ricc> 0 does not hold (see Addendum at the end of the paper confirming that).

In this paper we concentrate on the discrete case, that is, the case when $\mathbb{M}$ is a finite set. Typically, it is a more complicated case and so far no results have been published on that (except the ones for the product measure in [2]). It is sufficient to consider $\mathbb{M} = \{-1, +1\}$. We choose it also because it is an important case for many applications including those of statistical mechanics.

To prove $\mathbf{BBL}_2(\Lambda)$ for a general finite set $\Lambda \subset \mathbb{Z}^d$, we will need to show that it holds for measures associated to the one point sets. We begin by proving the following general fact.

LEMMA 4.1. *For any probability measure μ on $\mathbb{M} \equiv \{-1, +1\}$, there is a constant $C \in (0, \infty)$ such that we have*

$$\mathscr{U}(\mu f) \le \mu(\mathscr{U}(f)^2 + C(\nabla_0 f)^2)^{1/2} \tag{57}$$

for any function $0 \le f \le 1$, with $\nabla_0 f \equiv f - \nu_0 f$.

PROOF. Setting $\mu(\{-1\}) \equiv \alpha$ and $\mu(\{+1\}) \equiv \beta$, we will show that for any $a, b \in [0, 1]$ the following inequality holds:

$$\mathscr{U}(\alpha a + \beta b) \le \alpha(\mathscr{U}(a)^2 + \beta^2 C(b-a)^2)^{1/2} + \beta(\mathscr{U}(b)^2 + \alpha^2 C(b-a)^2)^{1/2}. \tag{58}$$

Squaring this inequality and bringing all the terms to the same side one gets

$$\begin{aligned} 0 \le{}& 2\beta\alpha(\mathscr{U}(a)^2 + \beta^2 C(b-a)^2)^{1/2} \times (\mathscr{U}(b)^2 + \alpha^2 C(b-a)^2)^{1/2} \\ &- [\mathscr{U}(\alpha a + \beta b)^2 - \alpha^2\mathscr{U}(a)^2 - \beta^2\mathscr{U}(b)^2 - 2(\alpha^2\beta^2)C(b-a)^2]. \end{aligned} \tag{59}$$

Since the sum of the first and the last term is nonnegative, we can multiply by it without changing the sign of the inequality. After simple transformations we get the following equivalent condition:

$$\begin{aligned} 0 \le{}& 4\beta^2\alpha^2(\mathscr{U}(a)^2\mathscr{U}(b)^2 + C(b-a)^2(\mathscr{U}(a)^2\alpha^2 + \mathscr{U}(b)^2\beta^2)) \\ &- A^2 + 4(\alpha^2\beta^2)AC(b-a)^2, \end{aligned} \tag{60}$$

where we have set

$$A \equiv \mathscr{U}(\alpha a + \beta b)^2 - \alpha^2\mathscr{U}(a)^2 - \beta^2\mathscr{U}(b)^2. \tag{61}$$

Using this we arrive at

$$C \ge C(a, b) \equiv \frac{A^2 - 4\beta^2\alpha^2\mathscr{U}(a)^2\mathscr{U}(b)^2}{(b-a)^2 4\beta^2\alpha^2\mathscr{U}(\alpha a + \beta b)^2}. \tag{62}$$

Using the definition of A, we note that

$$\begin{aligned} A^2 - 4\beta^2\alpha^2\mathscr{U}(a)^2\mathscr{U}(b)^2 ={}& (\mathscr{U}(\alpha a + \beta b)^2 - (\alpha\mathscr{U}(a) + \beta\mathscr{U}(b))^2) \\ &\times (\mathscr{U}(\alpha a + \beta b)^2 - (\alpha\mathscr{U}(a) - \beta\mathscr{U}(b))^2). \end{aligned} \tag{63}$$

Hence we can bound the right-hand side of (62) as follows:

$$C(a,b) \le \frac{\mathscr{U}(\alpha a+\beta b)^2-(\alpha\mathscr{U}(a)+\beta\mathscr{U}(b))^2}{4\beta^2\alpha^2(b-a)^2}. \tag{64}$$

Setting $x \equiv \alpha a+\beta b$ and $t \equiv b-a$, one can see that the function

$$F(t) \equiv \mathscr{U}(x)-(\alpha\mathscr{U}(x-\beta t)+\beta\mathscr{U}(x+\alpha t))$$

defined for $t \in (\max\{-x/\beta,(x-1)/\alpha\}, \min\{(1-x)/\beta, x/\alpha\})$, is a smooth function which equals to zero at $t=0$ together with its first derivative. Since $\mathscr{U}$ has a bounded second derivative on any closed interval $[\varepsilon, 1-\varepsilon]$, we can show that $C(a,b)$ is uniformly bounded in any such interval. On the other hand for sufficiently small $\varepsilon>0$ we have the following representation:

$$\mathscr{U}(y) = y\left(2\log\frac{1}{y}-2\log(2\pi)+O\left(\log\log\frac{1}{y}\right)\right)^{1/2} \tag{65}$$

for any $0<y<\varepsilon$ and $1-\varepsilon<y<1$, which follows from the detailed estimates on the distribution of the standard Gaussian measure (see, e.g., [7], Chapter VII, Section 6). Thus it is sufficient to show that the following function is bounded:

$$\frac{\mathscr{V}(\alpha a+\beta b)^2-(\alpha\mathscr{V}(a)+\beta\mathscr{V}(b))^2}{4\beta^2\alpha^2(b-a)^2}, \tag{66}$$

where $\mathscr{V}(y) \equiv y(2\log\frac{1}{y})^{1/2}$, for any $0<a<b<\varepsilon$. The case $0<b<a<\varepsilon$ is similar, whereas on the diagonal $a=b$, the l'Hospital rule together with the fact that $\mathscr{U}''\mathscr{U}=-1$ easily gives us an explicit bound $C(a,a) \le 1/4\beta\alpha$. [Since $\mathscr{U}$ is symmetric with respect to the midpoint of its domain, analogous estimates are true in $(1-\varepsilon,1)$.] By explicit computations for $0<a<\frac{b}{2}$, we have

$$\begin{aligned}
&\frac{\mathscr{V}(\alpha a+\beta b)^2-(\alpha\mathscr{V}(a)+\beta\mathscr{V}(b))^2}{(b-a)^2}\\
&\quad\le 8\alpha^2\frac{a^2}{b}\log\frac{1}{\alpha+\beta\frac{b}{a}}+8\beta^2\log\frac{1}{\alpha\frac{a}{b}+\beta}\\
&\qquad+16\alpha\beta\frac{a}{b}\log\frac{1}{\alpha\frac{a}{b}+\beta}+16\alpha\beta\frac{a}{b}\left(\log\frac{1}{b}\right)^{1/2}\left\{\left(\log\frac{1}{b}\right)^{1/2}-\left(\log\frac{1}{a}\right)^{1/2}\right\}.
\end{aligned} \tag{67}$$

Hence we get

$$\begin{aligned}
\frac{\mathscr{V}(\alpha a+\beta b)^2-(\alpha\mathscr{V}(a)+\beta\mathscr{V}(b))^2}{(b-a)^2} &\le 8\alpha^2\frac{a^2}{b}\log\frac{1}{\alpha}+8\beta^2\log\frac{1}{\beta}\\
&\quad+16\alpha\beta\frac{a}{b}\log\frac{1}{\beta}+16\alpha\beta\frac{a}{b}\log\frac{a}{b}.
\end{aligned} \tag{68}$$

Consider now the region $\frac{b}{2} < a < b$. Our function $F(t)$ satisfies $F(0) = 0$ and $F'(0) = 0$, and we have

$$\begin{aligned} F''(t) &= \beta\alpha(-\beta\mathcal{U}''(x - \beta t) - \alpha\mathcal{U}''(x + \alpha t)) \\ &= \beta\alpha\left(\frac{\alpha}{\mathcal{U}(x + \alpha t)} + \frac{\beta}{\mathcal{U}(x - \beta t)}\right). \end{aligned} \tag{69}$$

Hence

$$\begin{aligned} &\frac{\mathcal{U}(\alpha a + \beta b)^2 - (\alpha\mathcal{U}(a) + \beta\mathcal{U}(b))^2}{4\beta^2\alpha^2(b-a)^2} \\ &\qquad \le \frac{1}{2}\frac{F''(\vartheta t)}{4\beta^2\alpha^2} \cdot (\mathcal{U}(\alpha a + \beta b) + \alpha\mathcal{U}(a) + \beta\mathcal{U}(b)) \end{aligned} \tag{70}$$

with some $\vartheta \in [0, 1]$ Since on the interval $[0, \frac{1}{2}]$ the function $\mathcal{U}$ is increasing, we get the following estimate on the right-hand side of (70):

$$\frac{\mathcal{U}(\alpha a + \beta b)^2 - (\alpha\mathcal{U}(a) + \beta\mathcal{U}(b))^2}{4\beta^2\alpha^2(b-a)^2} \le \frac{1}{4\beta\alpha} \cdot \frac{\mathcal{U}(b)}{\mathcal{U}(a)}. \tag{71}$$

Finally using the fact that

$$\lim_{x\to 0} \frac{\mathcal{U}(x)}{x(2\log\frac{1}{x})^{1/2}} = 1 \tag{72}$$

and our present condition $\frac{b}{2} \le a \le b < \varepsilon$, with sufficiently small $\varepsilon > 0$, we get

$$\frac{\mathcal{U}(b)}{\mathcal{U}(a)} < 2. \tag{73}$$

This together with (71) implies that for $\frac{b}{2} \le a \le b < \varepsilon$, we have

$$\frac{\mathcal{U}(\alpha a + \beta b)^2 - (\alpha\mathcal{U}(a) + \beta\mathcal{U}(b))^2}{4\beta^2\alpha^2(b-a)^2} \le \frac{2}{4\beta\alpha}. \tag{74}$$

This ends the proof of Lemma 4.1. □

As a useful corollary we get the following property.

LEMMA 4.2. *There is a constant $C \in (0, \infty)$ such that, for any $\Lambda \subset \mathbb{Z}^d$ and $\omega \in \Omega$, we have*

$$\mathcal{U}(\mu_\Lambda^\omega f) \le \mu_\Lambda^\omega(\mathcal{U}(f)^2 + C(\nabla_i f)^2)^{1/2} \tag{75}$$

provided the function $0 \le f \le 1$ depends only on the one variable ω_i.

PROOF. The proof follows from Lemma 4.1 and the fact that by our assumption we have

$$\tfrac{1}{2}e^{-2\|\Phi\|} \le \mu_\Lambda^\omega\{\omega_i = \pm 1\} \le 1. \qquad \square$$

Given this lemma we are ready to prove the following main result of this section.

THEOREM 4.3. *For any $\Lambda \subset \mathbb{Z}^d$ there is a constant $C_\Lambda \in (0,\infty)$ such that for any $\omega \in \Omega$ and any function $0 \le f \le 1$ we have*

$$\mathscr{U}(\mu_\Lambda^\omega f) \le \mu_\Lambda^\omega (\mathscr{U}(f)^2 + C_\Lambda |\nabla_\Lambda f|_2^2)^{1/2}. \tag{76}$$

PROOF. Obviously by Lemma 4.2 the result is true with one point sets. Suppose the result holds for some $\Lambda \subset \mathbb{Z}^d$. We will show that this is also true for $\Lambda \cup \{i\}$ for any $i \in \mathbb{Z}^d$. To this end we observe that, setting $F \equiv \mu_\Lambda^\omega f$, by Lemma 4.2 we get

$$\mathscr{U}(\mu_{\Lambda\cup\{i\}}^\omega f) = \mathscr{U}(\mu_{\Lambda\cup\{i\}}^\omega F) \le \mu_{\Lambda\cup\{i\}}^\omega (\mathscr{U}(F)^2 + C(\nabla_i F)^2)^{1/2}. \tag{77}$$

Now by our inductive assumption we have

$$\mathscr{U}(F) = \mathscr{U}(\mu_\Lambda^\omega f) \le \mu_\Lambda^\omega (\mathscr{U}(f)^2 + C_\Lambda |\nabla_\Lambda f|_2^2)^{1/2} \tag{78}$$

On the other hand our local specification is regular, so by Theorem 3.2 we get

$$|\nabla_i F| = |\nabla_i (\mu_\Lambda^\omega f)| \le \sum_{j\in\Lambda\cup\{i\}} \alpha_{ij}^{(\Lambda)} \mu_\Lambda^\omega |\nabla_j f|. \tag{79}$$

Inserting (78) and (79) into the right-hand side of (77), using the Minkowski and Hölder inequalities, we arrive at the following inequality:

$$\mathscr{U}(\mu_{\Lambda\cup\{i\}}^\omega f) \le \mu_{\Lambda\cup\{i\}}^\omega (\mathscr{U}(f)^2 + C_{\Lambda\cup\{i\}} |\nabla_{\Lambda\cup\{i\}} f|_2^2)^{1/2} \tag{80}$$

with

$$C_{\Lambda\cup\{i\}} \le C_\Lambda + C \max_{i,j} \left(\sum_{k\in\Lambda\cup\{i\}} \alpha_{ik}^{(\Lambda)} \right) \alpha_{ij}^{(\Lambda)}. \tag{81}$$

This ends the proof of Theorem 4.3. □

This, together with the results of the previous sections, completes the proof of the following theorem.

THEOREM 4.4. *Suppose a local specification $\{\mu_\Lambda^\omega\}_{\Lambda\in\mathscr{F},\omega\in\Omega}$ corresponding to a potential Φ of finite range satisfies the strong mixing condition. Then the corresponding unique Gibbs measure μ satisfies the following $\mathbf{BBL}_2$ inequality with some coefficient $C \in (0,\infty)$,*

$$\mathscr{U}(\mu f) \le \mu(\mathscr{U}(f)^2 + C(\nabla f)_2^2)^{1/2} \tag{82}$$

for any function $0 \le f \le 1$ for which the right-hand side is finite.

REMARK 4.5. A careful reader could notice that, if the strong mixing condition is satisfied, one can utilize the idea of the proof of Theorem 4.3 to get yet another, (nice in some other way) proof of the $\mathbf{BBL}_2$ inequality for a Gibbs measure. (See also Remark 2.5.)

From this result, one gets the following corollary.

THEOREM 4.6. *Suppose a local specification* $\{\mu_\Lambda^\omega\}_{\Lambda\in\mathscr{F},\omega\in\Omega}$ *corresponding to a potential* Φ *of finite range satisfies the strong mixing condition. Then the corresponding unique Gibbs measure* μ *satisfies the logarithmic Sobolev inequality with the coefficient* $C \in (0,\infty)$,

$$\mu(f^2 \log|f|/\mu(f^2)^{1/2}) \le C\,\mu|\nabla f|_2^2 \tag{83}$$

for any function f *for which the right-hand side is finite.*

5. Summary. In this paper we have introduced a family of functional inequalities which have a form suitable for studying the isoperimetry in the infinite-dimensional setting. We have shown that such an inequality holds true for a Gibbs measure provided it is satisfied for a related finite-dimensional conditional expectation together with a strong mixing condition. In the special case $p = 2$ we have verified the required conditions and shown that $\mathbf{BBL}_2$ inequality is true for a large class of nontrivial measures. Besides other things, this helps to recover some well-known results concerning the logarithic Sobolev inequality [15] (possibly with better estimates on the relevant coefficients).

It is natural to suppose that the $\mathbf{BBL}_p$ inequalities with $p \in [1,2)$ will help us to understand better the sub-Gaussian measures which are rather poorly studied. To motivate that let us consider the boundary case $p = 1$. We begin from a double-sided Poisson measure $\nu(dx)$ on $\mathbb{M} = \mathbb{R}$ with density $\varphi_\nu(x) \equiv \frac{1}{2}e^{-|x|}$ and distribution F_ν. We use it to define a local specification $E_\Lambda^\omega \equiv E_{\Lambda,\Phi}^\omega$, $\Lambda \in \mathscr{F}$, $\omega \in \Omega \equiv \mathbb{R}^{\mathbb{Z}^d}$, corresponding to a smooth bounded potential of finite range. Let μ_Φ be a Gibbs measure corresponding to this potential. We have the following result.

THEOREM 5.1. *If the potential* Φ *is sufficiently small, then the unique Gibbs measure* μ_Φ *satisfies* $\mathbf{BBL}_1$ *with the isoperimetric function* $\mathscr{U}_1$ *of the double-sided Poisson measure.*

The proof follows from the fact that the double sided Poisson measure satisfies $\mathbf{BBL}_1$ with the function $\mathscr{U}_1(x) \equiv \varphi_\nu(F_\nu^{-1}(x)) \equiv \min(x, 1-x)$, $x \in [0,1]$, (see [3]) together with simple perturbation arguments which allow showing the desired inequality for a single site conditional expectaion E_i, $i \in \mathbb{Z}^d$. The strong mixing condition is easy to see in the present context.

It is interesting to conjecture that a similar result remains true when the Poisson measure is replaced by a measure $\nu_q(dx) \equiv \frac{1}{Z}e^{-|x|^q}dx$ with $q > 1$ and a corresponding isoperimetric function $\mathscr{U}_q$.

6. Addendum. Since this work was done, some interesting progress in this domain has appeared on which we would like to report briefly for the benefit of the reader.

In particular, we mention a very interesting preprint by Pierre Fougéres [8] in which he shows (by some semigroup technique using the finite curvature assumption) that in the case of diffusions the logarithmic Sobolev inequalities are actually equivalent to $\mathbf{BBL}_2$. (Recall that implication $\mathbf{BBL}_2 \Rightarrow \mathbf{LS}$ was proved in [5].) Since his method utilizes some optimization procedure, the coefficient at $\mathbf{BBL}_2$ does not need to coincide with the log-Sobolev coefficient, (contrary to the converse implication of [5]). On the other hand it could be used to show $\mathbf{BBL}_2$ in a finite dimension and, by our inductive procedure extended to the corresponding Gibbs measure, which in order (via the Bakry–Ledoux route) would yield an improved estimation for the logarithmic Sobolev constant.

In the discrete case in [8], the author obtains weaker inequalities than $\mathbf{BBL}_2$. In that case the advantage of our method is also that it applies to Markov chains and their stationary measures (which even in the case of a quite simple transition matrix may not be a Gibbs measure associated to any reasonable interaction potential).

Finally we recall that concentration of measure estimates and isoperimetry were studied extensively in the past in the case of product measures; see, for example, [17–19] and references therein (see also [2], [3] and [4] for further references), with application to a number of problems including, for example, the problem of bin packing, the traveling salesman problem, spin glasses, longest common/increasing sequances, etc. More recently it has been shown [4] (also in the case of product measures) that one can use $\mathbf{BBL}_2$ to obtain optimal constants in some concentration inequalities of Talagrand. Our work provides a possibility of the extension of such results to the case of nontrivial measures.

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