

STOCHASTIC CALCULUS WITH RESPECT TO GAUSSIAN PROCESSES

BY ELISA ALÒS,¹ OLIVIER MAZET² AND DAVID NUALART¹

*Universitat Autònoma de Barcelona, Universitat de Barcelona and
Universitat de Barcelona*

In this paper we develop a stochastic calculus with respect to a Gaussian process of the form $B_t = \int_0^t K(t, s) dW_s$, where W is a Wiener process and $K(t, s)$ is a square integrable kernel, using the techniques of the stochastic calculus of variations. We deduce change-of-variable formulas for the indefinite integrals and we study the approximation by Riemann sums. The particular case of the fractional Brownian motion is discussed.

1. Introduction. The stochastic integral with respect to the Brownian motion coincides with the adjoint of the derivative operator on the Wiener space. This property, established by Gaveau and Trauber in [10], has shed new light on the classical Itô calculus, leading to significant advances in this theory. For instance, the Clark–Ocone formula provides an explicit expression for Itô’s integral representation theorem in terms of the derivative operator. On the other hand, the adjoint of the derivative operator can be used as an anticipating stochastic integral.

We recall that the stochastic calculus of variations or Malliavin calculus is valid for an arbitrary Gaussian process (see [14] and [17]). Suppose, in particular, that $B = \{B_t, t \in [0, T]\}$ is a centered continuous Gaussian process of the form

$$(1) \quad B_t = \int_0^t K(t, s) dW_s,$$

where $W = \{W_t, t \in [0, T]\}$ is a Brownian motion and $K(t, s)$ is a square integrable kernel. The purpose of this paper is to use the stochastic calculus of variations in order to develop a stochastic calculus with respect to B , as in the case of the Brownian motion. The divergence operator δ^B with respect to B will be used as a stochastic integral and, under some regularity assumptions, this integral turns out to be the limit of Riemann sums defined in terms of the Wick product.

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We will consider two different types of kernels:

- (i) *Singular case:* $K(\cdot, s)$ has bounded variation on any interval $(u, T]$, $u > s$, but $\int_s^T |K|(dt, s)$ may be infinite.
- (ii) *Regular case:* The kernel satisfies $\int_0^T |K|((s, T], s)^2 ds < \infty$.

In both cases we establish an Itô formula for the process B itself and for stochastic integral processes of the form $X_t = \int_0^t u_s \delta B_s$, where the process u is adapted. In the case $X_t = B_t$ we obtain

$$(2) \quad F(B_t) = F(0) + \int_0^t F'(B_s) \delta B_s + \frac{1}{2} \int_0^t F''(B_s) dR_s,$$

where $R_s = E(B_s^2)$. In the singular case this formula requires the additional condition

$$(3) \quad \int_0^T \left(\int_s^T \|B_t - B_s\|_{L^2(\Omega)} |K|(dt, s) \right)^2 ds < \infty$$

for $F'(B_s)$ to belong to the domain of the divergence operator δ^B .

We note that the stochastic integral $\int_0^T u_s \delta B_s$ has zero mean. On the other hand, in the regular case and assuming $K(s^+, s) = 0$, we introduce another type of integral, which we call the Stratonovich integral and denote by $\int_0^T u_s \times dB_s$, that can be approximated by ordinary Riemann sums. This Stratonovich integral can be decomposed as the sum of the divergence $\int_0^T u_s \delta B_s$ plus a trace term. We also deduce change-of-variable formulas for the Stratonovich integral, which are analogous to those of ordinary calculus. Actually this integral is also of forward type because we assume $K(s^+, s) = 0$, which implies that B_t has no Brownian component and it is smoother than the Brownian motion.

An important example of processes of this form is the fractional Brownian motion of the Hurst parameter $H \in (0, 1)$, which has the covariance function:

$$E(B_t^H B_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}).$$

The process B^H can be represented as

$$B_t = \int_0^t K^H(t, s) dW_s,$$

where the kernel K^H is singular if $H < \frac{1}{2}$ and regular if $H > \frac{1}{2}$. Notice that condition (3) holds only in the case $H > \frac{1}{4}$.

The application of the stochastic calculus of variations to construct a stochastic calculus with respect to the fractional Brownian motion has been previously developed in [6] and [7]. In these papers Itô's formulas are established in the case $H > \frac{1}{2}$ for two types of integrals, one of them being equal to the divergence operator, and for the corresponding Stratonovich versions. Carmona and Coutin [3] have extended this approach to a general class of Gaussian processes. They consider a Stratonovich-type stochastic integral $\int_0^T u_s dB_s$, which can be decomposed into the sum of a divergence $\delta^B(u)$ term

plus a trace term involving the stochastic gradient of u . They prove the approximation of this integral by Riemann sums and they establish the Itô formula (2) for B_t in the regular case. In the case of the fractional Brownian motion with parameter $H < \frac{1}{2}$ the class of integrable processes (in the Stratonovich sense) has been characterized in [2] using classical fractional calculus.

Duncan, Hu and Pasik-Duncan [8] have introduced the stochastic integral $\int_0^T u_s \delta B_s^H$ as the limit of Riemann sums defined by means of the Wick product, and proved the Itô formula in the case $H > \frac{1}{2}$. Some applications of this formula are discussed.

An approach based on the pathwise Riemann–Stieltjes integration has been used by Lin [13] and Dai and Heyde [5] for the fractional Brownian motion with $H > \frac{1}{2}$. The integrator must have finite p -variation with $1/p + H > 1$. An extension of the Riemann–Stieltjes integral has been defined by Zähle [22] by means of integration by parts formulas and fractional derivatives. This pathwise integral coincides with the Stratonovich integral but the assumptions for its existence are different from those obtained using the stochastic calculus of variations.

The paper is organized as follows. Section 2 contains some preliminaries on the stochastic calculus of variations with respect to a Gaussian process of the form (1). The divergences associated with the processes B and W are related by the formula $\delta^B(u) = \delta^W(K^*u)$, where K^* is the adjoint of the operator K . Section 3 contains the proof of formula (2) in the singular case, the regular case being treated in Section 4. In Section 5 we show the Itô formula for an indefinite integral in the singular case and Section 6 is devoted to handle the regular case. In Section 7 we study the approximation of stochastic integrals (of divergence and Stratonovich type) by Riemann sums and in Section 8 we discuss the particular case of the fractional Brownian motion and related processes.

Throughout the paper C, C', \dots will denote constants that may be different from one formula to another one.

2. Preliminaries on the stochastic calculus of variations. Let $B = \{B_t, t \in [0, T]\}$ be a zero mean continuous Gaussian process with covariance function $E(B_t B_s) = R(t, s)$ such that $B_0 = 0$. We suppose that B is defined in a complete probability space (Ω, \mathcal{F}, P) and \mathcal{F} is generated by B . Let H_1 be the first Wiener chaos, that is, the closed subspace of $L^2(\Omega)$ generated by B . The reproducing kernel Hilbert space (RKHS) \mathcal{H} is defined as the closure of the linear span of the indicator functions $\{1_{[0, t]}, t \in [0, T]\}$ with respect to the scalar product $\langle 1_{[0, t]}, 1_{[0, s]} \rangle_{\mathcal{H}} = R(t, s)$. The mapping $1_{[0, t]} \mapsto B_t$ provides an isometry between \mathcal{H} and H_1 . We denote by $B(\varphi)$ the image in H_1 of an element $\varphi \in \mathcal{H}$.

We briefly recall some basic elements of the stochastic calculus of variations with respect to B . For a more complete presentation, see [17] and [18]. Let \mathcal{S} be the set of smooth and cylindrical random variables of the form

$$(4) \quad F = f(B(\varphi_1), \dots, B(\varphi_n)),$$

where $n \geq 1$, $f \in C_b^\infty(\mathbb{R}^n)$ (f and all its derivatives are bounded) and $\varphi_1, \dots, \varphi_n \in \mathcal{H}$. Given a random variable F of the form (4), we define its derivative as the \mathcal{H} -valued random variable given by

$$D^B F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(B(\varphi_1), \dots, B(\varphi_n)) \varphi_j.$$

The derivative operator D^B is a closable unbounded operator from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$ for any $p \geq 1$. In a similar way, the iterated derivative operator $D^{B,k}$ maps $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H}^{\otimes k})$. For any positive integer k and any real $p \geq 1$, we denote by $\mathbb{D}_B^{k,p}$ the closure of \mathcal{S} with respect to the norm defined by

$$\|F\|_{B,k,p}^p = \|F\|_{L^p(\Omega)}^p + \sum_{j=1}^k \|D^{B,j} F\|_{L^p(\Omega; \mathcal{H}^{\otimes j})}^p.$$

Henceforth the norm of $L^p(\Omega)$ will be denoted by $\|\cdot\|_p$. We denote by δ^B the adjoint of the derivative operator D^B . The domain of δ^B (denoted by $\text{Dom } \delta^B$) in L^2 is the set of elements $u \in L^2(\Omega; \mathcal{H})$ such that there exists a constant c verifying

$$\left| E\langle D^B F, u \rangle_{\mathcal{H}} \right| \leq c \|F\|_2$$

for all $F \in \mathcal{S}$. If $u \in \text{Dom } \delta^B$, $\delta^B(u)$ is the element in $L^2(\Omega)$ defined by the duality relationship

$$E(\delta^B(u)F) = E\langle D^B F, u \rangle_{\mathcal{H}}, \quad F \in \mathbb{D}_B^{1,2}.$$

If V is a separable Hilbert space, we can define in a similar way the spaces $\mathbb{D}_B^{k,p}(V)$ of V -valued random variables. We recall that the space $\mathbb{D}_B^{1,2}(\mathcal{H})$ of \mathcal{H} -valued random variables is included in the domain of δ^B , and for any element u in $\mathbb{D}_B^{1,2}(\mathcal{H})$ we have

$$E(\delta^B(u)^2) \leq E\|u\|_{\mathcal{H}}^2 + E\|D^B u\|_{\mathcal{H} \otimes \mathcal{H}}^2.$$

Furthermore, Meyer inequalities imply that for all $p > 1$ we have

$$(5) \quad \|\delta^B(u)\|_p \leq c_p \|u\|_{\mathbb{D}_B^{1,p}(\mathcal{H})}.$$

If u is a simple \mathcal{H} -valued random variable of the form

$$u = \sum_{j=1}^n F_j \varphi_j,$$

where $F_j \in \mathbb{D}_B^{1,2}$ and $\varphi_j \in \mathcal{H}$, then u belongs to the domain of δ^B and

$$\delta^B(u) = \sum_{j=1}^n \left(F_j B(\varphi_j) - \langle D^B F_j, \varphi_j \rangle_{\mathcal{H}} \right).$$

In the particular case where B is the Wiener process, that is, $R(t, s) = t \wedge s$, the space \mathcal{H} is $L^2([0, T])$ and the divergence operator δ^B is an extension of

the Itô integral in the sense that the set $L_a^2([0, T] \times \Omega)$ of square integrable and adapted processes is included in $\text{Dom } \delta^B$ and the operator δ^B restricted to $L_a^2([0, T] \times \Omega)$ coincides with the Itô stochastic integral (see [19]). This extension coincides with the stochastic integral introduced by Skorohod in [21] and it is also called the Skorohod integral.

In the general case, the divergence operator δ^B can also be interpreted as a generalized stochastic integral. In fact, notice that, for all $\varphi \in \mathcal{H}$, $B(\varphi) = \delta^B(\varphi)$, and, in particular,

$$\delta^B \left(\sum_{i=1}^n a_i \mathbf{1}_{[t_i, t_{i+1}]} \right) = \sum_{i=1}^n a_i (B(t_{i+1}) - B(t_i)).$$

Our purpose is to study the properties of the divergence operator as a stochastic integral. To do this, we need a representation of the elements of the RKHS \mathcal{H} as functions on $[0, T]$.

Suppose that the covariance $R(t, s)$ of the continuous Gaussian process B can be expressed as

$$(6) \quad R(t, s) = \int_0^{t \wedge s} K(t, r) K(s, r) dr,$$

where $K(t, s)$, $0 < s < t < T$, is a kernel satisfying

$$(7) \quad \|K\| = \sup_{t \in [0, T]} \int_0^t K(t, s)^2 ds < \infty.$$

Enlarging, if necessary, our probability space, we can find a Wiener process $W = \{W_t, t \in [0, T]\}$ such that

$$B_t = \int_0^t K(t, s) \delta W_s,$$

where we denote by δW_s the Itô differential. Notice that the RKHS \mathcal{H} is isometric to the closure in $L^2([0, T])$ of the linear span of the functions $\{K(t, \cdot) \mathbf{1}_{[0, t]}, t \in [0, T]\}$. Indeed,

$$R(t, s) = \langle \mathbf{1}_{[0, t]}, \mathbf{1}_{[0, s]} \rangle_{\mathcal{H}} = \langle K(t, \cdot) \mathbf{1}_{[0, t]}, K(s, \cdot) \mathbf{1}_{[0, s]} \rangle_{L^2([0, T])}.$$

On the other hand, the kernel K defines an operator in $L^2([0, T])$ given by

$$(Kh)(t) = \int_0^t K(t, s) h(s) ds,$$

and the function Kh is continuous and vanishes at 0 because

$$|(Kh)(t) - (Kh)(s)| \leq \|B_t - B_s\|_2 \|h\|_{L^2([0, T])}.$$

We denote by \mathcal{E} the set of step functions on $[0, T]$. Consider the linear operator K^* from \mathcal{E} to $L^2([0, T])$ defined by

$$(K^* \varphi)(s) = \varphi(s) K(T, s) + \int_s^T [\varphi(t) - \varphi(s)] K(dt, s).$$

If

$$(8) \quad \varphi = \sum_{i=1}^n a_i \mathbf{1}_{(s_i, s_{i+1}]},$$

where $a_i \in \mathbb{R}$, and $0 = s_1 < s_2 < \dots < s_{n+1} = T$, then

$$\begin{aligned} (K^* \varphi)(s) &= \sum_{i=1}^n a_i \mathbf{1}_{(s_i, s_{i+1}]}(s) K(T, s) \\ &\quad + \sum_{i=1}^{n-1} \mathbf{1}_{(s_i, s_{i+1}]}(s) \sum_{j=i+1}^n (a_j - a_i) (K(s_{j+1}, s) - K(s_j, s)). \end{aligned}$$

Notice that this operator can also be written as

$$(9) \quad (K^* \varphi)(s) = \sum_{i=1}^{n-1} \mathbf{1}_{(s_i, s_{i+1}]}(s) \left[a_i K(s_{i+1}, s) + \int_{s_{i+1}}^T \varphi(t) K(dt, s) \right].$$

However, this expression is not convenient because when we extend the operator K^* to continuous functions the first summand of (9) becomes $\varphi(s)K(s, s)$ and $K(s, s)$ may not be well defined.

The operator K^* is the adjoint of K in the following sense:

LEMMA 1. For any function $\varphi \in \mathcal{E}$ and $h \in L^2([0, T])$, we have

$$\int_0^T (K^* \varphi)(t) h(t) dt = \int_0^T \varphi(t) (Kh)(dt).$$

PROOF. Suppose that φ is a function of the form (8). We have, using the definition of Kh ,

$$\begin{aligned} \int_0^T \varphi(t) (Kh)(dt) &= \sum_{i=1}^n a_i [(Kh)(s_{i+1}) - (Kh)(s_i)] \\ &= a_n (Kh)(T) - \sum_{i=1}^{n-1} (a_{i+1} - a_i) (Kh)(s_{i+1}) \\ &= \int_0^T a_n K(T, s) h(s) ds \\ &\quad - \int_0^T \left(\sum_{i=1}^{n-1} (a_{i+1} - a_i) \mathbf{1}_{(0, s_{i+1}]}(s) K(s_{i+1}, s) \right) h(s) ds. \end{aligned}$$

Hence,

$$\begin{aligned}
\int_0^T \varphi(t)(Kh)(dt) &= \int_0^T \left[\sum_{i=1}^n a_i 1_{(s_i, s_{i+1}]}(s) K(T, s) \right. \\
&\quad \left. + \sum_{i=1}^{n-1} (a_{i+1} - a_i) 1_{(0, s_{i+1}]}(s) \times (K(T, s) - K(s_{i+1}, s)) \right] \\
&\quad \times h(s) ds \\
&= \int_0^T \varphi(s) K(T, s) h(s) ds \\
&\quad + \int_0^T \left(\sum_{i=1}^{n-1} 1_{(s_i, s_{i+1}]}(s) \sum_{j=i+1}^n (a_j - a_i) (K(s_{j+1}, s) - K(s_j, s)) \right) \\
&\quad \times h(s) ds \\
&= \int_0^T (K^* \varphi)(s) h(s) ds,
\end{aligned}$$

which completes the proof. \square

Replacing $h(s) ds$ by δW_s , the proof of the preceding lemma also shows that for any $\varphi \in \mathcal{E}$ the element $B(\varphi)$ of the first chaos can be written as

$$B(\varphi) = \int_0^T (K^* \varphi)(t) \delta W_t.$$

Hence, the RKHS \mathcal{H} can be represented as the closure of \mathcal{E} with respect to the norm $\|\varphi\|_{\mathcal{H}} = \|K^* \varphi\|_{L^2([0, T])}$. The operator K^* is an isometry between \mathcal{H} and a closed subspace of $L^2([0, T])$, that is,

$$(10) \quad \mathcal{H} = (K^*)^{-1}(L^2([0, T])).$$

Henceforth we will denote by $D, \delta, \mathbb{D}^{k, p}$ the operators and spaces associated with the Wiener process W . The equality (10) implies

$$(11) \quad \mathbb{D}_B^{1,2}(\mathcal{H}) = (K^*)^{-1}(\mathbb{L}^{1,2}),$$

where $\mathbb{L}^{1,2} = \mathbb{D}^{1,2}(L^2([0, T]))$. On the other hand, we have the following identity for any smooth random variable F and any \mathcal{H} -valued square integrable random variable u :

$$E\langle u, D^B F \rangle_{\mathcal{H}} = E\langle K^* u, DF \rangle_{L^2([0, T])}.$$

In fact, if $F = f(B_t)$, then

$$\begin{aligned}
E\langle u, D^B F \rangle_{\mathcal{H}} &= E\langle u, f'(B_t) 1_{[0, t]} \rangle_{\mathcal{H}} \\
&= E\langle K^* u, f'(B_t) K^* 1_{[0, t]} \rangle_{L^2([0, T])} \\
&= E\langle K^* u, f'(B_t) K(t, \cdot) \rangle_{L^2([0, T])} \\
&= E\langle K^* u, DF \rangle_{L^2([0, T])}.
\end{aligned}$$

As a consequence, we obtain

$$(12) \quad \text{Dom } \delta^B = (K^*)^{-1}(\text{Dom } \delta),$$

and $\delta^B(u) = \delta(K^*u)$ for any \mathcal{H} -valued random variable u in $\text{Dom } \delta^B$. We will make use of the notation $\delta(v) = \int_0^T v_s \delta W_s$ for any $v \in \text{Dom } \delta$. Hence, if $u \in \text{Dom } \delta^B$, then

$$\delta^B(u) = \int_0^T (K^*u)_s \delta W_s.$$

From (11) and (12) we deduce that $(K^*)^{-1}(\mathbb{L}^{1,2})$ is included in the domain of δ^B .

3. Gaussian processes with a singular kernel: Stochastic integral and Itô's formula. Suppose now that $K(t, s)$ satisfies the following condition:

(K1) $K(\cdot, s)$ has bounded variation on any interval $(u, T]$, $u > s$.

Consider the following seminorm on \mathcal{E} :

$$\|\varphi\|_K^2 = \int_0^T \varphi(s)^2 K(T, s)^2 ds + \int_0^T \left(\int_s^T |\varphi(t) - \varphi(s)| |K|(dt, s) \right)^2 ds.$$

The completion of \mathcal{E} with respect to this seminorm will be denoted by \mathcal{H}_K . The space \mathcal{H}_K is the class of functions φ on $[0, T]$ such that $\|\varphi\|_K^2 < \infty$ and it is included in $L^2([0, T]; K(T, s)^2 ds)$. Moreover, \mathcal{H}_K is continuously embedded in \mathcal{H} because $\|\varphi\|_{\mathcal{H}} \leq \sqrt{2}\|\varphi\|_K$.

Let $u = \{u_t, t \in [0, T]\}$ be a stochastic process in $\mathbb{D}^{1,2}(\mathcal{H}_K)$. That is, u verifies the following conditions:

$$(13) \quad E\|u\|_K^2 = E \int_0^T u_s^2 K(T, s)^2 ds + E \int_0^T \left(\int_s^T |u_t - u_s| |K|(dt, s) \right)^2 ds < \infty$$

and

$$(14) \quad \begin{aligned} E \int_0^T \|D_r u\|_K^2 dr &= E \int_0^T \int_0^T (D_r u_s)^2 K(T, s)^2 ds dr \\ &+ E \int_0^T \int_0^T \left(\int_s^T |D_r u_t - D_r u_s| |K|(dt, s) \right)^2 ds dr \\ &< \infty. \end{aligned}$$

These conditions imply that K^*u belongs to $\mathbb{L}^{1,2}$ and, as a consequence, u belongs to the domain of δ^B and $\delta^B(u) = \int_0^T (K^*u)_s \delta W_s$. For a process u in $\mathbb{D}^{1,2}(\mathcal{H}_K)$ we will make use of the notation $\delta^B(u) = \int_0^T u_s \delta B_s$, and, therefore, we can write

$$\int_0^T u_s \delta B_s = \int_0^T (K^*u)_s \delta W_s.$$

Notice that if u satisfies conditions (13) and (14), then $u1_{[0,t]}$ also satisfies these conditions for any $t \in [0, T]$. Moreover for $s \leq t$ we have

$$\begin{aligned} K^*(u1_{[0,t]})_s &= u_s K(T, s) + \int_{(s,t]} (u_r - u_s) K(dr, s) - \int_{(t,T]} u_s K(dr, s) \\ &= u_s K(t, s) + \int_{(s,t]} (u_r - u_s) K(dr, s), \end{aligned}$$

and for $s > t$ we clearly have $K^*(u1_{[0,t]})_s = 0$. We will denote $K^*(u1_{[0,t]})_s$ by $(K_t^*u)_s$, where K_t^* is the adjoint of the operator K in the interval $[0, t]$.

So, for a process u in $\mathbb{D}^{1,2}(\mathcal{H}_K)$ we can introduce the indefinite integral $X_t = \int_0^t u_s \delta B_s$, which will be given by

$$\begin{aligned} (15) \quad \int_0^t u_s \delta B_s &= \int_0^t u_s K(t, s) \delta W_s + \int_0^t \left(\int_s^t (u_r - u_s) K(dr, s) \right) \delta W_s \\ &= \int_0^t (K_t^*u)_s \delta W_s. \end{aligned}$$

In order to show an Itô formula for the Gaussian process B_t we will introduce the following additional conditions:

- (K2) $\int_0^T (\int_s^T \|B_t - B_s\|_2 |K|(dt, s))^2 ds < \infty$.
- (K3) The functions $R(s, s)$ and $\int_{t \wedge s}^s K(s, r) dr$ have bounded variation in $s \in [0, T]$ for any $t \in [0, T]$.

Let F be a twice continuously differentiable function satisfying the growth condition

$$(16) \quad \max\{|F(x)|, |F'(x)|, |F''(x)|\} \leq c \exp(\lambda|x|^2),$$

where c and λ are positive constants such that $\lambda < \frac{1}{4}(\sup_{0 \leq t \leq T} R_t)^{-1}$. This condition implies

$$E \left(\sup_{0 \leq t \leq T} |F(B_t)|^p \right) \leq c^p E e^{p\lambda \sup_t |B_t|^2} < \infty,$$

for all $p < \frac{1}{2\lambda}(\sup_{0 \leq t \leq T} R_t)^{-1}$, and the same property holds for F' and F'' . As a consequence of condition (K2), for any function F of this type, the process $F'(B_t)$ belongs to the space $L^2(\Omega; \mathcal{H}_K)$. Indeed, if $2 < p < \frac{1}{2\lambda}(\sup_{0 \leq t \leq T} R_t)^{-1}$, applying Hölder's inequality we obtain

$$\begin{aligned} E \|F'(B_t)\|_K^2 &= \int_0^T E F'(B_s)^2 K(T, s)^2 ds \\ &\quad + E \int_0^T \left(\int_s^T |F'(B_t) - F'(B_s)| |K|(dt, s) \right)^2 ds \\ &\leq E \left(\sup_{0 \leq t \leq T} |F'(B_t)|^2 \right) R(T, T) + c \left(E \left(\sup_{0 \leq t \leq T} |F''(B_t)|^p \right) \right)^{\frac{1}{p}} \\ &\quad \times \int_0^T \left(\int_s^T \|B_t - B_s\|_2 |K|(dt, s) \right)^2 ds \\ &< \infty. \end{aligned}$$

THEOREM 1. *Let F be a function of class $C^2(\mathbb{R})$ satisfying (16). Suppose that $B = \{B_t, t \in [0, T]\}$ is a zero mean continuous Gaussian process whose covariance function $R(t, s)$ is of the form (6), with a kernel $K(t, s)$ satisfying conditions (K1), (K2) and (K3). Then for each $t \in [0, T]$ the process $F'(B_s)1_{[0, t]}(s)$ belongs to $\text{Dom } \delta^B$ and the following formula holds:*

$$F(B_t) = F(0) + \int_0^t F'(B_s) \delta B_s + \frac{1}{2} \int_0^t F''(B_s) dR_s,$$

where $R_s = R(s, s)$.

PROOF. By the preceding remark $F'(B_t)$ belongs to $L^2(\Omega; \mathcal{H}_K)$. Then it suffices to show that

$$\begin{aligned} & E(GF(B_t)) - E(GF(0)) - \frac{1}{2} \int_0^t E(GF''(B_s)) dR_s \\ (17) \quad & = E \int_0^t D_s GK_t^*[F'(B_s)] ds \end{aligned}$$

for any random variable G in a total subset of $L^2(\Omega)$. In fact, the random variable $F(B_t) - F(0) - \frac{1}{2} \int_0^t F''(B_s) dR_s$ being square integrable, (17) implies that the process $F'(B_s)1_{[0, t]}(s)$ belongs to the domain of δ^B . Suppose that G is a random variable of the form $G = I_n(h^{\otimes n})$, where I_n denotes the multiple stochastic integral of order n with respect to W and h is a step function in $[0, T]$. The set of all these random variables forms a total subset of $L^2(\Omega)$. From hypothesis (K3) we deduce that KDG is of bounded variation. Thus we can apply Lemma 1 to the right-hand side of (17) and this equality is equivalent to

$$\begin{aligned} & E(GF(B_t)) - E(GF(0)) - \frac{1}{2} \int_0^t E(GF''(B_s)) dR_s \\ (18) \quad & = E \int_0^t F'(B_s)(KDG)(ds). \end{aligned}$$

In order to show (18) we will replace F by

$$F_k(x) = k \int_{-1}^1 F(x - y)\varepsilon(ky) dy,$$

where ε is a nonnegative smooth function supported by $[-1, 1]$ such that $\int_{-1}^1 \varepsilon(y) dy = 1$. The functions F_k are infinitely differentiable and their derivatives satisfy the growth condition (16) with some constants c_k and λ . Suppose first that G is a constant, that is, $n = 0$. Then the right-hand side of equality (18) vanishes. On the other hand, we can write

$$E(GF(B_t)) = G \int_{\mathbb{R}} F_k(y)p(R_t, y) dy,$$

where $p(\sigma, y) = (2\pi\sigma)^{-1/2} \exp(-y^2/2\sigma)$. We know that

$$(19) \quad \frac{\partial p}{\partial \sigma} = \frac{1}{2} \frac{\partial^2 p}{\partial y^2}.$$

As a consequence, integrating by parts, we obtain

$$\begin{aligned} E(GF_k(B_t)) - GF(0) &= \frac{1}{2}G \int_0^t dR_s \left(\int_{\mathbb{R}} F_k(y) \frac{\partial^2 p}{\partial y^2}(R_s, y) dy \right) \\ &= \frac{1}{2}G \int_0^t dR_s \left(\int_{\mathbb{R}} F_k''(y) p(R_s, y) dy \right) \\ &= \frac{1}{2}G \int_0^t dR_s E(F_k''(B_s)), \end{aligned}$$

which completes the proof of (18), when G is constant.

Suppose now that $n \geq 1$. In that case $E(G) = 0$ and we can write

$$\begin{aligned} E(GF_k(B_t)) &= E(F_k^{(n)}(B_t))(Kh)_t^n, \\ E(GF_k''(B_s)) &= E(F_k^{(n+2)}(B_s))(Kh)_s^n \end{aligned}$$

and

$$\begin{aligned} E \int_0^t F_k'(B_s)(KDG)(ds) &= n \int_0^t (Kh)(ds) E(F_k'(B_s) I_{n-1}(h^{\otimes(n-1)})) \\ &= n \int_0^t (Kh)(ds) E(F_k^{(n)}(B_s))(Kh)_s^{n-1} \\ &= \int_0^t E(F_k^{(n)}(B_s)) d(Kh)_s^n. \end{aligned}$$

Hence, it remains to show that

$$(20) \quad \begin{aligned} E(F_k^{(n)}(B_t))(Kh)_t^n &= \frac{1}{2} \int_0^t E(F_k^{(n+2)}(B_s))(Kh)_s^n dR_s \\ &\quad + \int_0^t E(F_k^{(n)}(B_s)) d(Kh)_s^n. \end{aligned}$$

For any $y \neq 0$ we can write, using again (19),

$$(Kh)_t^n p(R_t, y) = \frac{1}{2} \int_0^t (Kh)_s^n \frac{\partial^2 p}{\partial y^2}(R_s, y) dR_s + \int_0^t p(R_s, y) d(Kh)_s^n.$$

As a consequence, applying Fubini's theorem and integration by parts, we obtain

$$\begin{aligned} E(F_k^{(n)}(B_t))(Kh)_t^n &= \int_{\mathbb{R}} F_k^{(n)}(y)(Kh)_t^n p(R_t, y) dy \\ &= \frac{1}{2} \int_0^t dR_s \left(\int_{\mathbb{R}} F_k^{(n+2)}(y)(Kh)_s^n p(R_s, y) dy \right) \\ &\quad + \int_0^t d(Kh)_s^n \left(\int_{\mathbb{R}} F_k^{(n)}(y) p(R_s, y) dy \right), \end{aligned}$$

which completes the proof of (20) for the function F_k . Finally, it suffices to let k tend to ∞ . \square

4. Gaussian processes with a regular kernel: Stochastic integrals and Itô's formula. In this section we will impose the following condition on the kernel $K(t, s)$, which is stronger than (K1):

(K4) For all $s \in [0, T]$, $K(\cdot, s)$ has bounded variation on the interval $(s, T]$, and

$$\int_0^T |K|((s, T], s)^2 ds < \infty.$$

Notice that condition (K4) implies that $K(s^+, s) = K(T, s) - K((s, T], s)$ is square integrable in $[0, T]$. Moreover, conditions (K2) and (K3) hold. In fact, for any partition $0 < s_1 < \dots < s_{n+1} = T$ we obtain

$$\begin{aligned} \sum_{i=1}^n |R_{s_{i+1}} - R_{s_i}| &= \sum_{i=1}^n \left| \int_0^{s_{i+1}} K(s_{i+1}, r)^2 dr - \int_0^{s_i} K(s_i, r)^2 dr \right| \\ &= \sum_{i=1}^n \int_{s_i}^{s_{i+1}} K(s_{i+1}, r)^2 dr \\ &\quad + \sum_{i=1}^n \left| \int_0^{s_i} [K(s_{i+1}, r)^2 - K(s_i, r)^2] dr \right| \\ &= A_1 + A_2. \end{aligned}$$

The functions $|K(s, r)1_{[0, s]}(r)|$ are bounded by the square integrable function $k(r) = |K(T, r)| + |K|((r, T], r)$. Hence, for the term A_1 we have $A_1 \leq \int_0^T k(r)^2 dr$. On the other hand, for the term A_2 we can write

$$\begin{aligned} A_2 &\leq \sum_{i=1}^n \int_0^{s_i} |K|((s_i, s_{i+1}], r)|K(s_{i+1}, r) + K(s_i, r)| dr \\ &\leq 2 \int_0^T k(r)|K|((r, T], r) dr < \infty. \end{aligned}$$

By the same arguments the second part of hypothesis (K3) also holds.

In this case the operator K^* can be expressed as

$$(K^* \varphi)(s) = \varphi(s)K(s^+, s) + \int_s^T \varphi(t)K(dt, s),$$

where $\varphi \in \mathcal{E}$. We define the seminorm

$$\|\varphi\|_{K_r}^2 = \int_0^T \varphi(s)^2 K(s^+, s)^2 ds + \int_0^T \left(\int_s^T |\varphi(t)| |K|(dt, s) \right)^2 ds.$$

The completion of \mathcal{E} with respect to this seminorm will be denoted by \mathcal{H}_{K_r} , which is continuously embedded in \mathcal{H} because $\|\varphi\|_{\mathcal{H}} \leq \sqrt{2}\|\varphi\|_{K_r}$. The space $\mathbb{D}^{1,2}(\mathcal{H}_{K_r})$ is included in the domain of δ^B and for any u in this space we have $\delta^B(u) = \int_0^T (K^*u)_s \delta W_s$. The adjoint K_t^* of K in $[0, t]$ will be given by

$$(K_t^*u)_s = u_s K(s^+, s) + \int_{(s, t]} u_r K(dr, s),$$

and for a process u in $\mathbb{D}^{1,2}(\mathcal{H}_{K_r})$ the indefinite integral of u is

$$(21) \quad \begin{aligned} \int_0^t u_s \delta B_s &= \int_0^t (K_t^* u)_s \delta W_s \\ &= \int_0^t u_s K(s^+, s) \delta W_s + \int_0^t \left(\int_s^t u_r K(dr, s) \right) \delta W_s. \end{aligned}$$

Condition (16) implies that the process $F'(B_t)$ belongs to the space $\mathbb{D}^{1,2}(\mathcal{H}_{K_r})$. The following theorem can be proved by the same method as Theorem 1.

THEOREM 2. *Let F be a function of class $C^2(\mathbb{R})$ satisfying condition (16). Suppose that $B = \{B_t, t \in [0, T]\}$ is a zero mean continuous Gaussian process whose covariance function $R(t, s)$ is of the form (6), with a kernel $K(t, s)$ satisfying condition (K4). Then the process $F'(B_t)$ belongs to $\mathbb{D}^{1,2}(\mathcal{H}_{K_r})$ and for each $t \in [0, T]$ the following formula holds:*

$$(22) \quad F(B_t) = F(0) + \int_0^t F'(B_s) \delta B_s + \frac{1}{2} \int_0^t F''(B_s) dR_s.$$

Suppose that u is a process in the space $\mathbb{D}^{1,2}(\mathcal{H}_{K_r})$ and suppose that $K(s^+, s) = 0$. Then $s \mapsto \int_s^t u_r K(dr, s)$ is Stratonovich integrable with respect to W (see [18]), and we can write

$$\int_0^t \left(\int_s^t u_r K(dr, s) \right) dW_s = \int_0^t \left(\int_s^t u_r K(dr, s) \right) \delta W_s + \int_0^t \left(\int_s^t D_s u_r K(dr, s) \right) ds,$$

where dW_s denotes the Stratonovich differential. We define the Stratonovich integral of u with respect to B by

$$\int_0^t u_s dB_s = \int_0^t \left(\int_s^t u_r K(dr, s) \right) dW_s,$$

and, as a consequence,

$$\int_0^t u_s dB_s = \int_0^t u_s \delta B_s + \int_0^t \left(\int_s^t D_s u_r K(dr, s) \right) ds.$$

In particular, for $u_s = F'(B_s)$ we obtain

$$(23) \quad \begin{aligned} \int_0^t F'(B_s) dB_s &= \int_0^t F'(B_s) \delta B_s + \int_0^t \left(\int_s^t F''(B_r) K(r, s) K(dr, s) \right) ds \\ &= \int_0^t F'(B_s) \delta B_s + \frac{1}{2} \int_0^t F''(B_r) d \left(\int_0^r K(r, s)^2 ds \right). \end{aligned}$$

As a consequence, substituting (23) into (22), we obtain the following version of the Itô formula for the Stratonovich integral:

$$F(B_t) = F(0) + \int_0^t F'(B_s) dB_s.$$

5. Indefinite integrals and general Itô's formula in the singular case.

Suppose that $B = \{B_t, t \in [0, T]\}$ is a zero mean Gaussian process whose covariance function $R(t, s)$ is of the form (6), with a kernel $K(t, s)$ satisfying the following conditions, for some $\alpha > 0$:

- (i) $K(t, s)$ is differentiable in the variable t in $\{0 < s < t < T\}$, and both K and $\partial K/\partial t$ are continuous in $\{0 < s < t < T\}$;
- (ii) $|(\partial K/\partial t)(t, s)| \leq c(t - s)^{-\alpha-1}$;
- (iii) $\int_s^t K(t, u)^2 du \leq c(t - s)^{1-2\alpha}$.

Condition (i) implies (K1). Conditions (ii) and (iii) imply that B has Hölder continuous paths of order $\frac{1}{2} - \alpha - \varepsilon$ for all $\varepsilon > 0$. In fact, we have, for $t > s$,

$$\begin{aligned}
 E|B_t - B_s|^2 &= \int_s^t K(t, r)^2 dr + \int_0^s |K(t, r) - K(s, r)|^2 dr \\
 &\leq c(t - s)^{1-2\alpha} + \frac{c^2}{\alpha^2} \int_0^s |(s - r)^{-\alpha} - (t - r)^{-\alpha}|^2 dr \\
 &\leq c(t - s)^{1-2\alpha} + \frac{c^2}{\alpha^2} \int_0^s \left[1 - \left(\frac{t - s}{r} + 1 \right)^{-\alpha} \right]^2 r^{-2\alpha} dr.
 \end{aligned}
 \tag{24}$$

By means of the change of variable $(t - s)/r = v$ the last integral can be estimated by

$$(t - s)^{1-2\alpha} \int_0^\infty [1 - (v + 1)^{-\alpha}]^2 v^{2\alpha-2} dv,$$

and we obtain

$$E|B_t - B_s|^2 \leq C(t - s)^{1-2\alpha}.$$

On the other hand, condition (K2) holds if $\alpha < \frac{1}{4}$. Under the previous conditions we can derive the following Hölder continuity property for the indefinite integral.

PROPOSITION 1. *Suppose that the process $u = \{u_t, t \in [0, T]\}$ is λ -Hölder continuous in the norm of the space $\mathbb{D}^{1,p}$ for some $p \geq 2$, and $\lambda > \alpha$. Then u belongs to the space $\mathbb{D}^{1,p}(\mathcal{H}_K)$ and we have $E|X_t - X_s|^p \leq C|t - s|^{(p/2)(1-2\alpha)}$, where $X_t = \int_0^t u_s \delta B_s$.*

PROOF. Let $s \leq t$. The fact that u belongs to the space $\mathbb{D}^{1,p}(\mathcal{H}_K)$ is easy to verify. On the other hand, from (15) we can write

$$\begin{aligned}
 X_t - X_s &= \int_s^t u_r K(t, r) \delta W_r + \int_0^s u_r (K(t, r) - K(s, r)) \delta W_r \\
 &\quad + \int_s^t \left(\int_\sigma^t (u_r - u_\sigma) \frac{\partial K}{\partial r}(r, \sigma) dr \right) \delta W_\sigma \\
 &\quad + \int_0^s \left(\int_s^t (u_r - u_\sigma) \frac{\partial K}{\partial r}(r, \sigma) dr \right) \delta W_\sigma
 \end{aligned}$$

$$\begin{aligned}
&= \int_s^t u_r K(t, r) \delta W_r + \int_s^t \left(\int_\sigma^t (u_r - u_\sigma) \frac{\partial K}{\partial r}(r, \sigma) dr \right) \delta W_\sigma \\
&\quad + \int_0^s \left(\int_s^t u_r \frac{\partial K}{\partial r}(r, \sigma) dr \right) \delta W_\sigma.
\end{aligned}$$

Applying (5), $E|X_t - X_s|^p$ can be estimated by the following three terms:

$$E|X_t - X_s|^p \leq c_p(I_1 + I_2 + I_3),$$

where

$$\begin{aligned}
I_1 &= \left(\int_s^t \|u_r\|_{1,p}^2 K(t, r)^2 dr \right)^{p/2}, \\
I_2 &= \left(\int_s^t \left(\int_\sigma^t \|u_r - u_\sigma\|_{1,p} \left| \frac{\partial K}{\partial r}(r, \sigma) \right| dr \right)^2 d\sigma \right)^{p/2}
\end{aligned}$$

and

$$I_3 = \left(\int_0^s \left(\int_s^t \|u_r\|_{1,p} \left| \frac{\partial K}{\partial r}(r, \sigma) \right| dr \right)^2 d\sigma \right)^{p/2}.$$

Using condition (iii), we obtain

$$I_1 \leq \sup_r \|u_r\|_{1,p}^p c^{p/2} (t-s)^{(p/2)(1-2\alpha)}.$$

In order to handle the term I_2 we make use of the Hölder continuity of u and condition (ii). In this way we obtain

$$I_2 \leq C \left(\int_s^t \left(\int_\sigma^t (r-\sigma)^{\lambda-\alpha-1} dr \right)^2 d\sigma \right)^{p/2} = C'(t-s)^{(p/2)(1+2(\lambda-\alpha))}.$$

Finally, the term I_3 can be estimated as

$$I_3 \leq c^p \sup_r \|u_r\|_{1,p}^p \left(\int_0^s \left(\int_s^t (r-\sigma)^{-\alpha-1} dr \right)^2 d\sigma \right)^{p/2},$$

and by the same arguments as in the proof of (25) we obtain

$$I_3 \leq C \sup_r \|u_r\|_{1,p}^p (t-s)^{(p/2)(1-2\alpha)},$$

which completes the proof of the proposition. \square

Let $u = \{u_t, t \in [0, T]\}$ be a stochastic process satisfying the hypotheses of Proposition 1. Define

$$(26) \quad R(u)_s = \int_0^s (K_s^* u)_r^2 dr.$$

Notice that $R(u)$ is a continuous nonnegative function that vanishes at the origin.

Fix $\varepsilon > 0$, and for $t \leq T + \varepsilon$ we introduce the operator

$$(K^\varepsilon h)_t = \int_0^t K(t + \varepsilon, r)h(r) dr$$

and its adjoint

$$\begin{aligned} (K_t^{*, \varepsilon} \varphi)_s &= \varphi_s K(t + \varepsilon, s) + \int_s^t (\varphi_r - \varphi_s) \frac{\partial K}{\partial r}(r + \varepsilon, s) dr \\ (27) \quad &= \varphi_s K(s + \varepsilon, s) + \int_s^t \varphi_r \frac{\partial K}{\partial r}(r + \varepsilon, s) dr. \end{aligned}$$

Notice that if $\varphi : [0, T] \rightarrow V$ takes values in a separable Hilbert space V , for any $t \leq T + \varepsilon$ we have

$$\begin{aligned} &\int_0^t \|(K_t^{*, \varepsilon} \varphi)_s - (K_t^* \varphi)_s\|_V^2 ds \\ (28) \quad &\leq 2 \int_0^t \|\varphi_s\|_V^2 [K(t + \varepsilon, s) - K(t, s)]^2 ds \\ &\quad + 2 \int_0^t \left\| \int_s^t (\varphi_r - \varphi_s) \left(\frac{\partial K}{\partial r}(r + \varepsilon, s) - \frac{\partial K}{\partial r}(r, s) \right) dr \right\|_V^2 ds, \end{aligned}$$

which converges to 0 as $\varepsilon \rightarrow 0$, uniformly in $t \in [0, T - \varepsilon]$ provided φ is Hölder continuous of order $\lambda > \alpha$.

THEOREM 3. *Let F be a function of class $C_b^2(\mathbb{R})$. Suppose that $B = \{B_t, t \in [0, T]\}$ is a zero mean continuous Gaussian process whose covariance function $R(t, s)$ is of the form (6) with a kernel $K(t, s)$ satisfying conditions (i), (ii) and (iii) for some $\alpha < \frac{1}{4}$. Let $u = \{u_t, t \in [0, T]\}$ be an adapted process in the space $\mathbb{D}^{2,2}$ satisfying the following conditions:*

(C1) *The processes u and $D_r u$ are λ -Hölder continuous in the norm of the space $\mathbb{D}^{1,4}$ for some $\lambda > \alpha$, and the function*

$$\gamma_r = \sup_{0 \leq s \leq T} \|D_r u_s\|_{1,4} + \sup_{0 \leq s < t \leq T} \frac{\|D_r u_t - D_r u_s\|_{1,4}}{|t - s|^\lambda}$$

(C2) *satisfies $\int_0^T \gamma_r^p dr < \infty$ for some $p > 2/(1 - 4\alpha)$.*

$$\sup_{\varepsilon > 0} E \int_0^T \left| \frac{\partial}{\partial s} \int_0^s (K_s^{*, \varepsilon} u)_r^2 dr \right|^2 ds < \infty.$$

Set $X_t = \int_0^t u_s \delta B_s$. Then for each $t \in [0, T]$ the process $F'(X_s)u_s 1_{[0, t]}(s)$ belongs to $\text{Dom } \delta^B$ and the following formula holds:

$$(29) \quad \begin{aligned} F(X_t) &= F(0) + \int_0^t F'(X_s)u_s \delta B_s \\ &+ \int_0^t F''(X_s)u_s \left(\int_0^s \frac{\partial K}{\partial s}(s, r) \left(\int_0^s D_r(K_s^* u)_\theta dW_\theta \right) dr \right) ds \\ &+ \frac{1}{2} \int_0^t F''(X_s) \frac{\partial}{\partial s} \left(\int_0^s (K_s^* u)_r^2 dr \right) ds. \end{aligned}$$

Notice that condition (C2) implies that the function $R(u)$ has a distributional derivative in $L^2([0, T] \times \Omega)$. On the other hand, we shall provide sufficient conditions for (C2) to hold.

PROOF. For $t \leq T - \varepsilon$ we define the process

$$X_t^\varepsilon = \int_0^t (K_t^{*, \varepsilon} u)_s \delta W_s,$$

that is,

$$\begin{aligned} X_t^\varepsilon &= \int_0^t u_s K(t + \varepsilon, s) \delta W_s + \int_0^t \left(\int_s^t (u_r - u_s) \frac{\partial K}{\partial r}(r + \varepsilon, s) dr \right) \delta W_s \\ &= \int_0^t u_s K(s + \varepsilon, s) \delta W_s + \int_0^t \left(\int_0^s u_s \frac{\partial K}{\partial s}(s + \varepsilon, r) \delta W_r \right) ds. \end{aligned}$$

As a consequence, X_t^ε is a continuous semimartingale, and the classical Itô formula yields

$$\begin{aligned} F(X_t^\varepsilon) &= F(0) + \int_0^t F'(X_s^\varepsilon)u_s K(s + \varepsilon, s) \delta W_s \\ &+ \int_0^t F''(X_s^\varepsilon) \left(\int_0^s u_s \frac{\partial K}{\partial s}(s + \varepsilon, r) \delta W_r \right) ds \\ &+ \frac{1}{2} \int_0^t F''(X_s^\varepsilon)u_s^2 K(s + \varepsilon, s)^2 ds. \end{aligned}$$

Using the properties of the Skorohod integral with respect to W , we get

$$(30) \quad \begin{aligned} F(X_t^\varepsilon) &= F(0) + \int_0^t F'(X_s^\varepsilon)u_s K(s + \varepsilon, s) \delta W_s \\ &+ \int_0^t \left(\int_0^s F'(X_s^\varepsilon)u_s \frac{\partial K}{\partial s}(s + \varepsilon, r) \delta W_r \right) ds \\ &+ \int_0^t \left(\int_0^s D_r[F'(X_s^\varepsilon)]u_s \frac{\partial K}{\partial s}(s + \varepsilon, r) dr \right) ds \\ &+ \frac{1}{2} \int_0^t F''(X_s^\varepsilon)u_s^2 K(s + \varepsilon, s)^2 ds. \end{aligned}$$

We have

$$(31) \quad D_r[F'(X_s^\varepsilon)] = F''(X_s^\varepsilon) \left[(K_s^{*,\varepsilon}u)_r + \int_0^s D_r(K_s^{*,\varepsilon}u)_\sigma \delta W_\sigma \right].$$

Substituting (31) into (30), we obtain

$$\begin{aligned} F(X_t^\varepsilon) &= F(0) + \int_0^t F'(X_s^\varepsilon)u_s K(s + \varepsilon, s) \delta W_s \\ &\quad + \int_0^t \left(\int_0^s F'(X_s^\varepsilon)u_s \frac{\partial K}{\partial s}(s + \varepsilon, r) \delta W_r \right) ds \\ &\quad + \int_0^t F''(X_s^\varepsilon)u_s \left(\int_0^s (K_s^{*,\varepsilon}u)_r \frac{\partial K}{\partial s}(s + \varepsilon, r) dr \right) ds \\ &\quad + \int_0^t F''(X_s^\varepsilon)u_s \left(\int_0^s \left(\int_0^s D_r(K_s^{*,\varepsilon}u)_\sigma \delta W_\sigma \right) \frac{\partial K}{\partial s}(s + \varepsilon, r) dr \right) ds \\ &\quad + \frac{1}{2} \int_0^t F''(X_s^\varepsilon)u_s^2 K(s + \varepsilon, s)^2 ds, \end{aligned}$$

which can be written as

$$F(X_t^\varepsilon) = A_{1,\varepsilon} + A_{2,\varepsilon} + A_{3,\varepsilon},$$

where

$$\begin{aligned} A_{1,\varepsilon} &= F(0) + \int_0^t K_t^{*,\varepsilon} [F'(X_s^\varepsilon)u_s] \delta W_s, \\ A_{2,\varepsilon} &= \int_0^t F''(X_s^\varepsilon)u_s \left(\int_0^s \frac{\partial K}{\partial s}(s + \varepsilon, r) \left(\int_0^s D_r(K_s^{*,\varepsilon}u)_\sigma \delta W_\sigma \right) dr \right) ds \end{aligned}$$

and

$$\begin{aligned} A_{3,\varepsilon} &= \int_0^t F''(X_s^\varepsilon)u_s \left(\int_0^s (K_s^{*,\varepsilon}u)_r \frac{\partial K}{\partial s}(s + \varepsilon, r) dr \right) ds \\ &\quad + \frac{1}{2} \int_0^t F''(X_s^\varepsilon)u_s^2 K(s + \varepsilon, s)^2 ds. \end{aligned}$$

Notice that

$$A_{3,\varepsilon} = \frac{1}{2} \int_0^t F''(X_s^\varepsilon) \frac{\partial}{\partial s} \int_0^s (K_s^{*,\varepsilon}u)_r^2 dr.$$

In fact, from (27) we obtain

$$\begin{aligned} \frac{\partial}{\partial s} \int_0^s (K_s^{*,\varepsilon}u)_r^2 &= (K_s^{*,\varepsilon}u)_s^2 + 2 \int_0^s (K_s^{*,\varepsilon}u)_r \frac{\partial (K_s^{*,\varepsilon}u)_r}{\partial s} dr \\ &= u_s^2 K(s + \varepsilon, s)^2 + 2u_s \int_0^s (K_s^{*,\varepsilon}u)_r \frac{\partial K}{\partial s}(s + \varepsilon, r) dr. \end{aligned}$$

Now the proof will be decomposed into several steps.

Step 1. Let us show that

$$\alpha_{1,\varepsilon}^2 = E \int_0^t |K_t^{*,\varepsilon}(F'(X^\varepsilon)u)_s - K_t^*(F'(X)u)_s|^2 ds$$

converges to 0 as $\varepsilon \rightarrow 0$. We can write

$$\begin{aligned} \alpha_{1,\varepsilon}^2 &\leq E \int_0^t |K_t^{*,\varepsilon}((F'(X^\varepsilon) - F'(X))u)_s|^2 ds \\ &\quad + E \int_0^t |K_t^{*,\varepsilon}(F'(X)u)_s - K_t^*(F'(X)u)_s|^2 ds \\ &= \alpha_{11,\varepsilon}^2 + \alpha_{12,\varepsilon}^2. \end{aligned}$$

For the first term we have

$$\begin{aligned} \alpha_{11,\varepsilon}^2 &\leq 2 \int_0^t K(t+\varepsilon, s)^2 E(|F'(X_s^\varepsilon) - F'(X_s)|^2 |u_s|^2) ds \\ &\quad + 2 \int_0^t \left(\int_s^t (F'(X_r^\varepsilon) - F'(X_r))u_r \right. \\ &\quad \left. - (F'(X_s^\varepsilon) - F'(X_s))u_s^2 (r-s)^{-\alpha-1} dr \right)^2 ds \\ &= 2\alpha_{111,\varepsilon}^2 + 2\alpha_{112,\varepsilon}^2. \end{aligned}$$

The first summand in the preceding expression converges to 0 as $\varepsilon \rightarrow 0$ because

$$\alpha_{111,\varepsilon}^2 \leq \|K\| \|F''\|_\infty^2 \sup_{s \leq t} (\|X_s^\varepsilon - X_s\|_4^2 \|u_s\|_4^2),$$

and, by Meyer's inequality [see (5)],

$$\|X_s^\varepsilon - X_s\|_4 \leq c_4 \int_0^s \|(K_s^{*,\varepsilon}u)_r - (K_s^*u)_r\|_{1,4} dr,$$

which converges to 0 as $\varepsilon \rightarrow 0$, uniformly in $s \in [0, t]$, due to (28) and the Hölder continuity of order $\lambda > \alpha$ of the process u_s in the norm $\|\cdot\|_{1,4}$. In order to treat the term $\alpha_{112,\varepsilon}^2$ we write

$$\begin{aligned} &\|(F'(X_r^\varepsilon) - F'(X_r))u_r - (F'(X_s^\varepsilon) - F'(X_s))u_s\|_2 \\ &\leq \|(F'(X_r^\varepsilon) - F'(X_r))(u_r - u_s)\|_2 \\ &\quad + \|(F'(X_r^\varepsilon) - F'(X_r) - F'(X_s^\varepsilon) + F'(X_s))u_s\|_2 \\ &\leq \|F''\|_\infty \|X_r^\varepsilon - X_r\|_4 \|u_r - u_s\|_4 \\ &\quad + \|(F'(X_r^\varepsilon) - F'(X_r) - F'(X_s^\varepsilon) + F'(X_s))\|_4 \|u_s\|_4. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \alpha_{112, \varepsilon}^2 \leq & \|F''\|_\infty^2 \sup_{r \leq t} \|X_r^\varepsilon - X_r\|_4^2 \int_0^t \left(\int_s^t \|u_r - u_s\|_4 (r-s)^{-\alpha-1} dr \right)^2 ds \\ & + \sup_{s \leq t} \|u_s\|_4^2 \int_0^t \left(\int_s^t \|(F'(X_r^\varepsilon) - F'(X_r) - F'(X_s^\varepsilon) + F'(X_s))\|_4 \right. \\ & \left. \times (r-s)^{-\alpha-1} dr \right)^2 ds. \end{aligned}$$

By dominated convergence, it suffices to show that

$$\sup_\varepsilon \|(F'(X_r^\varepsilon) - F'(X_r) - F'(X_s^\varepsilon) + F'(X_s))\|_4 (r-s)^{-\alpha-1}$$

is bounded by an integrable function of the variable r , which is a consequence of Proposition 1. In fact, the proof of this proposition shows that

$$\sup_\varepsilon \|X_r^\varepsilon - X_s^\varepsilon\|_4 \leq C(r-s)^{1/2-\alpha}.$$

Step 2. We claim that the term

$$\alpha_{2, \varepsilon} = \left\| A_{2, \varepsilon} - \int_0^t F''(X_s) u_s \left(\int_0^s \frac{\partial K}{\partial s}(s, r) \left(\int_0^s D_r(K_s^* u)_\sigma \delta W_\sigma \right) dr \right) ds \right\|_2$$

converges to 0 as $\varepsilon \rightarrow 0$. We can write

$$\begin{aligned} \alpha_{2, \varepsilon} & \leq \left\| \int_0^t [F''(X_s^\varepsilon) - F''(X_s)] u_s \left(\int_0^s \frac{\partial K}{\partial s}(s+\varepsilon, r) \left(\int_0^s D_r(K_s^{*, \varepsilon} u)_\sigma \delta W_\sigma \right) dr \right) ds \right\|_2 \\ & + \left\| \int_0^t F''(X_s) u_s \left(\int_0^s \frac{\partial K}{\partial s}(s+\varepsilon, r) \left(\int_0^s D_r([K_s^{*, \varepsilon} - K_s^*] u)_\sigma \delta W_\sigma \right) dr \right) ds \right\|_2 \\ & + \left\| \int_0^t F''(X_s) u_s \left(\int_0^s \left[\frac{\partial K}{\partial s}(s+\varepsilon, r) - \frac{\partial K}{\partial s}(s, r) \right] \right. \right. \\ & \quad \left. \left. \times \left(\int_0^s D_r(K_s^* u)_\sigma \delta W_\sigma \right) dr \right) ds \right\|_2 \\ & = \alpha_{21, \varepsilon} + \alpha_{22, \varepsilon} + \alpha_{23, \varepsilon}. \end{aligned}$$

We have

$$\begin{aligned} \alpha_{21, \varepsilon} & \leq \int_0^t \|(F''(X_s^\varepsilon) - F''(X_s)) u_s\|_4 \\ & \quad \times \left(\int_0^s \left| \frac{\partial K}{\partial s}(s+\varepsilon, r) \right| \left\| \int_0^s D_r(K_s^{*, \varepsilon} u)_\sigma \delta W_\sigma \right\|_4 dr \right) ds \\ & \leq \left\{ \int_0^t \|(F''(X_s^\varepsilon) - F''(X_s)) u_s\|_4^2 ds \right. \\ & \quad \left. \times \int_0^t \left(\int_0^s (s-r)^{-\alpha-1} \left(\int_0^s \|D_r(K_s^{*, \varepsilon} u)_\sigma\|_{1,4}^2 d\sigma \right)^{1/2} dr \right)^2 ds \right\}^{1/2}. \end{aligned}$$

We know that $\int_0^t \|(F''(X_s^\varepsilon) - F''(X_s))u_s\|_4^2 ds$ converges to 0 as $\varepsilon \rightarrow 0$. Then it only remains to show that the last factor in the preceding expression is bounded uniformly in ε . We have

$$\int_0^s \|D_r(K_s^{*,\varepsilon}u_\sigma)\|_{1,4}^2 d\sigma = \int_0^r \|D_r(K_s^{*,\varepsilon}u)_\sigma\|_{1,4}^2 d\sigma + \int_r^s \|D_r(K_s^{*,\varepsilon}u)_\sigma\|_{1,4}^2 d\sigma.$$

Taking into account that the process u is adapted, we obtain, for $r > \sigma$,

$$D_r(K_s^{*,\varepsilon}u)_\sigma = \int_r^s D_r u_\theta \frac{\partial K}{\partial \theta}(\theta + \varepsilon, \sigma) d\theta.$$

Hence,

$$\begin{aligned} & \int_0^s (s-r)^{-\alpha-1} \left(\int_0^r \|D_r(K_s^{*,\varepsilon}u)_\sigma\|_{1,4}^2 d\sigma \right)^{1/2} dr \\ & \leq c \int_0^s (s-r)^{-\alpha-1} \left(\int_0^r \left(\int_r^s \|D_r u_\theta\|_{1,4}(\theta-\sigma)^{-\alpha-1} d\theta \right)^2 d\sigma \right)^{1/2} dr \\ & \leq c \int_0^s (s-r)^{-\alpha-1} k_1(r) \left(\int_0^r [(r-\sigma)^{-\alpha} - (s-\sigma)^{-\alpha}]^2 d\sigma \right)^{1/2} dr \\ & \leq c \int_0^s (s-r)^{-2\alpha-1/2} k_1(r) dr \\ & < \infty, \end{aligned}$$

where $k_1(r) = \sup_\theta \|D_r u_\theta\|_{1,4}$ due to condition (C1). On the other hand,

$$\begin{aligned} & \int_0^s (s-r)^{-\alpha-1} \left(\int_r^s \|D_r(K_s^{*,\varepsilon}u)_\sigma\|_{1,4}^2 d\sigma \right)^{1/2} dr \\ & \leq \int_0^s (s-r)^{-\alpha-1} \left(\int_r^s \|D_r u_\sigma\|_{1,4}^2 K(s+\varepsilon, \sigma)^2 d\sigma \right)^{1/2} dr \\ & \quad + \int_0^s (s-r)^{-\alpha-1} \left(\int_r^s \left(\int_\sigma^s \|D_r u_\theta - D_r u_\sigma\|_{1,4}(\theta-\sigma)^{-\sigma-1} d\theta \right)^2 d\sigma \right)^{1/2} dr \\ & \leq \int_0^s (s-r)^{-2\alpha-1/2} k_1(r) dr + c \int_0^s (s-r)^{\lambda-2\alpha-1/2} k_2(r) dr. \end{aligned}$$

where $k_2(r) = \sup_{0 \leq s < t \leq T} \frac{\|D_r u_s - D_r u_t\|_{1,4}}{|t-r|^\lambda}$ and again this is finite due to (C1). Let us now consider the second term:

$$\begin{aligned} \alpha_{22, \varepsilon} & \leq \|F''\|_\infty \sup_{s \leq t} \|u_s\|_4 \\ & \quad \times \int_0^t \left(\int_0^s \left| \frac{\partial K}{\partial s}(s+\varepsilon, r) \right| \left\| \int_0^s D_r \left([K_s^{*,\varepsilon} - K_s^*]u \right)_\sigma \delta W_\sigma \right\|_4 dr \right) ds \\ & \leq c \int_0^t \left(\int_0^s (s-r)^{-\alpha-1} \left(\int_0^s \|D_r([K_s^{*,\varepsilon} - K_s^*]u)_\sigma\|_{1,4} d\sigma \right) dr \right) ds. \end{aligned}$$

We have

$$\begin{aligned} & D_r([K_s^{*,\varepsilon} - K_s^*]u)_\sigma \\ &= \mathbf{1}_{\{r>\sigma\}} \int_r^s D_r u_\theta \left[\frac{\partial K}{\partial \theta}(\theta + \varepsilon, \sigma) - \frac{\partial K}{\partial \theta}(\theta, \sigma) \right] d\theta \\ &\quad + \mathbf{1}_{\{r\leq\sigma\}} D_r u_\sigma [K(s + \varepsilon, \sigma) - K(s, \sigma)] \\ &\quad + \mathbf{1}_{\{r\leq\sigma\}} \int_\sigma^s (D_r u_\theta - D_r u_\sigma) \left[\frac{\partial K}{\partial \theta}(\theta + \varepsilon, \sigma) - \frac{\partial K}{\partial \theta}(\theta, \sigma) \right] d\theta, \end{aligned}$$

and $\alpha_{22,\varepsilon}$ converges to 0 by dominated convergence, using the same estimates as in the first term. The term $\alpha_{23,\varepsilon}$ can be treated in a similar way.

Step 3. We claim that $A_{3,\varepsilon}$ converges to 0 in $L^1(\Omega)$ as $\varepsilon \rightarrow 0$ to the square integrable random variable

$$\frac{1}{2} \int_0^t F''(X_s) \frac{\partial}{\partial s} \int_0^s (K_s^* u)_r^2 dr.$$

Set

$$\alpha_{3,\varepsilon} = E \left| A_{3,\varepsilon} - \frac{1}{2} \int_0^t F''(X_s) \frac{\partial}{\partial s} \int_0^s (K_s^* u)_r^2 dr \right|.$$

We have

$$\begin{aligned} \alpha_{3,\varepsilon} &\leq \frac{1}{2} E \left| \int_0^t [F''(X_s^\varepsilon) - F''(X_s)] \left(\frac{\partial}{\partial s} \int_0^s (K_s^{*,\varepsilon} u)_r^2 dr \right) ds \right| \\ &\quad + \frac{1}{2} E \left| \int_0^t F''(X_s) \left(\frac{\partial}{\partial s} \int_0^s (K_s^{*,\varepsilon} u)_r^2 dr - \frac{\partial}{\partial s} \int_0^s (K_s^* u)_r^2 dr \right) ds \right| \\ &= \alpha_{31,\varepsilon} + \alpha_{32,\varepsilon}. \end{aligned}$$

The first summand can be estimated as follows, for any $\eta > 0$,

$$\begin{aligned} 2\alpha_{31,\varepsilon} &\leq \eta \int_0^t E \left| \frac{\partial}{\partial s} \int_0^s (K_s^{*,\varepsilon} u)_r^2 dr \right| ds + \int_0^t P \left\{ |F''(X_s^\varepsilon) - F''(X_s)| > \eta \right\}^{1/2} \\ &\quad \times \left[E \left| \frac{\partial}{\partial s} \int_0^s (K_s^{*,\varepsilon} u)_r^2 dr \right|^2 \right]^{1/2} ds, \end{aligned}$$

and letting first $\varepsilon \rightarrow 0$ and then $\eta \rightarrow 0$, we get the desired convergence for this term, due to condition (C2). Finally, let C_s be a smooth step process such that

$$E \int_0^t |F''(X_s) - C_s|^2 ds < \varepsilon.$$

Then it suffices to show that

$$E \left| \int_0^t C_s \left(\frac{\partial}{\partial s} \int_0^s (K_s^{*,\varepsilon} u)_r^2 dr - \frac{\partial}{\partial s} \int_0^s (K_s^* u)_r^2 dr \right) ds \right| \text{ tends to zero as } \varepsilon \rightarrow 0,$$

which follows from the fact that $\int_0^s (K_s^{*,\varepsilon} u)_r^2 dr$ converges to $\int_0^s (K_s^* u)_r^2 dr$ in the norm of $L^2([0, T] \times \Omega)$ as $\varepsilon \rightarrow 0$.

Step 4. The family of stochastic processes

$$\left\{ K_t^{*,\varepsilon} (F'(X^\varepsilon)u)_s, s \in [0, t] \right\}$$

converges in $L^2([0, T] \times \Omega)$ to $K_t^* (F'(X)u)_s$ as $\varepsilon \rightarrow 0$, by Step 1. Moreover, Steps 2 and 3 imply that the square integrable random variable

$$\begin{aligned} R = & F(X_t) - F(0) - \int_0^t F''(X_s)u_s \left(\int_0^s \frac{\partial K}{\partial s}(s, r) \left(\int_0^s D_r(K_s^* u)_\theta \delta W_\theta \right) dr \right) ds \\ & - \frac{1}{2} \int_0^t F''(X_s) \frac{\partial}{\partial s} \left(\int_0^s (K_s^* u)_r^2 dr \right) ds \end{aligned}$$

satisfies

$$\lim_{\varepsilon \downarrow 0} E \left(G \int_0^t K_t^{*,\varepsilon} (F'(X^\varepsilon)u)_s \delta W_s \right) = E(GR)$$

for any smooth random variable G . Hence, by Lemma 1 of [1], we conclude that $K_t^* (F'(X)u)_s$ is Skorohod integrable and $R = \int_0^t K_t^* (F'(X)u)_s \delta W_s$. \square

The next proposition gives us sufficient conditions under which hypothesis (C2) holds.

PROPOSITION 2. Assume that the kernel $K(t, s)$ satisfies conditions (i), (ii) and (iii) for some $\alpha < \frac{1}{4}$ and that u is an adapted process in $\mathbb{D}^{2,2}$ which satisfies condition (C1). Suppose, moreover, that

(iv)

$$\left| \frac{\partial}{\partial s} \left(\int_0^s K^2(s + \varepsilon, r) dr \right) \right| < s^{-2\alpha},$$

(v)

$$\frac{\partial K^2}{\partial s}(s + \varepsilon, r) < (s - r)^{-2\alpha - 1},$$

and that the process u is ρ -Hölder continuous in the norm of the space $L^4(\Omega)$ for some $\rho > 2\alpha$. Then condition (C2) holds.

PROOF. We can write

$$\begin{aligned} \int_0^s (K_s^{*,\varepsilon} u)_r^2 dr &= \int_0^s u_r^2 K^2(s + \varepsilon, r) dr \\ &+ 2 \int_0^s u_r K(s + \varepsilon, r) \left(\int_r^s (u_\theta - u_r) \frac{\partial K}{\partial \theta}(\theta + \varepsilon, r) d\theta \right) dr \\ &+ \int_0^s \left(\int_r^s (u_\theta - u_r) \frac{\partial K}{\partial \theta}(\theta + \varepsilon, r) d\theta \right)^2 dr. \end{aligned}$$

Differentiating with respect to the variable s yields

$$\begin{aligned} \frac{\partial}{\partial s} \int_0^s (K_s^{*,\varepsilon} u)_r^2 dr &= u_s^2 K^2(s + \varepsilon, s) + \int_0^s u_r^2 \frac{\partial K^2}{\partial s}(s + \varepsilon, r) dr \\ &\quad + 2 \int_0^s u_r K(s + \varepsilon, r) (u_s - u_r) \frac{\partial K}{\partial s}(s + \varepsilon, r) dr \\ &\quad + 2 \int_0^s \left(\int_r^s u_r (u_\theta - u_r) \frac{\partial K}{\partial s}(s + \varepsilon, r) \frac{\partial K}{\partial \theta}(\theta + \varepsilon, r) d\theta \right) dr \\ &\quad + 2 \int_0^s \left(\int_r^s (u_\theta - u_r) \frac{\partial K}{\partial \theta}(\theta + \varepsilon, r) d\theta \right) (u_s - u_r) \frac{\partial K}{\partial s}(s + \varepsilon, r) dr \\ &= u_s^2 K^2(s + \varepsilon, s) + u_s \int_0^s u_r \frac{\partial K^2}{\partial s}(s + \varepsilon, r) dr \\ &\quad + 2u_s \int_0^s \left(\int_r^s (u_\theta - u_r) \frac{\partial K}{\partial \theta}(\theta + \varepsilon, r) d\theta \right) \frac{\partial K}{\partial s}(s + \varepsilon, r) dr. \end{aligned}$$

Now we add and subtract the term $u_s^2 \int_0^s (\partial K^2 / \partial s)(s + \varepsilon, r)$, obtaining

$$\begin{aligned} \frac{\partial}{\partial s} \int_0^s (K_s^{*,\varepsilon} u)_r^2 dr &= u_s^2 K^2(s + \varepsilon, s) + u_s^2 \int_0^s \frac{\partial K^2}{\partial s}(s + \varepsilon, r) dr \\ &\quad + u_s \int_0^s (u_s - u_r) \frac{\partial K^2}{\partial s}(s + \varepsilon, r) dr \\ &\quad + 2u_s \int_0^s \left(\int_r^s (u_\theta - u_r) \frac{\partial K}{\partial \theta}(\theta + \varepsilon, r) d\theta \right) \frac{\partial K}{\partial s}(s + \varepsilon, r) dr \\ &= u_s^2 \frac{\partial}{\partial s} \int_0^s K^2(s + \varepsilon, r) dr + u_s \int_0^s (u_s - u_r) \frac{\partial K^2}{\partial s}(s + \varepsilon, r) dr \\ &\quad + 2u_s \int_0^s \left(\int_r^s (u_\theta - u_r) \frac{\partial K}{\partial \theta}(\theta + \varepsilon, r) d\theta \right) \frac{\partial K}{\partial s}(s + \varepsilon, r) dr. \end{aligned}$$

Integrating the square of the preceding expression in s and taking the expectation, the result follows easily from conditions (iv) and (v). \square

6. Indefinite integrals and general Itô’s formula in the regular case.

In the regular case treated in Section 3, the process B_t admits the following decomposition:

$$B_t = \int_0^t K(s^+, s) \delta W_s + \int_0^t K((s, t], s) \delta W_s,$$

where the first summand is a Gaussian martingale, and the second process is expressed in terms of the kernel $K_1(t, s) = K((s, t], s)$, which vanishes as $t \downarrow s$. We are mainly interested in the stochastic calculus with respect to a process $\int_0^t K((s, t], s) \delta W_s$, which includes the case of the fractional Brownian motion of the Hurst parameter $H > \frac{1}{2}$. For this reason in this section we will assume that $K(s^+, s) = 0$.

Suppose that $B = \{B_t, t \in [0, T]\}$ is a zero mean Gaussian process whose covariance function $R(t, s)$ is of the form (6), with a kernel $K(t, s)$ satisfying condition (i) and

(ii')

$$\left| \frac{\partial K}{\partial t}(t, s) \right| \leq c(t-s)^{\alpha-1} s^{-\alpha},$$

(iii')

$$\int_s^t K(t, u)^2 du \leq c(t-s)^{1+2\alpha} \quad \text{for some } 0 < \alpha \leq \frac{1}{2}.$$

Conditions (i) and (ii') imply (K4). Conditions (ii') and (iii') imply that B has Hölder continuous paths of order $\frac{1}{2} + \alpha - \varepsilon$ for all $\varepsilon > 0$. In fact, using the Hölder inequality, we have

$$\begin{aligned} E|B_t - B_s|^2 &= \int_s^t K(t, r)^2 dr + \int_0^s |K(t, r) - K(s, r)|^2 dr \\ &\leq C(t-s)^{1+2\alpha}. \end{aligned}$$

Under the preceding conditions we can derive the following Hölder continuity property for the indefinite integral.

PROPOSITION 3. *Suppose that the process $u = \{u_t, t \in [0, T]\}$ is bounded in the norm of the space $\mathbb{D}^{1,p}$ for some $p \geq 2$. Then u belongs to the space $\mathbb{D}^{1,p}(\mathcal{H}_{K_r})$ and we have $E|X_t - X_s|^p \leq C|t-s|^{(p/2)(1+2\alpha)}$, where $X_t = \int_0^t u_s \delta B_s$.*

PROOF. The fact that u belongs to the space $\mathbb{D}^{1,p}(\mathcal{H}_{K_r})$ is easy to verify. On the other hand, we can write, for $s \leq t$,

$$X_t - X_s = \int_0^s \left(\int_s^t u_\sigma K(d\sigma, r) \right) \delta W_r + \int_s^t \left(\int_r^t u_\sigma K(d\sigma, r) \right) \delta W_r,$$

and we can estimate $E|X_t - X_s|^p$ by $C \sup_r \|u_r\|_{1,p}^p (t-s)^{(p/2)(1+2\alpha)}$, as in the proof of Proposition 1. \square

Let $u = \{u_t, t \in [0, T]\}$ be a stochastic process satisfying the hypotheses of Proposition 3. In this case we have

$$(K_t^* u)_s = \int_s^t u_r \frac{\partial K}{\partial r}(r, s) dr,$$

and the function $R(u)_s$ defined in (26) has now the following expression:

$$R(u)_s = \int_0^s \left(\int_r^s u_\theta \frac{\partial K}{\partial \theta}(\theta, r) d\theta \right)^2 dr.$$

THEOREM 4. *Let F be a function of class $C_b^2(\mathbb{R})$. Suppose that $B = \{B_t, t \in [0, T]\}$ is a zero mean continuous Gaussian process whose covariance function $R(t, s)$ is of the form (6), with a kernel $K(t, s)$ satisfying conditions (i), (ii') and (iii'). Let $u = \{u_t, t \in [0, T]\}$ be an adapted process bounded in the norm of the space $\mathbb{D}^{2,4}$. Set $X_t = \int_0^t u_s \delta B_s$. Then the process $F'(X_t)$ belongs to $\mathbb{D}^{1,2}(\mathcal{H}_{K_r})$ and for each $t \in [0, T]$ the following formula holds:*

$$\begin{aligned}
 (32) \quad F(X_t) &= F(0) + \int_0^t F'(X_s) u_s \delta B_s \\
 &+ \int_0^t F''(X_s) u_s \left(\int_0^s \frac{\partial K}{\partial s}(s, r) \left(\int_0^s D_r(K_s^* u)_\theta \delta W_\theta \right) dr \right) ds \\
 &+ \frac{1}{2} \int_0^t F''(X_s) \frac{\partial}{\partial s} \left(\int_0^s (K_s^* u)_r^2 dr \right) ds.
 \end{aligned}$$

PROOF. Define the process

$$X_t^\varepsilon = \int_0^t \left(\int_s^t u_r \frac{\partial K}{\partial r}(r + \varepsilon, s) dr \right) \delta W_s = \int_0^t \left(\int_0^s u_s \frac{\partial K}{\partial s}(s + \varepsilon, r) \delta W_r \right) ds$$

for any $t \leq T - \varepsilon$. The process X_t^ε has bounded variation paths and we can write

$$F(X_t^\varepsilon) = F(0) + \int_0^t F'(X_s^\varepsilon) \left(\int_0^s u_s \frac{\partial K}{\partial s}(s + \varepsilon, r) \delta W_r \right) ds.$$

Using the properties of the Skorohod integral with respect to W yields

$$\begin{aligned}
 (33) \quad F(X_t^\varepsilon) &= F(0) + \int_0^t \left(\int_0^s F'(X_s^\varepsilon) u_s \frac{\partial K}{\partial s}(s + \varepsilon, r) \delta W_r \right) ds \\
 &+ \int_0^t \left(\int_0^s D_r[F'(X_s^\varepsilon)] u_s \frac{\partial K}{\partial s}(s + \varepsilon, r) dr \right) ds.
 \end{aligned}$$

We have

$$(34) \quad D_r[F'(X_s^\varepsilon)] = F''(X_s^\varepsilon) \left[K_{r,s}^\varepsilon(u) + \int_0^s D_r[K_{\sigma,s}^\varepsilon(u)] \delta W_\sigma \right],$$

where

$$K_{s,r}^\varepsilon(u) = \int_r^s u_\theta \frac{\partial K}{\partial \theta}(\theta + \varepsilon, r) d\theta.$$

Thus, substituting (34) into (33), we obtain

$$\begin{aligned}
 F(X_t^\varepsilon) &= F(0) + \int_0^t \left(\int_r^t F'(X_s^\varepsilon) u_s \frac{\partial K}{\partial s}(s + \varepsilon, s) ds \right) \delta W_r \\
 &+ \int_0^t F''(X_s^\varepsilon) u_s \frac{\partial K}{\partial s}(s + \varepsilon, r) \left(\int_0^s \left(\int_0^s D_r[K_{\sigma,s}^\varepsilon(u)] \delta W_\sigma \right) dr \right) ds \\
 &+ \int_0^t F''(X_s^\varepsilon) u_s \left(\int_0^s K_{s,r}^\varepsilon(u) \frac{\partial K}{\partial s}(s + \varepsilon, r) dr \right) ds \\
 &= F(0) + A_{1,\varepsilon} + A_{2,\varepsilon} + A_{3,\varepsilon}.
 \end{aligned}$$

Notice that

$$A_{3,\varepsilon} = \frac{1}{2} \int_0^t F''(X_s^\varepsilon) \frac{\partial}{\partial s} \int_0^s (K_{s,r}^\varepsilon(u))^2 dr.$$

As in the proof of Theorem 3, it suffices to show that the following terms converge to 0 as $\varepsilon \rightarrow 0$:

$$\begin{aligned} \alpha_{1,\varepsilon}^2 &= E \int_0^t |K_t^{*,\varepsilon}(F'(X^\varepsilon)u)_s - K_t^*(F'(X)u)_s|^2 ds, \\ \alpha_{2,\varepsilon} &= \left\| A_{2,\varepsilon} - \int_0^t F''(X_s)u_s \left(\int_0^s \frac{\partial K}{\partial s}(s,r) \left(\int_0^s D_r[K_{\sigma,s}^\varepsilon(u)] \delta W_\sigma \right) dr \right) ds \right\|_2 \end{aligned}$$

and

$$\alpha_{3,\varepsilon} = E \left| A_{3,\varepsilon} - \frac{1}{2} \int_0^t F''(X_s) \frac{\partial}{\partial s} \int_0^s (K_s^*u)_r^2 dr \right|.$$

We can write

$$\begin{aligned} \alpha_{1,\varepsilon}^2 &\leq E \int_0^t |K_t^{*,\varepsilon}((F'(X^\varepsilon) - F'(X))u)_s|^2 ds \\ &\quad + E \int_0^t |K_t^{*,\varepsilon}(F'(X)u)_s - K_t^*(F'(X)u)_s|^2 ds \\ &= \alpha_{11,\varepsilon}^2 + \alpha_{12,\varepsilon}^2. \end{aligned}$$

For the first term we have

$$\begin{aligned} \alpha_{11,\varepsilon}^2 &\leq C \int_0^t \int_s^t \|F'(X_r^\varepsilon) - F'(X_r)\|_4^2 \|u_r\|_4^2 (r-s)^{\alpha-1} s^{-\alpha} dr ds \\ &\leq C' \|F''\|_\infty^2 \sup_{r \leq t} \|u_r\|_4^2 \int_0^t \|X_r^\varepsilon - X_r\|_4^2 dr. \end{aligned}$$

We have

$$\begin{aligned} \|X_s^\varepsilon - X_s\|_4^2 &\leq C \int_0^s \|K_{s,r}^\varepsilon(u) - (K_s^*u)_r\|_{1,4}^2 dr \\ &= C \int_0^s \left\| \int_r^s u_\theta \left(\frac{\partial K}{\partial \theta}(\theta + \varepsilon, r) - \frac{\partial K}{\partial \theta}(\theta, r) \right) d\theta \right\|_{1,4}^2 dr, \end{aligned}$$

which converges to 0 as $\varepsilon \rightarrow 0$, by dominated convergence, because the process u_s is bounded in the norm $\|\cdot\|_{1,4}$. Hence, the term $\alpha_{11,\varepsilon}$ converges to 0 as $\varepsilon \rightarrow 0$. The term $\alpha_{12,\varepsilon}$ can be treated in a similar way.

For the term $\alpha_{2,\varepsilon}$ we can write

$$\begin{aligned} \alpha_{2,\varepsilon} &\leq \left\| \int_0^t [F''(X_s^\varepsilon) - F''(X_s)] u_s \left(\int_0^s \frac{\partial K}{\partial s}(s+\varepsilon, r) \left(\int_0^s D_r [K_{s,\sigma}^\varepsilon(u)] \delta W_\sigma \right) dr \right) ds \right\|_2 \\ &\quad + \left\| \int_0^t F''(X_s) u_s \left(\int_0^s \frac{\partial K}{\partial s}(s+\varepsilon, r) \left(\int_0^s D_r [K_{s,\sigma}^\varepsilon(u) - (K_s^* u)_\sigma] \delta W_\sigma \right) dr \right) ds \right\|_2 \\ &\quad + \left\| \int_0^t F''(X_s) u_s \left(\int_0^s \left[\frac{\partial K}{\partial s}(s+\varepsilon, r) - \frac{\partial K}{\partial s}(s, r) \right] \left(\int_0^s D_r (K_s^* u)_\sigma \delta W_\sigma \right) dr \right) ds \right\|_2 \\ &= \alpha_{21,\varepsilon} + \alpha_{22,\varepsilon} + \alpha_{23,\varepsilon}. \end{aligned}$$

We have

$$\begin{aligned} \alpha_{21,\varepsilon}^4 &\leq C \left(\int_0^t E \left| (F''(X_s^\varepsilon) - F''(X_s)) u_s \right|^4 ds \right) \\ (35) \quad &\quad \times \int_0^t \left(\int_0^s (s-r)^{\alpha-1} E \left| \int_0^s D_r [K_{s,\sigma}^\varepsilon(u)] \delta W_\sigma \right|^4 dr \right) ds. \end{aligned}$$

Taking into account that the process u is adapted, we obtain

$$D_r [K_{s,\sigma}^\varepsilon(u)] = \int_{r \vee \sigma}^s D_r u_\theta \frac{\partial K}{\partial \theta}(\theta + \varepsilon, \sigma) d\theta,$$

and, hence,

$$(36) \quad E \left| \int_0^s D_r [K_{s,\sigma}^\varepsilon(u)] \delta W_\sigma \right|^4 \leq C \int_0^s \int_{r \vee \sigma}^s \|D_r u_\theta\|_{1,4}^4 (\theta - \sigma)^{\alpha-1} \sigma^{-\alpha} d\theta d\sigma.$$

As a consequence, substituting (36) into (35) yields

$$\alpha_{21,\varepsilon} \leq C \sup_{s \leq t} \|u_s\|_{2,4} \left(E \int_0^t \left| (F''(X_s^\varepsilon) - F''(X_s)) u_s \right|^4 ds \right)^{1/4},$$

which converges to 0 as $\varepsilon \rightarrow 0$. The second and third terms are treated similarly.

The term $\alpha_{3,\varepsilon}$ can be estimated as follows:

$$\begin{aligned} \alpha_{3,\varepsilon} &\leq \frac{1}{2} E \left| \int_0^t [F''(X_s^\varepsilon) - F''(X_s)] \left(\frac{\partial}{\partial s} \int_0^s K_{s,r}^\varepsilon(u)^2 dr \right) ds \right| \\ &\quad + \frac{1}{2} E \left| \int_0^t F''(X_s) \left(\frac{\partial}{\partial s} \int_0^s K_{s,r}^\varepsilon(u)^2 dr - \frac{\partial}{\partial s} \int_0^s (K_s^* u)_r^2 dr \right) ds \right| \\ &= \alpha_{31,\varepsilon} + \alpha_{32,\varepsilon}. \end{aligned}$$

We have

$$2\alpha_{31,\varepsilon} \leq \left(E \int_0^t [F''(X_s^\varepsilon) - F''(X_s)]^2 ds \right)^{1/2} \left(E \int_0^T \left| \frac{\partial}{\partial s} \int_0^s K_{s,r}^\varepsilon(u)^2 dr \right|^2 ds \right)^{1/2},$$

which converges to 0 as $\varepsilon \rightarrow 0$, because the second factor can be bounded by a constant:

$$E \int_0^T \left| \frac{\partial}{\partial s} \int_0^s K_{s,r}^\varepsilon(u)^2 dr \right|^2 ds \leq C \sup_{s \leq T} E |u_s|^4.$$

Finally, by dominated convergence, the term $\alpha_{32,\varepsilon}$ also converges to 0 as $\varepsilon \rightarrow 0$. \square

Let us now write the Itô formula in terms of the Stratonovich integral. Define

$$Y_t = \int_0^t u_s dB_s = \int_0^t u_s \delta B_s + \int_0^t \int_0^r D_s u_r \frac{\partial K}{\partial r}(r, s) ds dr.$$

A straightforward extension of the Itô formula to the process Y_t yields

$$\begin{aligned} (37) \quad F(Y_t) &= F(0) + \int_0^t F'(Y_s) u_s \delta B_s + \int_0^t F'(Y_r) \left(\int_0^r D_s u_r \frac{\partial K}{\partial r}(r, s) ds \right) dr \\ &\quad + \int_0^t \int_0^r F''(Y_r) u_r \left(\int_0^r \int_0^\sigma D_s D_\theta u_\sigma \frac{\partial K}{\partial r}(\sigma, \theta) d\theta d\sigma \right) \frac{\partial K}{\partial r}(r, s) ds dr \\ &\quad + \int_0^t F''(Y_s) u_s \left(\int_0^s \frac{\partial K}{\partial s}(s, r) \left(\int_0^s D_r (K_s^* u)_\theta \delta W_\theta \right) dr \right) ds \\ &\quad + \frac{1}{2} \int_0^t F''(Y_s) \frac{\partial}{\partial s} \left(\int_0^s (K_s^* u)_r^2 dr \right). \end{aligned}$$

From the relationship between the Skorohod integral and the Stratonovich integral we deduce that

$$\begin{aligned} (38) \quad &\int_0^t F'(Y_s) u_s \delta B_s \\ &= \int_0^t F'(Y_s) u_s dB_s - \int_0^t \int_0^r F'(Y_r) D_s u_r \frac{\partial K}{\partial r}(r, s) ds dr \\ &\quad - \int_0^t F''(Y_r) u_r \left(\int_0^r (K_r^* u)_s \frac{\partial K}{\partial r}(r, s) ds \right) dr \\ &\quad - \int_0^t \int_0^r F''(Y_r) u_r \left(\int_0^r D_s (K_r^* u)_\theta \delta W_\theta \right) \frac{\partial K}{\partial r}(r, s) ds dr \\ &\quad - \int_0^t \int_0^r F''(Y_r) u_r \times \left(\int_0^r \int_0^\sigma D_s D_\theta u_\sigma \frac{\partial K}{\partial r}(\sigma, \theta) d\theta d\sigma \right) \frac{\partial K}{\partial r}(r, s) ds dr. \end{aligned}$$

Sustituting (38) into (37) yields

$$F(Y_t) = F(0) + \int_0^t F'(Y_s) u_s dB_s.$$

7. Approximation by Riemann sums. In this section we study the approximation of the stochastic integral $\int_0^T u_t \delta B_t$ by Riemann sums. Let us recall the notion of the Wick product.

DEFINITION 1. Consider two square integrable random variables of the form $F = \sum_{n=0}^\infty I_n^B(f_n)$ and $G = \sum_{n=0}^\infty I_n^B(g_n)$, where I_n^B denotes the multiple stochastic integral of order n with respect to the Gaussian process B . The Wick product is the random variable defined by

$$F \diamond G = \sum_{n,m=0}^\infty I_{n+m}^B(f_n \tilde{\otimes} g_m),$$

provided the sum converges in $L^2(\Omega)$, where $f_n \tilde{\otimes} g_m$ denotes the symmetrization of the tensor product of f_n and g_m .

Given an element $\varphi \in \mathcal{H}$ and a random variable $F \in L^2(\Omega)$, the Wick product $F \diamond B(\varphi)$ exists if and only if φF belongs to the domain of δ^B and in this case $F \diamond B(\varphi) = \delta^B(\varphi F)$. This fact is an immediate consequence of the characterization of the domain of δ^B in terms of the Wiener chaos expansion.

Suppose $\pi = \{0 = s_0 < s_1 < \dots < s_{n+1} = T\}$ is a partition of $[0, T]$ and define the mesh of π as $|\pi| = \max_{i=0, \dots, n} |s_{i+1} - s_i|$. Then we have the following approximation result for singular kernels:

PROPOSITION 4. *Suppose that the kernel $K(t, s)$ satisfies conditions (i), (ii) and (iii). Let u be a process that is λ -Hölder continuous in the norm of $\mathbb{D}^{1,2}$ with $\lambda > \alpha$. Then we have*

$$\lim_{|\pi| \downarrow 0} \sum_{i=0}^n u_{s_i} \diamond (B_{s_{i+1}} - B_{s_i}) = \int_0^T u_t \delta B_t,$$

where the convergence is in $L^2(\Omega)$.

PROOF. Set $u_s^\pi = \sum_{i=0}^n u_{s_i} \mathbf{1}_{[s_i, s_{i+1})}(s)$. We know that

$$(39) \quad \int_0^T u_t^\pi \delta B_t = \sum_{i=0}^n u_{s_i} \diamond (B_{s_{i+1}} - B_{s_i}).$$

Then the result follows from the convergence of u^π to u in the norm of $\mathbb{D}^{1,2}(\mathcal{H}_K)$. \square

In the regular case, that is, when the kernel satisfies (K4), Proposition 4 holds assuming only the continuity of u in the norm of $\mathbb{D}^{1,2}$.

Let us now consider the convergence of the ordinary Riemann sums to the Stratonovich integral, and in this case we will restrict ourselves to the regular case assuming $K(s^+, s) = 0$. Actually, the Stratonovich integral is less interesting in the singular case because the trace condition is very restrictive (see [2]).

PROPOSITION 5. *Suppose that the kernel $K(t, s)$ satisfies condition (K4) and $K(s^+, s) = 0$. If u is an adapted process continuous in the norm of $\mathbb{D}^{1,2}$, verifying*

$$(40) \quad \lim_{n \rightarrow \infty} \int_0^T \sup_{s, s' \in (r, r+1/n] \cap [0, T]} E |D_r u_s - D_r u_{s'}|^2 dr = 0,$$

then we have

$$\lim_{|\pi| \downarrow 0} \sum_{i=0}^n u_{s_i} (B_{s_{i+1}} - B_{s_i}) = \int_0^T u_t dB_t,$$

where the convergence is in $L^2(\Omega)$.

PROOF. We know that

$$\int_0^T u_t dB_t = \int_0^T u_t \delta B_t + \int_0^T \left(\int_r^T D_r u_s K(ds, r) \right) dr.$$

On the other hand,

$$\begin{aligned} \int_0^T u_t^\pi \delta B_t &= \sum_{i=0}^n u_{s_i} \diamond (B_{s_{i+1}} - B_{s_i}) \\ &= \sum_{i=0}^n u_{s_i} (B_{s_{i+1}} - B_{s_i}) + \sum_{i=0}^n \langle D^B u_{s_i}, \mathbf{1}_{(s_i, s_{i+1}]} \rangle_{\mathcal{H}}. \end{aligned}$$

The second summand of the preceding expression can be written as

$$\begin{aligned} A_\pi &= \sum_{i=0}^n \langle D^B u_{s_i}, \mathbf{1}_{(s_i, s_{i+1}]} \rangle_{\mathcal{H}} \\ &= \sum_{i=0}^n \langle Du_{s_i}, K^* \mathbf{1}_{(s_i, s_{i+1}]} \rangle_{L^2([0, T])} \\ &= \sum_{i=0}^n \int_0^T D_r u_{s_i} (K(s_{i+1}, r) - K(s_i, r)) dr \\ &= \sum_{i=0}^n \int_0^T D_r u_{s_i} K((s_i, s_{i+1}], r) dr. \end{aligned}$$

As a consequence, we can write

$$\begin{aligned} &E \left| A_\pi - \int_0^T \left(\int_r^T D_r u_s K(ds, r) \right) dr \right|^2 \\ &\leq TE \int_0^T \left(\sum_{i=0}^n \int_{s_i}^{s_{i+1}} |D_r u_s - D_r u_{s_i}| |K|(ds, r) \right)^2 dr \\ &\leq T \int_0^T |K|((r, T], r) \left(\sum_{i=0}^n \int_{s_i}^{s_{i+1}} E |D_r u_s - D_r u_{s_i}|^2 |K|(ds, r) \right) dr \\ &\leq T \int_0^T |K|((r, T], r)^2 dr \left(\int_0^T \sup_{s, s' \in (r, r+|\pi|] \cap [0, T]} E |D_r u_s - D_r u_{s'}|^2 dr \right), \end{aligned}$$

which converges to 0 as $|\pi| \rightarrow 0$, due to condition (40). This completes the proof of the proposition. \square

If the kernel $K(t, s)$ satisfies conditions (i) and (ii'), then (40) can be replaced by the hypothesis

$$\sup_{s \leq T} \int_0^s E |D_r u_s|^{2q} dr < \infty$$

for some $q > 1/\alpha$.

REMARK 1. The divergence operator is local in the space $\mathbb{D}^{1,2}(\mathcal{H})$. As a consequence, using a standard localization argument, we can generalize the preceding Itô formulas to the case of functions F that are twice (or three times) continuously differentiable, but without any growth restriction on the derivatives or the function itself.

8. Itô formulas for fractional Brownian motion. In this section we will discuss the particular case of the fractional Brownian motion. We first briefly recall some elements of the deterministic fractional calculus that are useful to characterize the RKHS space of the fractional Brownian motion with parameter $H = \frac{1}{2} - \alpha \in (0, \frac{1}{2})$ and related processes. We refer to [20] for a complete survey of this subject. The right-sided fractional Riemann–Liouville integral of order $\alpha \in (0, 1)$ of an integrable function f on $[0, T]$ is given at almost all s by

$$I_{T-}^\alpha f(s) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_s^T (r - s)^{\alpha-1} f(r) dr.$$

We will denote by $I_{T-}^\alpha(L^2)$ the class of functions f in $L^2([0, T])$ which may be represented as an I_{T-}^α -integral of some function $\phi \in L^2([0, T])$. If $f \in I_{T-}^\alpha(L^2)$, the function ϕ such that $f = I_{T-}^\alpha \phi$ is unique in L^2 and it agrees with the right-sided Riemann–Liouville derivative of f of order α given by

$$D_{T-}^\alpha f(s) = \frac{(-1)^\alpha}{\Gamma(1 - \alpha)} \left(\frac{f(s)}{(T - s)^\alpha} - \alpha \int_s^T \frac{f(r) - f(s)}{(r - s)^{\alpha+1}} dr \right).$$

Note that $I_{T-}^\alpha(L^2)$ is a Hilbert space with the scalar product

$$\langle f, g \rangle_{\alpha, 2} = \langle f, g \rangle_2 + \langle D_{T-}^\alpha f, D_{T-}^\alpha g \rangle_2.$$

A function $f \in L^2([0, T])$ belongs to $I_{T-}^\alpha(L^2)$ if and only if

$$\int_0^T \frac{|f(s)|^2}{(T - s)^{2\alpha}} ds < \infty,$$

and the integral

$$\int_{s+\varepsilon}^T \frac{f(r) - f(s)}{(r - s)^{\alpha+1}} dr$$

converges in $L^2([0, T])$ as $\varepsilon \rightarrow 0$, as a function of s .

Consider now the case of the fractional Brownian motion. We recall that the fractional Brownian motion (fBm) of Hurst parameter $H \in (0, 1)$ (see, for instance, [15]) is as a centered Gaussian process $B^H = \{B_t^H, 0 \leq t \leq T\}$ with covariance

$$(41) \quad R_H(t, s) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}).$$

It is proved in [16] that this process can be expressed as $B_t^H = \int_0^t s^{H-1/2} dY_s$, where Y_s is the Gaussian process defined by

$$Y_t = c_H \int_0^t (t - s)^{H-1/2} s^{1/2-H} dW_s,$$

c_H is a normalizing constant given by

$$c_H = \left(\frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2})\Gamma(2 - 2H)} \right)^{1/2}$$

and W is a Wiener process. As a consequence, using the results obtained in Section 2, we can deduce the following integral representation for the fractional Brownian motion:

$$B_t^H = \int_0^t K_H(t, s) dW_s,$$

where $K_H(t, s)$ is the kernel

$$(42) \quad \begin{aligned} K_H(t, s) &= c_H(t - s)^{H-1/2} \\ &+ c_H\left(\frac{1}{2} - H\right) \int_s^t (u - s)^{H-3/2} \left(1 - \left(\frac{s}{u}\right)^{1/2-H}\right) du. \end{aligned}$$

From (42) we obtain (see also [3])

$$(43) \quad \frac{\partial K_H}{\partial t}(t, s) = c_H\left(H - \frac{1}{2}\right) \left(\frac{s}{t}\right)^{1/2-H} (t - s)^{H-3/2}.$$

Notice that if $H > \frac{1}{2}$ then the kernel $K_H(t, s)$ is regular and if $H < \frac{1}{2}$ this kernel is singular.

Regular case $H > \frac{1}{2}$. If $H > \frac{1}{2}$ the kernel K_H has the simpler expression

$$(44) \quad K_H(t, s) = c_H\left(H - \frac{1}{2}\right) s^{1/2-H} \int_s^t (u - s)^{H-3/2} u^{H-1/2} du.$$

From (43) it follows that the operator K_H^* is given by

$$(K_H^* h)_s = c_H\left(H - \frac{1}{2}\right) \frac{\Gamma(H - \frac{1}{2})}{(-1)^{1/2-\alpha}} s^{-\alpha} I_{T^-}^\alpha(h_\alpha)s,$$

where h_α denotes the function $h_\alpha(x) = x^\alpha h(x)$ and $\alpha = H - \frac{1}{2}$.

On the other hand, the kernel $K_H(t, s)$ satisfies condition (K4), and Theorem 2 provides the following version of the Itô formula:

THEOREM 5. *Let B^H be a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Let F be a function of class $C^2(\mathbb{R})$ such that*

$$\max\{|F(x)|, |F'(x)|, |F''(x)|\} \leq ce^{\lambda x^2}$$

for any $\lambda < \frac{1}{4}T^{-2H}$. Then the process $F'(B_t)$ belongs to $\mathbb{D}^{1,2}(\mathcal{H})$ and for each $t \in [0, T]$ the following formula holds:

$$(45) \quad F(B_t^H) = F(0) + \int_0^t F'(B_s^H) \delta B_s^H + \frac{1}{H} \int_0^t F''(B_s^H) s^{2H-1} ds.$$

Moreover, the following version of the Itô formula for the Stratonovich integral holds:

$$F(B_t^H) = F(0) + \int_0^t F'(B_s^H) dB_s^H.$$

On the other hand, the kernel $K_H(t, s)$ satisfies conditions (i), (ii') and (iii') with $\alpha = H - \frac{1}{2}$. By Proposition 3, if a process $u = \{u_t, t \in [0, T]\}$ is bounded in the norm of the space $\mathbb{D}^{1,p}$ for some $p \geq 2$, then u belongs to the space $\mathbb{D}^{1,p}(\mathcal{H})$ and the indefinite integral $X_t = \int_0^t u_s \delta B_s^H$ satisfies $E|X_t - X_s|^p \leq C|t - s|^{(p/2)(H+1/2)}$. Furthermore, if u is bounded in $\mathbb{D}^{2,4}$, then the Itô formula (32) holds for $F(X_t)$, provided F belongs to $C_b^2(\mathbb{R})$.

Singular case $H < \frac{1}{2}$. In this case the kernel can be written as

$$(46) \quad K_H(t, s) = c_H(t - s)^{H-1/2} + s^{H-1/2} F_1\left(\frac{t}{s}\right),$$

where

$$F_1(z) = c_H\left(\frac{1}{2} - H\right) \int_0^{z-1} \theta^{H-3/2} (1 - (\theta + 1)^{H-1/2}) d\theta.$$

PROPOSITION 6. *The RKHS space \mathcal{H} is the space $I_{T-}^\alpha(L^2)$ and the operator K^* is given by*

$$(K^*h)_s = c_H s^\alpha D_{T-}^\alpha (h_{-\alpha})_s,$$

where $h_{-\alpha}$ denotes the function $h_{-\alpha}(x) = x^{-\alpha}h(x)$.

PROOF. We have, using (46),

$$\begin{aligned} (K^*h)_s &= c_H \left[(T-s)^{-\alpha} h_s - \alpha \int_s^T (h_r - h_s)(r-s)^{-\alpha-1} dr \right] + s^{-\alpha-1} \int_s^T h_r F_1\left(\frac{r}{s}\right) dr \\ &= c_H (T-s)^{-\alpha} h_s - c_H \alpha \int_s^T (h_r - h_s)(r-s)^{-\alpha-1} dr \\ &\quad + \alpha c_H \int_s^T h_r (r-s)^{-\alpha-1} \left(1 - \left(\frac{r}{s}\right)^{-\alpha}\right) dr. \end{aligned}$$

Notice that

$$\begin{aligned} & \int_0^T \left(\int_s^T h_r (r-s)^{-\alpha-1} \left(1 - \left(\frac{r}{s} \right)^{-\alpha} \right) dr \right)^2 ds \\ & \leq C \int_0^T \int_s^T h_r^2 (r-s)^{-\alpha-1} \left(1 - \left(\frac{r}{s} \right)^{-\alpha} \right) dr ds \\ & \leq C' \int_0^T h_r^2 dr. \end{aligned}$$

Hence, K^*h is square integrable if and only if h belongs to $I_{T-}^\alpha(L^2)$. Finally, a simple computation yields

$$\begin{aligned} (K^*h)_s &= c_H (T-s)^{-\alpha} h_s - c_H s^\alpha \alpha \int_s^T (r^{-\alpha} h_r - s^{-\alpha} h_s) (r-s)^{-\alpha-1} dr \\ &= c_H s^\alpha D_{T-}^\alpha (r^{-\alpha} h)_s. \end{aligned} \quad \square$$

We claim that this kernel satisfies conditions (i), (ii) and (iii) of Section 5 with $\alpha = \frac{1}{2} - H$. Condition (i) is clear from the expression (46). Property (43) implies (ii) with the constant $c = c_H (\frac{1}{2} - H)$. Finally, (iii) is true because for any $s < t$ we have

$$\int_s^t K_H(t, r)^2 dr \leq E |B_t^H - B_s^H|^2 = |t - s|^{1-2\alpha}.$$

As a consequence, the process B^H satisfies condition (K2) if $H > \frac{1}{4}$. On the other hand, K_H also satisfies condition (K3). Indeed, $R_H(s, s) = s^{1-2\alpha}$ is increasing and from (46) it follows that $\int_t^s K_H(s + \varepsilon, r) dr$ has bounded variation, uniformly in ε . Thus, by Theorem 1, if $H > \frac{1}{4}$ and F satisfies the assumptions of Theorem 5, the process $F'(B_t)$ belongs to the domain of δ^B , and the Itô formula (45) holds.

Fix $p \geq 2$. By Proposition 1, if u is a Hölder continuous process in the norm $\|\cdot\|_{1,p}$ of order larger than $\frac{1}{2} - H$, then the indefinite stochastic integral $X_t = \int_0^t u_s \delta B_s^H$ is Hölder continuous of order H in the norm $\|\cdot\|_p$. This means that the indefinite stochastic integrals possess the same order of continuity as the fBm.

Note that conditions (iv) and (v) of Proposition 2 are satisfied. By Proposition 2, the Itô formula (29) holds for the indefinite stochastic integral $X_t = \int_0^t u_s \delta B_s^H$ if $H > \frac{1}{4}$, u is an adapted process in $\mathbb{D}^{2,2}$ which satisfies condition (C1) for $\alpha = \frac{1}{2} - H$ and u is Hölder continuous in the norm $\|\cdot\|_p$ of order larger than $1 - 2H$.

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E. ALÒS
 DEPARTAMENT DE MATEMÀTIQUES
 UNIVERSITAT DE AUTÒNOMA DE BARCELONA
 08193 BELLATERRA
 SPAIN
 E-MAIL: alos@cerber.mat.ub.es

O. MAZET
 D. NUALART
 FACULTAT DE MATEMÀTIQUES
 UNIVERSITAT DE BARCELONA
 GRAN VIA 585
 08007 BARCELONA
 SPAIN