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STRONGLY SUPERMEDIAN KERNELS AND REVUZ MEASURES[1]

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In the frame of Borel right Markov processes, we investigate, following an analytical point of view, the Revuz correspondence between classes of potential kernels and their associated measures, improving upon the results of Revuz, Azéma, Getoor and Sharpe, Fitzsimmons, Fitzsimmons and Getoor and Dellacherie, Maisonneuve and Meyer. In the probabilistic approach of the problem, the kernels that occur are the potential operators of different types of homogeneous random measures. We completely characterize the hypothesis (B) of Hunt in terms of Revuz measures.

Introduction. The aim of this paper is to investigate, following an analytical point of view, the well-known Revuz correspondence between different classes of "potential kernels" and their associated measures. In the probabilistic approach of this problem (developed by Revuz, Azéma, Getoor and Sharpe, Fitzsimmons, Fitzsimmons and Getoor and Dellacherie, Maisonneuve and Meyer), these kernels are precisely the potential operators associated with the corresponding classes of homogeneous random measures. Our new approach allows us to dig deeper and improve upon the results obtained up to now in this direction.

Let $\mathscr{U} = (U_\alpha)_{\alpha>0}$ be the resolvent of a transient Borel right process with state space X and let ξ be a $\mathscr{U}$-excessive measure. With every strongly supermedian kernel V on X we associate the so-called *Revuz measure* ν_V^ξ by

$$\nu_V^\xi(f) = L(\xi, Vf) := \sup\Big\{\nu(Vf)/\nu \circ U \le \xi\Big\}.$$

In the classical case, if $g(x, y)$ is "the Green function" with respect to a reference measure ξ [i.e., $Uf(x) = \int g(x, y)f(y)\,d\xi(y)$], then for each positive measure μ on X such that G_μ is the excessive kernel on X given by $G_\mu f(x) = \int g(x, y)f(y)\,d\mu(y)$, the Revuz measure of G_μ is μ. Moreover, the equality $\nu_{G_\mu}^{t\cdot\xi}(f) = \nu_{G_\mu}^{\xi}(tf)$ (which holds for every co-excessive function t), written in the form

$$L(t\cdot\xi, Vf) = L\big(\xi, V(tf)\big),$$

where $V = G_\mu$, gives the *Revuz formula*.

In our general frame, the proper kernel V will be strongly supermedian (hence not necessarily excessive) and the measure ξ will be $\mathscr{U}$-excessive

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(no more a reference measure). In the papers dealing with this subject, the following two problems are considered:

1. Characterize those measures on X which can be the Revuz measures associated with different classes of kernels.
2. When does the Revuz measure completely determine its generating kernel?

Concerning the first problem, we give results not only for measures charging no ξ-semipolar set but also for measures charging no ξ-polar set (see Sections 3 and 4) or even charging no set which is both ξ-polar and ρ-negligible (where $\rho \circ U$ is the potential component of ξ). This is the reason that the wider class (than that of excessive kernels) of strongly supermedian kernels should be taken into account.

It turns out that the second problem is intimately related to the Motoo–Mokobodzki property for kernels (Section 2) and the hypothesis (B) of Hunt for the *balayage* operation (Section 5).

In Section 2 we present results about the *regular strongly supermedian kernels*, completing the work of Mokobodzki [29] and [30] and Feyel [19]. This class of kernels V corresponds to the potential operators of the random measures k which are homogeneous on $[0, \infty)$ (cf. [17], [20] and [32]),

$$Vf(x) = E^x\left(\int_{[0,\infty)} f \circ \mathsf{X}_t k(dt)\right).$$

Following the terminology of Azéma [2], V is the potential kernel of a d-additive functional $(\mathsf{A}_t)_{t\geq 0}$,

$$Vf(x) = E^x\left(\int_{[0,\infty)} f \circ \mathsf{X}_t \, d\mathsf{A}_t\right).$$

We remark that each regular strongly supermedian kernel V enjoys the Motoo–Mokobodzki property [i.e., or every strongly supermedian function s such that $s \prec Vf$, with Vf bounded, is of the form $s = V(gf)$, where $0 \leq g \leq 1$] and V is uniquely determined by $V1$ if it is bounded; we have denoted by $\prec$ the specific order in the cone of strongly supermedian functions.

In Section 3 we show that the Revuz measures of regular strongly supermedian kernels are exactly the measures charging no set that is both ξ-polar and ρ-negligible and, moreover, these measures completely determine their generating kernels. This result extends a similar one of Azéma [2]. Note that Fitzsimmons [20] proved an analogous assertion, in terms of Palm measures.

In Sections 2 and 3 we also consider the *semiregular excessive kernels*, which are the excessive kernels V on X of the form $Vf = \widehat{Wf}$, where W is a regular strongly supermedian kernel and $\hat{s}$ denotes the $\mathscr{U}$-excessive regularization of the strongly supermedian function s. These kernels V correspond to the potential operators associated with the random measures k which are homogeneous on $[0, \infty)$ in the following way:

$$Vf(x) = E^x\left(\int_{(0,\infty)} f \circ \mathsf{X}_t k(dt)\right)$$

(see [17]). The Revuz measures of the semiregular excessive kernels are characterized by the property of charging no ξ-polar set. In addition, as in the previous case, the problem (2) has an affirmative answer. This is the analytic version of a probabilistic result from [17].

In Section 4 we study the class of *natural excessive kernels*, which are the proper excessive kernels V such that $B^G Vf = Vf$ for each Ray open set G and positive measurable function f, vanishing outside G. If $\mathscr{U}$ is the resolvent of a special standard process on X (cf. [24] and [32]), then the natural excessive kernels correspond to the potential operators of the natural additive functionals (in the terminology of [12]); see also [17], [24] and [26]. The natural excessive kernels have been considered in [2], [11] and [22]. We characterize the case when each measure charging no ξ-polar set is the Revuz measure of a natural excessive kernel. For this class of kernels the answer to the problem (2) is negative. We show that we have a positive answer for the problem (2) if and only if the Motoo–Mokobodzki property (with respect to ξ) holds for each natural excessive kernel (see Theorem 4.3).

In Section 5 we study the hypothesis (B) of Hunt (i.e., $B^G B^K s = B^K s$ for each $\mathscr{U}$-excessive function s and Ray open set G with $G \supset K$). We prove that the hypothesis (B) of Hunt is equivalent to the fact that every semiregular excessive kernel on X is natural. Consequently, we show that if the hypothesis (B) of Hunt holds with respect to ξ, then each semiregular excessive kernel V enjoys the following Motoo–Mokobodzki property: if s is a regular strongly supermedian function such that $\hat{s} \prec Vf$, with Vf bounded, then $\hat{s} = V(gf)$ ξ-q.e., where $0 \leq g \leq 1$. This improves a result of Azéma [2], obtained under restrictive assumptions [Hunt process satisfying the hypothesis (B) of Hunt]. Notice also that for each excessive measure ξ there exists a semipolar subset M of X such that every semiregular excessive kernel on $X \setminus M$ is equal ξ-quasi everywhere with a natural excessive kernel.

The results from this paper have been partially announced in [9].

1. Preliminaries.

Throughout the paper, $\mathscr{U} = (U_\alpha)_{\alpha>0}$ will be a proper sub-Markovian resolvent of kernels on a Lusin measurable space $(X, \mathscr{B})$ such that the set $\mathscr{E}_{\mathscr{U}}$ of all $\mathscr{B}$-measurable $\mathscr{U}$-excessive functions on X which are $\mathscr{U}$-almost everywhere ($\mathscr{U}$-a.e.) finite is min-stable, contains the positive constant functions and generates the σ-algebra $\mathscr{B}$ (see, e.g., [18]). The *fine topology* is the topology on X generated by $\mathscr{E}_{\mathscr{U}}$. We suppose that the set X is *semisaturated* with respect to $\mathscr{U}$; that is, every $\mathscr{U}$-excessive measure dominated by a potential is also a potential (see [5]). This property is equivalent to the existence of a (Borel) right process on X, having $\mathscr{U}$ as associated resolvent. We denote by Exc the set of all $\mathscr{U}$-excessive measures on X.

If $A \subset X$ and s is a $\mathscr{U}$-excessive function on X (i.e., s is universally measurable and $\alpha U_\alpha s \nearrow s$ when $\alpha \nearrow \infty$), then the *réduite of s on A* is the function $R^A s$ on X defined by

$$R^A s := \inf\{t/t \ \mathscr{U}\text{-excessive}, \ s \leq t \text{ on } A\}.$$

If, moreover, $A \in \mathscr{B}$, then $R^A s$ is universally measurable (cf. [4]) and we denote by $B^A s$ its $\mathscr{U}$-excessive regularization.

Let θ be a finite measure on X. We say that a set $M \in \mathscr{B}$ is *θ-polar* if $\theta(B^M 1) = 0$. An arbitrary subset of X is called θ-polar if it is a subset of a $\mathscr{B}$-measurable θ-polar set. A property is said to hold *θ-quasi everywhere* (*θ-q.e.*) if the set where it does not hold is θ-polar and θ-negligible.

We say that a positive numerical function f on X is *nearly analytic* (resp. *nearly Borel*) if for each finite measure λ on X there exist two positive analytic (resp. $\mathscr{B}$-measurable) numerical functions g, h on X such that $g \leq f \leq h$ and the set $[g < h]$ is λ-polar and λ-negligible. It is known that every $\mathscr{U}$-excessive function on X is nearly Borel.

Recall that a set $M \in \mathscr{B}$ is *thin* at a point $x \in X$ if there exists $s \in \mathscr{E}_{\mathscr{U}}$ such that $B^M s(x) < s(x)$. An arbitrary subset of X is called thin at x if it is a subset of a $\mathscr{B}$-measurable set which is thin at x. A subset of X is said to be *totally thin* if it is thin at each point of X. A *semipolar* set is a countable union of totally thin sets. A set $A \in \mathscr{B}$ is termed *θ-semipolar* if it is of the form $A = A_0 \cup A_1$, where $A_0, A_1 \in \mathscr{B}$ with A_0 θ-polar and A_1 semipolar. A subset of X is called θ-semipolar if it is a subset of a $\mathscr{B}$-measurable θ-semipolar set. Recall that if A is a nearly analytic θ-semipolar set, then there exists a finite measure μ on X, carried by A, such that $\mu \leq_{\varepsilon_{\mathscr{U}}} \theta$ and such that for every $\mathscr{B}$-measurable subset M of A we have $\mu(M) = 0$ if and only if M is θ-polar and θ-negligible. Such a measure μ will be called a *Dellacherie measure* associated with θ and A (see [10]).

For each nearly analytic subset A of X and every $\mathscr{U}$-excessive function s on X, the *réduite* $R^A s$ of s on A is nearly analytic (see [8]) and we have

$$\mu(R^A s) = \inf\{\mu(t)/t\ \mathscr{U}\text{-excessive},\ s \leq t \text{ on } A\}$$

for each finite measure μ on X with $\mu(s) < \infty$. On the other hand, for such a measure μ one has $\mu(R^A s) = \sup\{\nu(s1_A)/\nu \leq_{\varepsilon_{\mathscr{U}}} \mu\}$.

If μ is a finite measure on X, then the functional on $\mathscr{E}_{\mathscr{U}}$ given by $s \longmapsto \mu(R^A s)$ is additive, increasing and σ-continuous in order from below and therefore, since X is semisaturated, there exists a measure on X denoted by R^A_μ such that $\mu(R^A s) = R^A_\mu(s)$ for all $s \in \mathscr{E}_{\mathscr{U}}$. If we denote by $\overline{A}^f$ the fine closure of A, then $\overline{A}^f$ is also nearly analytic and for every finite measure μ on X the measure R^A_μ is carried by $\overline{A}^f$ and we have $R^A s = R^{\overline{A}^f} s$, $R^A_\mu = R^{\overline{A}^f}_\mu$.

We denote by R^A the kernel on X such that $R^A f(x) = R^A_{\varepsilon_x}(f)$ for each bounded $\mathscr{B}$-measurable function f on X. Note that generally the function $R^A f$ is not $\mathscr{B}$-measurable even if A is $\mathscr{B}$-measurable but it is nearly analytic.

If A is a nearly analytic set and s is a $\mathscr{U}$-excessive function, then the $\mathscr{U}$-excessive regularization $B^A s$ of $R^A s$, called the *balayage of s on A*, is the infimum in the set of all $\mathscr{U}$-excessive functions of the subset $\{t/t\ \mathscr{U}\text{-excessive}, s \leq t \text{ on } A\}$. We also have $B^A s = R^A s$ on $X \setminus A$ and the set $[B^A s < R^A s]$ is semipolar. As before, for every finite measure μ on X there exists a measure on X, denoted by B^A_μ, such that $\mu(B^A s) = B^A_\mu(s)$ for each $s \in \mathscr{E}_{\mathscr{U}}$. We denote by B^A the kernel on X such that $B^A f(x) = B^A_{\varepsilon_x}(f)$ for every bounded $\mathscr{B}$-measurable function f on X. The function $B^A f$ is nearly Borel.

Recall now some considerations concerning the Ray topology on X. Since the initial kernel U of the resolvent $\mathscr{U} = (U_\alpha)_{\alpha>0}$ is proper, there exists a bounded sub-Markovian resolvent $\mathscr{V} = (V_\alpha)_{\alpha>0}$ on X such that $\mathscr{E}_{\mathscr{U}} = \mathscr{E}_{\mathscr{V}}$. A *Ray cone* will be a subcone $\mathscr{R}$ of the bounded $\mathscr{U}$-excessive functions which is min-stable, separable in the uniform norm, generates the σ-algebra $\mathscr{B}$ and, moreover, $1 \in \mathscr{R}$, $V((\mathscr{R} - \mathscr{R})_+) \subset \mathscr{R}$, $V_\alpha(\mathscr{R}) \subset \mathscr{R}$, $\alpha > 0$. A *Ray topology* is the topology on X generated by a Ray cone. We consider the Ray compactification Y of X with respect to $\mathscr{R}$ (see, e.g., [18]). Since $(X, \mathscr{B})$ is a Lusin measurable space, it follows that X is a Borel subset of Y and $\mathscr{B}(Y)|_X = \mathscr{B}$, where $\mathscr{B}(Y)$ denotes the σ-algebra of all Borel subsets of Y. We denote by $\mathscr{F}(Y)$ [resp. $\mathscr{F}(X)$, $\mathscr{F}^*(X)$] the set of all positive numerical Borel-measurable (resp. $\mathscr{B}$-measurable, universally measurable) functions on Y (resp. on X).

In the remainder of the paper θ will be a finite measure on X. Also, we need sometimes to fix a bounded (Ray continuous) $\mathscr{U}$-excessive function p_0 of the form $p_0 = Uf_0$, where f_0 is $\mathscr{B}$-measurable and $0 < f_0 \le 1$.

Recall that a positive numerical function s on X is called *strongly supermedian* (with respect to $\mathscr{E}_{\mathscr{U}}$) if for each two finite measures μ, ν on X we have

$$\mu \le_{\varepsilon_{\mathscr{U}}} \nu \Rightarrow \int^* s\, d\mu \le \int^* s\, d\nu.$$

It is known that (cf. [10] and [19]) if s is a strongly supermedian function which is nearly analytic, then it is nearly Borel. Also, if f is a numerical function on X which is nearly Borel, then it will be strongly supermedian if and only if $R^K f \le f$ for every Ray compact subset K of X.

If f is a positive numerical function on X, then the *réduite of* f will be the function Rf defined by

$$Rf = \inf\{s/s \ge f, s \text{ strongly supermedian}\}.$$

It is known that (see [18]) if f is analytic, then Rf is also analytic and $\lambda(Rf) = \sup\{\mu(f)/\mu \le_{\varepsilon_{\mathscr{U}}} \lambda\}$ for each finite measure λ on X. By Proposition 1.2 in [8] it follows that for every nearly analytic set A and $\mathscr{U}$-excessive function s on X we have $R^A s = R(s1_A)$. If, moreover, A is finely closed, then $R^A s = R(s1_A)$ for each strongly supermedian function s (cf. [10]). Since for each $\mathscr{U}$-excessive function s and nearly analytic set A we have $R^A s = R^{\overline{A}^f} s$ and $R(s1_A) = R(s1_{\overline{A}^f})$, we may consider this last formula as an extension of the preceding one. We note also that if s is a strongly supermedian function on X and K is a Ray compact subset of X such that $s|_K$ is bounded and Ray upper semicontinuous, then (see, e.g., [13]) $R(s1_K) = R^K s$ and there exists a decreasing sequence (s_n) in $\mathscr{R}$ with $R^K s = \inf_n s_n$.

We denote by $\mathscr{S}$ the convex cone of all strongly supermedian nearly Borel finite functions on X. Obviously, for each sequence (s_n) in $\mathscr{S}$ we have $\inf_n s_n \in \mathscr{S}$. If, in addition, the sequence is increasing, then $\sup_n s_n \in \mathscr{S}$, provided it is finite.

The following result is a version of Theorem 3.3 in [10].

THEOREM 1.1. *The convex cone $\mathscr{S}$ is a cone of potentials on X. In particular, for each sequence in $\mathscr{S}$ there exists its infimum with respect to the specific order.*

PROOF. It is sufficient to show that, for each $s, t \in \mathscr{S}$, $t \le s$, we have $R(s-t) \in \mathscr{S}$ and $R(s-t) \prec s$ ($\prec$ denotes the specific order in $\mathscr{S}$). Since for each positive analytic function f on X the function Rf is a strongly supermedian analytic function, it follows that $R(s-t)$ belongs to $\mathscr{S}$. If μ is a finite measure on X, then $\mu(R(s-t)) = \sup\{\nu(s-t)/\nu \le_{\varepsilon_{\mathscr{U}}} \mu\}$ (see [18]). For $s \in \mathscr{S}$ we denote by $\tilde{s}$ the θ-supermedian functional defined by $\tilde{s}(\nu) = \nu(s)$, where θ is a positive finite measure on X (see [10]). It is known that the set of all θ-supermedian functionals is a cone of potentials and from the previous considerations we get $R(\tilde{s} - \tilde{t}\,) = R(\widetilde{s-t})$. Hence for $\mu \le_{\varepsilon_{\mathscr{U}}} \theta$ we have $\mu(s - R(s-t)) = (\tilde{s} - R(\widetilde{s-t}))(\mu) = (\tilde{s} - R(\tilde{s} - \tilde{t}))(\mu) \le (\tilde{s} - R(\tilde{s} - \tilde{t}))(\theta) = (\tilde{s} - R(\widetilde{s-t}))(\theta)$ and therefore the function $s - R(s-t)$ belongs to $\mathscr{S}$. □

2. Strongly supermedian kernels.

DEFINITION. A kernel $V : \mathscr{F}(Y) \longrightarrow \mathscr{F}^*(X)$ is called *strongly supermedian* (resp. *excessive*) if for every $f \in \mathscr{F}(Y)$ the function Vf on X is nearly Borel strongly supermedian (resp. excessive).

A strongly supermedian kernel V on X is called *regular* if it is proper and for every $f \in \mathscr{F}(X)$ and each strongly supermedian function s on X we have $Vf \le s$ whenever $Vf \le s$ on $[f > 0]$. An excessive kernel V on X is called *semiregular* if there exists a regular strongly supermedian kernel W on X such that $Vf = \widehat{Wf}$ for all $f \in \mathscr{F}(X)$. (For a strongly supermedian function s we have, as usual, denoted by $\hat{s}$ its $\mathscr{U}$-excessive regularization.) In this case we write $V = \widehat{W}$.

REMARKS. (i) The notion of a semiregular excessive kernel is suggested by the paper [2].

(ii) Let V be a proper strongly supermedian kernel on X. Then V will be regular if and only if for all $f \in \mathscr{F}(X)$ and each $\mathscr{U}$-excessive function s on X we have $Vf \le s$ whenever $Vf \le s$ on $[f > 0]$.

(iii) If V is a regular strongly supermedian kernel, then, for every $u \in \mathscr{E}_{\mathscr{U}}$, $u > 0$, there exists $f \in \mathscr{F}(X)$, $0 < f \le 1$, with $Vf \le u$.

(iv) If V is a strongly supermedian kernel and f is a positive universally measurable function on Y, then Vf is a nearly Borel strongly supermedian function on X.

(v) If V is a regular strongly supermedian kernel and g is a finite positive universally measurable function on X, then the kernel $g \cdot V$ on X, defined by $g \cdot V(f) := V(gf)$, $f \in \mathscr{F}(X)$, is also a regular strongly supermedian kernel.

(vi) If (V_n) is a sequence of regular strongly supermedian (resp. excessive) kernels, then $\sum_n V_n$ is also a regular strongly supermedian (resp. excessive) kernel, provided it is proper.

(vii) If V is a bounded strongly supermedian kernel on X, then it is regular if and only if there exists a sub-Markovian resolvent $\mathscr{V}$ having V as its initial kernel and such that every strongly supermedian function (or only every $\mathscr{U}$-excessive function) is $\mathscr{V}$-supermedian. Consequently, each regular strongly supermedian kernel V possesses the following Motoo–Mokobodzki property: for each $s \in \mathscr{S}$ and $f \in \mathscr{F}$ such that Vf is bounded and $s \prec Vf$ there exists $g \in \mathscr{F}$, $0 \leq g \leq 1$, with $s = V(gf)$ (cf. [18] and [28]).

LEMMA 2.1. *If $s \in \mathscr{S}$ is bounded, then there exists a strongly supermedian kernel V on Y such that $V1 = s$ and*

$$V(1_F) = \curlywedge \{s \curlywedge R^{G\cap X} s / G \text{ open}, G \supset F\}$$

for each Ray closed subset F of Y.

PROOF. If F is a closed subset of Y, then we put $s_F := \curlywedge\{s \curlywedge R^{G\cap X} s/G$ open, $G \supset F\}$. Since for every $A \in \mathscr{B}$ we have $s \curlywedge R^A s = \curlyvee\{t \in \mathscr{S}/t \prec s, R^A t = t\}$ it follows that if $A_1, A_2 \in \mathscr{B}$ and $A_1 \subset A_2$, then $s \curlywedge R^{A_1} s \prec s \curlywedge R^{A_2} s$. Therefore, if F_1, F_2 are two closed subsets of Y, then we get $s_{F_1\cup F_2} = \curlywedge\{s \curlywedge R^{G\cap X} s/G$ open, $G \supset F_1 \cup F_2\} \prec \curlywedge\{s \curlywedge R^{(G_1\cup G_2)\cap X} s/G_1, G_2$ open, $G_i \supset F_i, i = 1, 2\} \prec \curlywedge\{s \curlywedge R^{G_1\cap X} s + s \curlywedge R^{G_2\cap X} s/G_1, G_2$ open, $G_i \supset F_i, i = 1, 2\} = s_{F_1} + s_{F_2}$. On the other hand, if, moreover, $F_1 \cap F_2 = \varnothing$, then there exist two open sets G_1, G_2 in Y such that $\overline{G}_1 \cap \overline{G}_2 = \varnothing$ and $G_1 \supset F_1$, $G_2 \supset F_2$. Consequently, we obtain $s_{F_i} \prec s \curlywedge R^{G_i\cap X} s \prec s \curlywedge R^{\overline{G}_i\cap X} s$, $i = 1, 2$. If $u := s_{F_1} \curlywedge s_{F_2}$, then $R^{\overline{G}_i\cap X} u = u$, $i = 1, 2$, and, u being bounded, we deduce that $u = 0$. Hence $s_{F_1} + s_{F_2} = s_{F_1} \curlyvee s_{F_2} \prec s_{F_1\cup F_2}$, $s_{F_1\cup F_2} = s_{F_1} + s_{F_2}$. From the preceding considerations it follows that for every $x \in X$ the map $F \longmapsto s_F(x)$, defined on the set of all closed subsets of Y, is additive on pairwise disjoint sets and right continuous. It is easy to see that this map is strongly subadditive. Therefore, by standard arguments from capacity theory, we deduce that the map $A \overset{\theta_x}{\longmapsto} \sup\{s_F(x)/F \subset A, F \text{ compact}\}$ is a measure on Y. If for all $x \in X$ and $f \in \mathscr{F}(Y)$ we set $Vf(x) := \int f\, d\theta_x$, we conclude that V defines a strongly supermedian kernel on Y such that $V1 = s$ and $V(1_F) = s_F$. □

THEOREM 2.2. *Let V, W be two regular strongly supermedian kernels such that $Vf \prec Wf$ for all $f \in \mathscr{F}(X)$* [*or only for one function $g_0 \in \mathscr{F}(X)$, $0 < g_0 \leq 1$ for which Wg_0 is bounded*]. *Then there exists $g \in \mathscr{F}^*(X)$, $0 \leq g \leq 1$, such that $V = g \cdot W$.*

PROOF. Let $g_0 \in \mathscr{F}(X)$, $0 < g_0 \leq 1$, such that Wg_0 is bounded and $Vg_0 \prec Wg_0$. The function Vg_0 being excessive with respect to the kernel $g_0 \cdot W$, by the Motoo–Mokobodzki property we deduce that there exists $g \in \mathscr{F}^*(X)$, $0 \leq g \leq 1$, with $Vg_0 = W(g_0 g)$. Since $g_0 \cdot V$ and $g_0 g \cdot W$ are two bounded regular supermedian kernels such that $g_0 \cdot V1 = g_0 g \cdot W1$, we conclude that $g_0 \cdot V = g_0 g \cdot W$ and, finally, $V = g \cdot W$. $\square$

THEOREM 2.3. *Let V be a strongly supermedian kernel on X such that $V1 \in \mathscr{S}$. Then V is regular if and only if for every Ray closed* (*or only Ray compact*) *subset F of X we have $V(1_F) = V1 \curlywedge R^F V1$.*

PROOF. Assume that V is regular. Then for every Ray closed subset F of X we have $R^F V(1_F) = V(1_F)$ and therefore $V(1_F) \prec V1$, $V(1_F) \prec R^F V1$ and, consequently, $V(1_F) \prec V1 \curlywedge R^F V1$. If we set $u := V1 \curlywedge R^F V1 \in \mathscr{S}$ and K is a Ray compact subset of $X \setminus F$, then $R^K(u \curlywedge V(1_K)) = u \curlywedge V(1_K)$, $R^F(u \curlywedge V(1_K)) = u \curlywedge V(1_K)$. Therefore, from $K \cap F = \varnothing$, we get $u \curlywedge V(1_K) = 0$. We conclude that $u \curlywedge V(1_{X\setminus F}) = 0$, $u \prec V(1_K)$. Conversely, suppose that for every Ray compact subset F of X we have $V(1_F) = V1 \curlywedge R^F V1$ and let $f \in \mathscr{F}(X)$ bounded, $s \in \mathscr{E}_{\mathscr{U}}$ such that $Vf \leq s$ on $[f > 0]$. Since $Vf = \sup\{V(f1_K)/K \subset [f > 0],\ K \text{ Ray compact}\}$ and $R^K V(1_K) = V(1_K)$, it follows that $R^K V(f1_K) = V(f1_K)$ and, finally, $s \geq V(f1_K)$, $K \subset [f > 0]$, K Ray compact, $s \geq Vf$. $\square$

THEOREM 2.4. *Let $s \in \mathscr{S}$ be Ray upper semicontinuous such that there exists a Ray compact subset K of X with $R^K s = s$. Then there exists a regular strongly supermedian kernel V on X such that $V1 = V(1_K) = s$.*

PROOF. By Lemma 2.1 there exists a strongly supermedian kernel V on Y such that $V1 = s$ and $V(1_F) = \curlywedge\{s \curlywedge R^{G\cap X}s/G \text{ open},\ G \supset F\}$ for each closed subset F of Y. It will be sufficient to show that $V(1_F) = s \curlywedge R^F s$ for every Ray compact subset F of X. Indeed, in this case $V(1_K) = s$ and therefore V is a kernel on X and by Theorem 2.3 it follows that V is regular. Obviously, for every Ray compact subset F of X and each open subset G of Y, $G \supset F$, we get $s \curlywedge R^{F\cap K}s \prec s \curlywedge R^F s \prec R^{G\cap X}s$. Let now G' be an open subset of Y with $G' \supset F \cap K$. Since the set $F\setminus G'$ is Ray compact and $(F\setminus G') \cap K = \varnothing$, there exists an open set G_1 in Y, $G_1 \supset F \setminus G'$, with $\overline{G}_1 \cap K = \varnothing$. We have $s \curlywedge R^{(G'\cup G_1)\cap X}s \preceq s \curlywedge R^{G'\cap X}s + s \curlywedge R^{G_1\cap X}s$. If we set $u := s \curlywedge R^{\overline{G}_1\cap X}s$, then we obtain $R^K u = u$, $R^{\overline{G}_1\cap X}u = u$ and, consequently, $u = 0$. Hence $s \curlywedge R^{(G'\cup G_1)\cap X}s \prec s \curlywedge R^{G'\cap X}s$, $\curlywedge\{s \curlywedge R^{G'\cap X}s/G' \text{ open},\ G' \supset F\} \prec s \curlywedge R^{G\cap X}s$, for every open set G, $G \supset F \cap K$. Since s is Ray upper semicontinuous there exists a decreasing sequence (G'_n) of open sets in Y, $G'_n \supset F \cap K$, such that $\inf_n R^{G'_n\cap X}s = R^{F\cap K}s$. We deduce that $\curlywedge_n (s \curlywedge R^{G'_n\cap X}s) \prec s \curlywedge R^{F\cap K}s$ and, finally, $\curlywedge\{s \curlywedge R^{G\cap X}s/G \text{ open},\ G \supset F\} \prec s \curlywedge R^{F\cap K}s$. $\square$

DEFINITION. An element $s \in \mathscr{S}$ is called *regular* if for each increasing sequence (s_n) in $\mathscr{S}$ with $\sup_n s_n = s$ we have $\inf_n R(s - s_n) = 0$.

It is easy to see that the set $\mathscr{S}_r$ of all regular elements from $\mathscr{S}$ is a convex cone solid in $\mathscr{S}$ with respect to the specific order and such that $\sum_n s_n \in \mathscr{S}_r$ for every sequence (s_n) in $\mathscr{S}_r$ with $\sum_n s_n \in \mathscr{S}$. Also each function from $\mathscr{S}_r$ is the sum of a sequence of bounded elements of $\mathscr{S}_r$.

THEOREM 2.5. *If $s \in \mathscr{S}$, then the following statements are equivalent*:

(i) *s is regular.*

(ii) *There exists a regular strongly supermedian kernel V on X such that $V1 = s$.*

PROOF. (i) $\Rightarrow$ (ii). Since there exists a sequence (s_n) of bounded functions from $\mathscr{S}$ such that $s = \sum_n s_n$, we may suppose that s is bounded. By Lemma 2.1 there exists a strongly supermedian kernel on Y with $V1 = s$ and $V(1_F) = \curlywedge\{s \curlywedge R^{G\cap X}s / G \text{ open}, G \supset F\}$ for each closed subset F of Y. We show that V is a kernel on X. Indeed, let K be a compact subset of $Y \setminus X$ and let (F_n) be a decreasing sequence of closed subsets of Y such that $\bigcap_n F_n = K$ and $F_{n+1} \subset \overset{\circ}{F}_n$. We have $V(1_K) = \curlywedge_n(s \curlywedge R^{F_n \cap X}s)$ and therefore $V(1_K) = R^{F_n\cap X}V(1_K)$ for all n. It follows that $V(1_K)$ is $\mathscr{U}$-excessive. The function s being regular, we deduce that $V(1_K)$ is also regular and, consequently, by [10] we get $V(1_K) = 0$.

We prove now that V is a regular strongly supermedian kernel on X. Let θ be a finite measure on X and let $\mathscr{T}$ be the set of all $t \in \mathscr{S}$, $t \prec s$, for which there exists a regular strongly supermedian kernel V_t on X with $V_t 1 = t$. Since $\mathscr{T}$ is specifically increasing (cf. Theorem 2.2), there exists $t_0 \in \mathscr{T}$ such that for all $u \in \mathscr{T}$, $u \preceq s - t_0$, we have $\theta(u) = 0$. We show that $\theta(s - t_0) = 0$. Indeed, if $s_0 := s - t_0$ and $\widehat{s_0}$ is its $\mathscr{U}$-excessive regularization, then the set $[s_0 \neq \widehat{s_0}]$ is θ-semipolar. Let μ_θ be a Dellacherie measure on $[s_0 \neq \widehat{s_0}]$ such that $\mu_\theta \leq_{\mathscr{E}_{\mathscr{U}}} \theta$. If K is a Ray compact subset of $[s_0 \neq \widehat{s_0}]$ with $\mu_\theta(K) = 0$, then K is θ-polar and θ-negligible. Suppose that $[s_0 \neq \widehat{s_0}]$ is not θ-polar and θ-negligible. Then there exists a Ray compact subset K of $[s_0 \neq \widehat{s_0}]$ such that K is not θ-polar and θ-negligible and $s_0|_K$ is finite and Ray continuous. From $s_0 \prec s_0 \curlywedge R^K s_0 + s_0 \curlywedge R^{X\setminus K}s_0$ and since $s_0 \curlywedge R^K s_0 \in \mathscr{T}$ (cf. Theorem 2.4), we deduce that $\theta(s_0 \curlywedge R^K s_0) = 0$ and thus $s_0 = R^{X\setminus K}s_0$ θ-q.e. This implies that $s_0 = \widehat{s_0}\theta$-q.e. on K, which is a contradiction. By $s_0 \curlywedge \widehat{s_0} \in \mathscr{T}$, $s_0 \curlywedge \widehat{s_0} = s_0 = \widehat{s_0}$ θ-q.e., we conclude that $\theta(s_0) = 0$. From the previous considerations we obtain $s = t_0$ θ-q.e. If $f \in \mathscr{F}(X)$ and $s' \in \mathscr{S}$ is such that $Vf \leq s'$ on $[f > 0]$, then $V_{t_0} f \leq s'$ on $[f > 0]$ and therefore $V_{t_0} f \leq s'$ on X, $Vf \leq s'$ θ-q.e. The measure θ being arbitrary, we conclude that $Vf \leq s'$ on X.

(ii) $\Rightarrow$ (i). Let V be a regular strongly supermedian kernel on X such that $V1 = s$ and let (s_n) be an increasing sequence in $\mathscr{S}$ such that $\sup_n s_n = s$. For each $\varepsilon > 0$ let us put $A_{n,\varepsilon} := [s \leq s_n + \varepsilon]$. Since $A_{n,\varepsilon} \nearrow X$ we get $V(1_{A_{n,\varepsilon}}) \nearrow V1 = s$. From $V(1_{A_{n,\varepsilon}}) \leq s_n + \varepsilon$ for all n, we deduce that $s - s_n \leq V(1_{X\setminus A_{n,\varepsilon}}) + \varepsilon$ and therefore $\inf_n R(s - s_n) \leq \varepsilon$. Letting ε tend to 0, we get $\inf_n R(s - s_n) = 0$. $\square$

COROLLARY 2.6. *If $s \in \mathscr{S}$ is Ray upper semicontinuous and such that there exists a Ray compact set K with $s = R^K s$, then s is regular.*

3. Revuz measures associated with regular strongly supermedian kernels. For the next two sections we fix a $\mathscr{U}$-excessive measure $\xi = \rho \circ U + h$ with h harmonic.

DEFINITION. Let V be a strongly supermedian kernel on X. The positive measure on X defined by

$$\nu_V^{\xi}(M) = L\big(\xi, V(1_M)\big) = \sup\big\{\nu\big(V(1_M)\big)/\nu \circ U \leq \xi\big\}, \quad M \in \mathscr{B},$$

is called the *Revuz measure* of V (with respect to ξ).

REMARKS. (i) If V is a regular strongly supermedian kernel, then its Revuz measure ν_V^{ξ} is σ-finite. In particular, $\nu_{\widehat{V}}^{\xi}$ is also σ-finite.

(ii) If two strongly supermedian (resp. excessive) kernels coincide ξ-q.e. and ρ-a.e. (resp. ξ-q.e.), then they have the same Revuz measure.

PROPOSITION 3.1. *Let V be a regular strongly supermedian kernel. Then*:

(i) *The Revuz measure ν_V^{ξ} charges no set that is both ξ-polar and ρ-negligible.*

(ii) *The Revuz measure $\nu_{\widehat{V}}^{\xi}$ charges no ξ-polar set.*

PROOF. Obviously, we may assume that $V1 \leq 1$. Let $M \in \mathscr{B}$ be a ξ-polar set. From $V(1_M) \leq R^M 1$ it follows that $\widehat{V}(1_M) \leq B^M 1$ and therefore $\nu_{\widehat{V}}^{\xi}(M) = L(\xi, \widehat{V}(1_M)) \leq L(\xi, B^M 1) = 0$. If, in addition, M is ρ-negligible and μ is a finite measure on X such that $\mu \circ U \leq \xi$, then, by Theorem 4.1 in [7], we have $\mu(M) = 0$ and, consequently, $\mu(V(1_M)) \leq \mu(R^M 1) = \mu(B^M 1) \leq L(\xi, B^M 1) = 0$. We conclude that $\nu_V^{\xi}(M) = \sup\{\mu(V(1_M))/\mu \text{ finite}, \mu \circ U \leq \xi\} = 0$. □

THEOREM 3.2. *Let V_1, V_2 be two regular strongly supermedian kernels. Then the following assertions hold*:

(i) *If $\nu_{V_1}^{\xi} = \nu_{V_2}^{\xi}$, then $V_1 = V_2$ ξ-q.e. and ρ-a.e.*

(ii) *If $\nu_{\widehat{V}_1}^{\xi} = \nu_{\widehat{V}_2}^{\xi}$, then $\widehat{V}_1 = \widehat{V}_2$ ξ-q.e.*

PROOF. We may assume that the measures $\nu_{V_1}^{\xi}, \nu_{V_2}^{\xi}$ are finite. Since $V := V_1 + V_2$ is a regular strongly supermedian kernel on X, by Theorem 2.2 there exist $g_1, g_2 \in \mathscr{F}^*(X)$, $g_1, g_2 \leq 1$, such that $V_1 = g_1 \cdot V$, $V_2 f = g_2 \cdot V$.

(i) By hypothesis, we have $L(\xi, V(fg_1)) = L(\xi, V(fg_2))$ for all $f \in \mathscr{F}(X)$. We may assume, in addition, that $\inf(g_1, g_2) = 0$. In this case we get $0 = L(\xi, V(g_2 1_{[g_1>0]})) = L(\xi, Vg_1)$, $0 = L(\xi, V(g_1 1_{[g_2>0]})) = L(\xi, Vg_2)$ and, consequently, $Vg_1 = Vg_2 = 0$ ξ-q.e. and ρ-a.e., $V_1 = V_2$ ξ-q.e. and ρ-a.e.

(ii) As in the proof of the first assertion, we may assume that $\inf(g_1, g_2) = 0$ and we deduce that $L(\xi, \widehat{V}g_1) = L(\xi, \widehat{V}g_2)$, $\widehat{V}g_1 = \widehat{V}g_2$ ξ-q.e. □

COROLLARY 3.3. *Let V_1, V_2 be two regular strongly supermedian kernels on X. Then the following assertions hold*:

(i) *If $V_1 f \leq V_2 f$ for all $f \in \mathscr{F}(X)$, then there exists $g \in \mathscr{F}^*(X)$, $g \leq 1$, such that $V_1 = g \cdot V_2$.*

(ii) *If* $\widehat{V}_1 f \leq \widehat{V}_2 f$ *ξ-q.e. for all* $f \in \mathscr{F}(X)$, *then there exists* $g \in \mathscr{F}^*(X)$, $g \leq 1$, *such that* $\widehat{V}_1 = g \cdot \widehat{V}_2$ *ξ-q.e.*

PROOF. (i) By Theorem 2.2 it is sufficient to show that $V_1 f \prec V_2 f$ for all $f \in \mathscr{F}(X)$ such that $V_2 f$ is bounded. Let θ be a finite measure on X. By hypothesis, there exists $g \in \mathscr{F}^*(X)$, $0 \leq g \leq 1$, such that $\theta(V_1 f) = \theta(V_2(fg))$ for all $f \in \mathscr{F}(X)$. Theorem 3.2 implies that $V_1 = g \cdot V_2$ θ-q.e. If $V_2 f$ is finite, then we have $V_2 f - V_1 f = V_2(f(1-g))$ θ-q.e. and therefore $V_2 f - V_1 f \in \mathscr{S}_\theta$. The measure θ being arbitrary, we conclude that $V_1 f \prec V_2 f$.

(ii) Since, by hypothesis, we have $\nu^\xi_{\widehat{V}_1} \leq \nu^\xi_{\widehat{V}_2}$, there exists $g \in \mathscr{F}(X)$, $g \leq 1$, such that $L(\xi, \widehat{V}_1 f) = L(\xi, \widehat{V}_2(fg))$ for all $f \in \mathscr{F}(X)$. Assertion (ii) follows now by Theorem 3.2. □

THEOREM 3.4. (i) *A subset* $M \in \mathscr{B}$ *will be ξ-polar and ρ-negligible if and only if* $\nu^\xi_V(M) = 0$ *for each regular strongly supermedian kernel* V.

(ii) *Every σ-finite measure charging no set that is both ξ-polar and ρ-negligible is the Revuz measure of a regular strongly supermedian kernel on* X.

PROOF. (i) By Proposition 3.1 we have $\nu^\xi_V(M) = 0$ for every $M \in \mathscr{B}$ which is both ξ-polar and ρ-negligible. Conversely, let $M \in \mathscr{B}$ and let K be a Ray compact subset of M. From Theorem 2.4 applied to $s := R^K p_0$, it follows that there exists a regular strongly supermedian kernel V on X such that $V1 = V(1_K) = s$. By hypothesis, $0 = \nu^\xi_V(K) \geq \rho(V(1_K)) = \rho(s)$ and therefore K is ρ-negligible. Since $0 = \nu^\xi_V(K) = L(\xi, V(1_K)) \geq L(\xi, \widehat{V}(1_K)) \geq (\xi, B^K p_0)$, we conclude that K is also ξ-polar.

(ii) Let ν be a finite measure on X charging no set that is both ξ-polar and ρ-negligible. By Corollary 2.4 in [11] ν may be written in the form $\nu = \nu' + \nu''$, where ν' charges no ξ-semipolar set and ν'' is carried by a ξ-semipolar set $A \in \mathscr{B}$. From Theorem 4.3(iii) in [11] there exists a regular excessive kernel V' with $\nu' = \nu^\xi_{V'}$. Let θ be a finite measure on X such that $\theta = f \cdot (\xi + \rho)$, $0 < f \leq 1$. Then a subset of X will be ξ-polar (resp. ξ-polar and ρ-negligible) if and only if it is θ-polar (resp. θ-polar and θ-negligible). Further, let μ be a Dellacherie measure associated with θ and the ξ-semipolar set A; that is, a subset of A will be ξ-polar and ρ-negligible if and only if it is μ-negligible. Assertion (i) and Lemma 2.5 in [11], applied on the measurable space $(A, \mathscr{B}|_A)$ for the set $\mathscr{M} = \{\nu^\xi_V / V$ regular strongly supermedian kernel, ν^ξ_V finite$\}$ and for the measure μ, imply that there exists a sequence (W_n) of regular strongly supermedian kernels such that $\mu = \sum_n \nu^\xi_{W_n}$. Since the kernel $W := \sum_n W_n$ is strongly supermedian and $W1$ is finite ξ-q.e. and ρ-a.e., it follows that $V := 1_{[W1<\infty]} \cdot W$ is a (proper) regular strongly supermedian kernel and $V = W$ ξ-q.e. and ρ-a.e. Therefore $\mu = \nu^\xi_V$. Since $\nu'' \ll \mu$ we conclude that $\nu'' = \nu^\xi_{V''}$ with $V'' := g \cdot V$, g being a $\mathscr{B}$-measurable positive real function on X. □

REMARK. Theorem 3.4 gives an analytic solution to a problem solved by Azéma (see [2]), in the particular case $\xi = \rho \circ U$, by probabilistic methods. Note that the d-functionals of Azéma correspond to the regular strongly supermedian kernels (see [20], page 490, and [32]). Another probabilistic approach to this problem, covering the general case $\xi = \rho \circ U + h$, has been given by Fitzsimmons in [20], where the d-functionals are replaced by the homogeneous random measures.

THEOREM 3.5. *The following assertions hold*:

(i) *A subset* $M \in \mathscr{B}$ *will be* ξ*-polar if and only if* $\nu_W^\xi(M) = 0$ *for each semiregular excessive kernel* W *on* X.

(ii) *Every* σ*-finite measure charging no* ξ*-polar set is the Revuz measure of a semiregular excessive kernel on* X.

PROOF. (i) If $M \in \mathscr{B}$ is ξ-polar and W is a semiregular excessive kernel on X, then by Proposition 3.1 we have $\nu_W^\xi(M) = 0$. Conversely, suppose that $M \in \mathscr{B}$ is such that $\nu_W^\xi(M) = 0$ for each semiregular excessive kernel W on X. Let K be a Ray compact subset of M and let $s := R^K p_0$. Since s is a regular strongly supermedian function and $R^K s = s$, by Theorem 2.5, there exists a regular strongly supermedian kernel V on X with $V1 = V(1_K) = s$. From $\widehat{V}1 = \widehat{V}(1_K) = B^K p_0$, $\nu_{\widehat{V}}^\xi(K) = 0$ we get $L(\xi, B^K p_0) = \nu_{\widehat{V}}^\xi(K) = 0$ and, consequently, K is ξ-polar.

(ii) Let ν be a finite measure on X charging no ξ-polar set. By Corollary 2.4 in [11] ν may be written in the form $\nu = \nu' + \nu''$, where ν' charges no ξ-semipolar set and ν'' is carried by a ξ-semipolar set $A \in \mathscr{B}$. From Theorem 4.3(iii) in [11] there exists a regular excessive kernel V' with $\nu' = \nu_{V'}^\xi$. We consider a Dellacherie measure μ associated with $f \cdot \xi$ and the ξ-semipolar set A, where f is a bounded strictly positive $\mathscr{B}$-measurable function such that $f \cdot \xi$ is a finite measure. Finally, by (i) and Lemma 2.5 in [11], arguing as in the proof of assertion (ii) of Theorem 3.4, there exists a semiregular excessive kernel V'' such that $\nu'' = \nu_{V''}^\xi$. □

REMARK. A probabilistic version of Theorem 3.5 has been proved by Dellacherie, Maisonneuve and Meyer [17].

The next result shows that assertion (ii) in Theorem 3.5 may be obtained directly from Theorem 3.4.

PROPOSITION 3.6. *Let* V *be a regular strongly supermedian kernel. Then the Revuz measure* $\nu_{\widehat{V}}^\xi$ *is of the form*

$$\nu_{\widehat{V}}^\xi = f \cdot \nu_V^\xi,$$

where f *is a* $\mathscr{B}$*-measurable function which is strictly positive* ξ*-q.e. In particular, if* ν *is a* σ*-finite measure on* X *charging no* ξ*-polar set and* V *is a regular* *strongly supermedian kernel with* $\nu = \nu_V^\xi$, *then there exists a strictly positive* $\mathscr{B}$*-measurable function* f *such that* $\nu_{\widehat{V}}^\xi = f \cdot \nu$, *or, equivalently,* ν *is the Revuz measure of the semiregular excessive kernel* $\widehat{f^{-1} \cdot V}$.

PROOF. Suppose that there exists $M \in \mathscr{B}$ such that $\nu(M) > 0$ and $f = 0$ on M. If we set $M_0 := \{x \in M / V(1_M)(x) = 0\}$, then $V(1_{M_0}) \equiv 0$ and therefore $V(1_M) = V(1_{M \setminus M_0}) > 0$ on $M \setminus M_0$, $\nu(M_0) = 0$. Hence we may suppose that M possesses the property that $V(1_M) > 0$ on M. Since $f = 0$ on M we get $\nu_{\widehat{V}}^\xi(M) = \nu_V^\xi(1_M f) = 0$ and, consequently, $\xi(\widehat{V(1_M)}) = 0$. On the other hand, from $V(1_M) > 0$ on M we conclude that $\xi(\widehat{R^M 1}) = 0$; that is, the set M is ξ-polar. □

4. Natural excessive kernels and Revuz measures.

DEFINITION. An excessive kernel V on X is called *natural* if for each Ray open set G and $f \in \mathscr{F}(X)$ vanishing outside G we have $B^G V f = V f$.

PROPOSITION 4.1. *Let* V *be an excessive kernel on* X *such that* $s := V1 \in \mathscr{E}_{\mathscr{U}}$ *and there exists a sequence* $(s_n) \subset \mathscr{E}_{\mathscr{U}}$, $s_n \leq 1$, *for all* n, *with* $s = \sum_n s_n$. *Then* V *is natural if and only if*

$$V(1_F) = \curlywedge \{B^G s / G \textit{ Ray open}, \ G \supset F\}$$

for each Ray closed (*or only Ray compact*) *subset* F *of* X.

PROOF. Suppose that V is natural. Let F be a Ray closed subset of X and G a Ray open set, $F \subset G$. Since $1_F = 0$ on $X \setminus G$ we have $B^G V(1_F) = V(1_F)$ and therefore, from $V(1_F) \prec V1 = s$, we get $V(1_F) \prec B^G s$. Consequently,

$$V(1_F) \prec \curlywedge \{B^G s / G \text{ Ray open}, \ G \supset F\}.$$

On the other hand, if we put $r := \curlywedge \{B^G s / G \text{ Ray open}, \ G \supset F\}$, then $r \prec s$. From $r = B^G r$ for each Ray open set G, $G \supset F$, it follows that for every Ray compact subset K of $X \setminus F$ we have $V(1_K) \curlywedge r = 0$. We deduce that $V(1_{X \setminus F}) \curlywedge r = 0$ and so $r \prec V(1_F)$. Conversely, suppose that $V(1_F) = \curlywedge \{B^G s / G \text{ Ray open}, \ G \supset F\}$ for each Ray compact set F. If $f \in \mathscr{F}(X)$ is bounded and vanishes outside a Ray open set G, $G \supset F$, then we get

$$V f = \curlyvee \left\{ V(f 1_F) / F \text{ Ray compact}, \ F \subset G \right\}.$$

Since, by hypothesis, we have $B^G V(f 1_K) = V(f 1_K)$, we conclude that $V f = B^G V f$. □

THEOREM 4.2. *The following assertions are equivalent*:

(i) *If* K *is a Ray compact subset of* X *which is not* ξ*-polar, then there exists a bounded* $\mathscr{U}$*-excessive function* s *on* X *such that* $\xi(s) \neq 0$ *and* $B^G s = s$ *for every Ray open subset* G *of* X *with* $G \supset K$.

(ii) *Every σ-finite measure charging no ξ-polar set is the Revuz measure of a proper natural excessive kernel.*

(iii) *A Borel subset of X will be ξ-polar if and only if it is negligible with respect to each Revuz measure ν_V^ξ of a proper natural excessive kernel V.*

PROOF. (i) $\Rightarrow$ (iii). Let $M \in \mathscr{B}$ and suppose that there exists a Ray compact subset K of M which is not ξ-polar. If s is a bounded $\mathscr{U}$-excessive function such that $\xi(s) \neq 0$ and $B^G s = s$ for each Ray open subset G of X with $G \supset K$, then by Lemma 2.1 there exists an excessive kernel V_s on Y such that $V_s(1_F) = \curlywedge\{B^{G\cap X}s/G \text{ open in } Y, G \supset F\}$ for each closed subset F of Y. Since $B^{G\cap X}s = s$ for every open subset G of Y with $G \supset K$, we get $V_s(1_K) = s$. Therefore V_s is a bounded natural excessive kernel on X such that $\nu^\xi_{V_s}(K) \neq 0$.

(iii) $\Rightarrow$ (ii). Let μ be a finite measure on X charging no ξ-polar set. By Lemma 2.5 in [11], reasoning as in the proof of Theorem 3.5(ii), there exists a sequence (V_n) of natural excessive kernels on X such that $\mu = \sum_n \nu^\xi_{V_n}$. Let $f_n \in \mathscr{F}(X)$, $0 < f_n \leq 1$, be such that $V_n f_n \leq 1$. It follows that the kernel $W := \sum_n f_n/2^n \cdot V_n$ is natural and $\mu \ll \nu^\xi_W$. We conclude that $\mu = \nu^\xi_V$, where $V := g \cdot W$, for a suitable function $g \in \mathscr{F}(X)$.

(ii) $\Rightarrow$ (i). Let K be a Ray compact set which is not ξ-polar. If K is not ξ-semipolar, then (see [8]) there exists a bounded regular excessive kernel W on X such that $W(1_K) = W1$ and $\xi(W1) \neq 0$. Obviously, we have $B^G W1 \geq B^K W1 = W1$ for each Ray open set G with $G \supset K$. If K is ξ-semipolar, then there exists a Dellacherie measure θ on K such that for each Borel subset M of K we have $\theta(M) = 0$ if and only if M is ξ-polar. By hypothesis (ii) there exists a proper natural excessive kernel Von X with $\theta = \nu^\xi_V$. If $g_0 \in \mathscr{F}(X)$, $0 < g_0 \leq 1$, is such that Vg_0 is bounded, then we have $L(\xi, Vg_0) = L(\xi, V(g_0 1_K))$, $\xi(V(g_0 1_K)) = \xi(Vg_0) \neq 0$. Since the kernel V is natural we also have $B^G V(g_0 1_K) = V(g_0 1_K)$ for every Ray open set G with $G \supset K$. $\square$

THEOREM 4.3. *The following assertions are equivalent*:

(i) *Two bounded natural excessive kernels on X having the same σ-finite Revuz measure* (*with respect to ξ*) *are equal ξ-q.e.*

(ii) *For each bounded natural excessive kernel V on X, having σ-finite Revuz measure, and every $\mathscr{U}$-excessive function s with $s \prec V1$, there exists $g \in \mathscr{F}$, $g \leq 1$, such that $s = Vg$ ξ-q.e.* (*i.e., V verifies the Motoo–Mokobodzki property with respect to ξ*).

PROOF. (i) $\Rightarrow$ (ii). Let V be a bounded natural excessive kernel on X, having σ-finite Revuz measure, and $s \in \mathscr{E}_{\mathscr{U}}$, $s \prec V1$. Since s is of potential type on X (see [11]), there exists a unique natural excessive kernel V_s on X with $V_s 1 = s$. By Proposition 4.1, from $V_s 1 \prec V1$ we deduce that $V_s f \prec Vf$ for all bounded $f \in \mathscr{F}(X)$ and therefore $\nu^\xi_{V_s} \leq \nu^\xi_V$. Hence there exists $g \in \mathscr{F}(X)$, $g \leq 1$, such that $\nu^\xi_{V_s} = \nu^\xi_{g\cdot V}$. From (i) we get $V_s = g \cdot V$ ξ-q.e. and therefore $s = V_s 1 = Vg$ ξ-q.e.

(ii) $\Rightarrow$ (i). Let V, W be two bounded natural excessive kernels on X such that $\nu_V^\xi = \nu_W^\xi$ and such that ν_V^ξ is σ-finite. From $V1 \prec (V+W)1$, $W1 \prec (V+W)1$ it follows that there exists $g_1, g_2 \in \mathscr{F}(X)$, $g_1, g_2 \le 1$, with $V1 = (V+W)g_1$, $W1 = (V+W)g_2$ ξ-q.e. Consequently, we have $V = g_1 \cdot (V+W)$, $W = g_2 \cdot (V+W)$ ξ-q.e. Finally, we deduce that $\nu^\xi_{g_1\cdot(V+W)} = \nu^\xi_{g_2\cdot(V+W)}$, $g_1 = g_2$ ν^ξ_{V+W}-a.e, $(V+W)g_1 = (V+W)(g_1 \wedge g_2) = (V+W)g_2$ ξ-q.e., $V1 = W1$ ξ-q.e. □

DEFINITION. An excessive kernel V on X is called *ξ-natural* if $B^G Vf = Vf$ ξ-q.e. for each Ray open set G and $f \in \mathscr{F}(X)$ vanishing outside G.

PROPOSITION 4.4. *Each ξ-natural excessive kernel V such that $V1$ is finite ξ-a.e. is equal ξ-q.e. with a natural excessive kernel.*

PROOF. Let V be a ξ-natural excessive kernel such that $V1 < \infty$ ξ-a.e. Let $\mathscr{G}$ be a countable base for the Ray topology on X. For each $G \in \mathscr{G}$ let $F_G := [B^G V(1_G) < V(1_G)]$ and let $F := \bigcup\{F_G/G \in \mathscr{G}\} \cup (X\setminus\overline{[V1 < \infty]}^f)$. By hypothesis, F is ξ-polar and therefore the set $E := [B^F 1 = 0]$ is absorbent, $X\setminus F \supset E$ and $X\setminus E$ is also ξ-polar. Hence $B^G V(f1_G) = V(f1_G)$ on E for all $f \in \mathscr{F}(X)$ and $G \in \mathscr{G}$. Consequently, the preceding equality holds on E for every Ray open set G. We define the kernel W by $Wf := B^E Vf$. Since E is absorbent we have $Wf \prec Vf$ for all bounded $f \in \mathscr{F}(X)$ and thus $B^G W(f1_G) = W(f1_G)$ on X for every Ray open set G. We conclude that W is a natural excessive kernel and $V = W\xi$-q.e. □

5. Hypothesis (B) of Hunt. Recall (cf. [25]) that *the hypothesis* (B) *of Hunt* holds on $F \in \mathscr{B}$ if for every $M \in \mathscr{B}$, $M \subset F$ and each Ray open set G, $G \supset M$, we have $B^G B^M s = B^M s$ for all $s \in \mathscr{E}_{\mathscr{U}}$.

DEFINITION. If θ is a σ-finite measure on X and $F \in \mathscr{B}$, we say that *the hypothesis* (B) *of Hunt with respect to* θ holds on F if $B^G B^M s = B^M s$ θ-q.e. for every $s \in \mathscr{E}_{\mathscr{U}}$, $M \in \mathscr{B}$, $M \subset F$ and each Ray open set G, $G \supset M$.

LEMMA 5.1 (Chung [16]). *Let μ be a finite measure on X charging no semipolar set and let H, $E \in \mathscr{B}$. Then the following assertions are equivalent*:

(i) *The measure $\mu \circ B^H$ charges no semipolar subset of $H\setminus E$.*
(ii) *For every Borel-measurable* (*or only Ray compact*) *subset M of $H\setminus E$ and all $s \in \mathscr{E}_{\mathscr{U}}$ we have $\mu(B^H B^M s) = \mu(B^M s)$.*

PROOF. (i) $\Rightarrow$ (ii). Since for every $M \in \mathscr{B}$, $M \subset H\setminus E$ and $p \in \mathscr{E}_{\mathscr{U}}$, the set $[B^M p < R^M p]$ is a semipolar subset of M, we get $\mu \circ B^H(B^M p) = \mu \circ B^H(R^M p)$. From $B^H s = R^H s$ μ-a.e. for every $s \in \mathscr{E}_{\mathscr{U}}$ it follows that $B^H f = R^H f$ μ-a.e. for all $f \in \mathscr{F}(X)$. Consequently, $B^H R^M p = R^H R^M p$ μ-a.e. and therefore

$$(\mu \circ B^H)(B^M p) = (\mu \circ B^H)(R^M p) = \mu(B^H R^M p) = \mu(R^H R^M p).$$

Finally, by $R^H R^M p = R^M p$ for all $p \in \mathscr{E}_{\mathscr{U}}$, we conclude that $\mu(B^H B^M p) = \mu(B^M p)$.

(ii) $\Rightarrow$ (i). Let M be a finely closed subset of $H \backslash E$ which is totally thin. By hypothesis and since $B^H(R^M p_0) = R^H R^M p_0 = R^M p_0$ μ-a.e., we get $\mu \circ B^H(B^M p_0) = \mu \circ B^H(R^M p_0)$. From $M = \{x \in X / B^M p_0(x) < R^M p_0(x)\}$ we conclude that $\mu \circ B^H(M) = 0$. □

COROLLARY 5.2. *Let $F \in \mathscr{B}$ and let θ be a σ-finite measure on X. Then the following assertions are equivalent*:

(i) *The hypothesis* (B) *of Hunt with respect to θ holds on F.*

(ii) *For every Ray compact subset K of F and $s \in \mathscr{E}_{\mathscr{U}}$ we have $B^G B^K s = B^K s$ θ-q.e., whenever G is Ray open, $G \supset K$.*

COROLLARY 5.3. *If ξ is an excessive measure on X and $M \in \mathscr{B}$, then there exists a semipolar set $E \in \mathscr{B}$ such that for all $C \in \mathscr{B}$, $C \subset M \backslash E$ and $s \in \mathscr{E}_{\mathscr{U}}$ we have $B^M B^C s = B^C s$ ξ-q.e.*

PROOF. Let μ be a finite measure on X, charging no semipolar set and having the same negligibile sets as ξ. We take $E \in \mathscr{B}$ such that $\mu \circ B^M(E) = \sup\{\mu \circ B^M(K)/K \in \mathscr{B}$ semipolar$\}$. By Lemma 5.1, for all $C \in \mathscr{B}$, $C \subset M \backslash E$ and $s \in \mathscr{E}_{\mathscr{U}}$ we have $B^M B^C s = B^C s$ μ-a.e. or, equivalently, the equality holds ξ-q.e. □

THEOREM 5.4. *Let θ be a σ-finite measure on X. Then there exists $F \in \mathscr{B}$ such that $X \backslash F$ is semipolar and the hypothesis* (B) *of Hunt with respect to θ holds on F.*

PROOF. We may assume that θ is finite. Let $\mathscr{G}$ be a countable base for the Ray topology on X which is closed to finite union. By Corollary 5.3 applied to $\xi := \theta \circ U$, for every $G \in \mathscr{G}$ we find a semipolar set $E_G \in \mathscr{B}$ such that $B^G B^M s = B^M s$ θ-q.e. for all $s \in \mathscr{E}_{\mathscr{U}}$ and $M \in \mathscr{B}$, $M \subset G \backslash E_G$. If we put $F := X \backslash \bigcup\{E_G / G \in \mathscr{G}\}$, then $F \in \mathscr{B}$ and $X \backslash F$ is semipolar. Let now $M \in \mathscr{B}$, $M \subset F$ and G be a Ray open set, $G \supset M$. If $(G_n)_n$ is an increasing sequence in $\mathscr{G}$ such that $\bigcup_n G_n = G$, then from $B^{G_n} B^{M \cap G_n} s = B^{M \cap G_n} s$ θ-q.e. for all n and $s \in \mathscr{E}_{\mathscr{U}}$ we conclude that $B^G B^M s = B^M s$ θ-q.e. □

REMARKS. (i) There are examples of sub-Markovian resolvents for which the hypothesis (B) of Hunt does not hold although it holds with respect to a suitable excessive measure (see [16]).

(ii) If θ is a reference measure for the resolvent $\mathscr{U}$, then there exists a semipolar subset E of X such that the hypothesis (B) of Hunt holds on $X \backslash E$ (see [14]).

THEOREM 5.5. *If $F \in \mathscr{B}$ and θ is a σ-finite measure on X, then the following statements are equivalent*:

(i) *The hypothesis* (B) *of Hunt with respect to θ holds on F.*

(ii) *For each semiregular excessive kernel* $\widehat{W}$ *on* X *with* $\widehat{W}(1_{X\setminus F}) = 0$ *there exists a proper natural excessive kernel* V *on* F *such that* $V = \widehat{W}$ θ*-q.e.*

PROOF. Let K be a Ray compact subset of F and let G be a Ray open subset of X, $G \supset K$.

(i) $\Rightarrow$ (ii). Let $\mu := \theta \circ U_\alpha$, where $\alpha > 0$, and let W be a regular strongly supermedian kernel on X such that $\widehat{W}(1_{X\setminus F}) = 0$. If $f \in \mathscr{F}(X)$ is bounded and $\widehat{W}(f)$ is also bounded, then, by Lemma 5.1, we have $\mu(\widehat{W}(f1_K)) = \mu(W(f1_K)) = \mu(R^G W(f1_K)) = \mu(R^G \widehat{W}(f1_K)) = \mu(B^G \widehat{W}(f1_K))$. We deduce that $\widehat{W}(f1_K) = B^K \widehat{W}(f1_K)$ θ-q.e. and, by Proposition 4.4, assertion (ii) follows.

(ii) $\Rightarrow$ (i). Since $R^K p_0$ is an upper semicontinuous strongly supermedian function on X and $R^K(R^K p_0) = R^K p_0$, then by Theorem 2.4 there exists a regular strongly supermedian kernel W on X such that $W1 = R^K p_0 = W(1_K)$. From $B^K p_0 = \widehat{W}(1_K)$ and, by hypothesis, we conclude that θ-q.e. we have $B^G B^K p_0 = B^G \widehat{W}(1_K) = \widehat{W}(1_K) = B^K p_0$. □

COROLLARY 5.6. *The following statements are equivalent for a set* $F \in \mathscr{B}$:

(i) *The hypothesis* (B) *of Hunt holds on* F.

(ii) *Every semiregular excessive kernel* W *on* X *with* $W(1_{X\setminus F}) = 0$ *is natural.*

REMARK. By Theorem 5.5, Proposition 4.1 and Corollary 3.3 it follows that if the hypothesis (B) of Hunt with respect to ξ holds on X, then each semiregular excessive kernel enjoys the following Motoo–Mokobodzki property: if s is a regular strongly supermedian function such that $\widehat{s} \prec Vf$, where $f \in \mathscr{F}$ with Vf bounded, then there exists $g \in \mathscr{F}$, $0 \le g \le 1$, such that $\widehat{s} = V(gf)$ ξ-q.e. This improves a result of Azéma [2].

THEOREM 5.7. *Let* $\xi \in \mathrm{Exc}$ *and let* $F \in \mathscr{B}$. *Then the following statements are equivalent*:

(i) *The hypothesis* (B) *of Hunt with respect to* ξ *holds on* F.

(ii) *A subset* $M \in \mathscr{B}$ *of* F *is* ξ*-polar if and only if* $\nu_V^\xi(M) = 0$ *for each semiregular* ξ*-natural excessive kernel on* X.

(iii) *Every* σ*-finite measure on* F *charging no* ξ*-polar set is the Revuz measure of a semiregular* ξ*-natural excessive kernel.*

PROOF. (i) $\Rightarrow$ (ii) follows by Theorems 5.5 and 3.5(i).

(ii) $\Rightarrow$ (iii) holds by Lemma 2.5 in [11], arguing as in the proof of Theorem 3.5(ii).

(iii) $\Rightarrow$ (i) follows by Theorem 5.5 and the uniqueness property given by Theorem 3.2(ii). □

THEOREM 5.8. *If* $F \in \mathscr{B}$, *then the following statements are equivalent*:

(i) *The hypothesis* (B) *of Hunt holds on* F.

(ii) *If* $\xi \in \mathrm{Exc}$, *then a subset* M *of* F, $M \in \mathscr{B}$, *will be* ξ*-polar if and only if* $\nu_V^{\xi}(M) = 0$ *for every semiregular natural excessive kernel* V *on* X.

(iii) *If* $\xi \in \mathrm{Exc}$, *then every* σ*-finite measure* μ *on* F *charging no* ξ*-polar set is the Revuz measure of a semiregular natural excessive kernel on* X.

PROOF. The assertion follows by Theorem 5.7, Corollary 5.6 and Theorem 3.5. □

REMARKS. (i) If the hypothesis (B) of Hunt holds on X, then every semiregular excessive kernel on X is natural. From this fact it follows that Theorem 5.8 improves a result from [11], where each σ-finite measure charging no ξ-polar set is represented as the Revuz measure of a proper natural excessive kernel. This new representation enjoys, in addition, the uniqueness property.

(ii) Let $\xi \in \mathrm{Exc}$ be such that the hypothesis (B) of Hunt with respect to ξ holds on X. Assume further that if two proper natural excessive kernels have the same σ-finite Revuz measure (with respect to ξ), then they coincide ξ-q.e. (Note that this last assumption is fulfilled under "weak duality hypotheses"; see [11] and [24]). Then each proper natural excessive kernel is equal ξ-q.e. to a semiregular excessive kernel. In particular, if hypothesis (L) of Meyer holds, then every proper natural excessive kernel is semiregular. This statement improves a result of Azéma [2], obtained under more restrictive assumptions.

REFERENCES

[1] AZÉMA, J. (1969). Noyau potentiel associé à une fonction excessive d'un processus de Markov. *Ann. Inst. Fourier* (*Grenoble*) **19** 495–526.

[2] AZÉMA, J. (1973). Théorie générale des processus et retournement du temps. *Ann. Sci. École Norm. Sup.* **6** 459–519.

[3] BEZNEA, L. and BOBOC, N. (1994). Duality and biduality for excessive measures. *Rev. Roumaine Math. Pures Appl.* **39** 419–438.

[4] BEZNEA, L. and BOBOC, N. (1994). Excessive functions and excessive measures: Hunt's theorem on balayages, quasi-continuity. *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.* **430** 77–92.

[5] BEZNEA, L. and BOBOC, N. (1995). On the integral representation for excessive measures. *Rev. Roumaine Math. Pures Appl.* **40** 725–734.

[6] BEZNEA, L. and BOBOC, N. (1996). Kuran's regularity criterion and localization in excessive structures. *Bull. London Math. Soc.* **28** 273–282.

[7] BEZNEA, L. and BOBOC, N. (1996). Quasi-boundedness and subtractivity: applications to excessive measures. *Potential Anal.* **5** 467–485.

[8] BEZNEA, L. and BOBOC, N. (1996). Once more about the semipolar sets and regular excessive functions. In *Potential Theory—ICPT 94* 255–274. de Gruyter, Berlin.

[9] BEZNEA, L. and BOBOC, N. (1998). Noyaux fortement surmédians et mesures de Revuz. *C. R. Acad. Sci. Paris Sér. I Math.* **327** 139–142.

[10] BEZNEA, L. and BOBOC, N. (1999). Feyel's techniques on the supermedian functionals and strongly supermedian functions. *Potential Anal.* **10** 347–372.

[11] BEZNEA, L. and BOBOC, N. (2000). Excessive kernels and Revuz measures. *Probab. Theory Related Fields* **117** 267–288.

[12] BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory*. Academic, New York.

[13] BOBOC, N. and BUCUR, GH. (1976). *Conuri convexe de funcţii continue pe spaţii compacte*. Editura Academiei, Bucharest.

[14] BOBOC, N. and BUCUR, GH. (1986). Sur une hypothèse de G. A. Hunt. *C. R. Acad. Sci. Paris Sér. I Math*. **302** 701–703.

[15] BOBOC, N., BUCUR, GH. and CORNEA, A. (1981). *Order and Convexity in Potential Theory: H-Cones. Lecture Notes in Math*. **853**. Springer, Berlin.

[16] CHUNG, K. L. (1981). *Lectures from Markov Processes to Brownian Motion*. Springer, Berlin.

[17] DELLACHERIE, C., MAISONNEUVE, B. and MEYER, P. A. (1992). *Probabilités et Potentiel*, Chaps. 17–24. Hermann, Paris.

[18] DELLACHERIE, C. and MEYER, P. A. (1987). *Probabilités et Potentiel* Chaps. 12–16. Hermann, Paris.

[19] FEYEL, D. (1981). Représentation des fonctionnelles surmédianes. *Z. Wahrsch. Verw. Gebiete* **58** 183–198.

[20] FITZSIMMONS, P. J. (1987). Homogeneous random measures and a weak order for the excessive measures of a Markov process. *Trans. Amer. Math. Soc.* **303** 431–478.

[21] FITZSIMMONS, P. J. and GETOOR, R. K. (1996). Smooth measures and continuous additive functionals of right Markov processes. In *Itô's Stochastic Calculus and Probability Theory* 31–49. Springer, Berlin.

[22] GARCIA ALVAREZ, M. A. (1973). Représentation des noyaux excessifs. *Ann. Inst. H. Poincaré Probab. Statist.* **9** 277–283.

[23] GETOOR, R. K. (1975). *Markov Processes: Ray Processes and Right Processes. Lecture Notes in Math*. **440**. Springer, Berlin.

[24] GETOOR, R. K. and SHARPE, M. J. (1984). Naturality, standardness and weak duality for Markov processes. *Z. Wahrsch. Verw. Gebiete* **67** 1–62.

[25] HUNT, G. A. (1957). Markoff processes and potentials I. *Illinois J. Math.* **1** 44–93.

[26] MEYER, P. A. (1962). Fonctionnelles multiplicatives et additives de Markov. *Ann. Inst. Fourier* (*Grenoble*) **12** 125–230.

[27] MEYER, P. A. (1973). Note sur l'interprétation des mesures d'équilibre. *Lecture Notes in Math*. **381** 191–211. Springer, Berlin.

[28] MOKOBODZKI, G. (1970). Densité relative de deux potentiels comparables. *Lecture Notes in Math*. **124** 170–194. Springer, Berlin.

[29] MOKOBODZKI, G. (1979). Ensembles compacts de fonctions fortement surmédianes. *Lecture Notes in Math*. **713** 178–193. Springer, Berlin.

[30] MOKOBODZKI, G. (1984). Compactification relative à la topologie fine en théorie du potentiel. *Lecture Notes in Math*. **1096** 450–473. Springer, Berlin.

[31] REVUZ, D. (1970). Mesures associées aux fonctionelles additives de Markov I. *Trans. Amer. Math. Soc.* **148** 501–531.

[32] SHARPE, M. J. (1988). *General Theory of Markov Processes.* Academic Press, New York.

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