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ETERNAL ADDITIVE COALESCENTS AND CERTAIN BRIDGES WITH EXCHANGEABLE INCREMENTS

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Aldous and Pitman have studied the asymptotic behavior of the additive coalescent processes using a nested family random forests derived by logging certain inhomogeneous continuum random trees. Here we propose a different approach based on partitions of the unit interval induced by certain bridges with exchangeable increments. The analysis is made simple by an interpretation in terms of an aggregating server system.

1. Introduction. This work is motivated by some recent papers [12], [4, 5] and [8] on additive coalescents. First, recall that these random processes describe the evolution of the ranked sequence of masses in a system of clusters in which each pair of clusters, say with masses m_i and m_j, merges as a single cluster with mass $m_i + m_j$ at rate $\kappa(m_i, m_j) = m_i + m_j$, independently of the other pairs. We refer to [2] for a survey on such (and also other) coalescent models and their applications.

Roughly, a natural problem in this setting is to consider asymptotics when one starts with a large number of small clusters. In this direction, Evans and Pitman [12] have first shown that the so-called standard additive coalescent arises at the limit as $n \to \infty$ of the additive coalescent process started at time $-\frac{1}{2}\log n$ with n clusters, each with mass $1/n$. By splitting the continuum random tree along its skeleton, Aldous and Pitman [4] have then constructed a fragmentation process that is connected to the standard additive coalescent by a simple deterministic time-change. A different construction of the same fragmentation process has been presented in [8], by considering the partitions of the unit interval induced by a standard Brownian excursion with drift, where the drift coefficient coincides with the time parameter of the fragmentation process. Finally, Aldous and Pitman [5] have characterized the entrance boundary of the additive coalescent (which corresponds to the so-called eternal additive coalescents) and made the connection with a family of inhomogeneous continuum random trees; see also Camarri and Pitman [9].

Our purpose here is to present a simpler approach to investigate the asymptotic behavior of additive coalescents, which also provides a different representation of the eternal processes. Again, partitions of the unit interval induced by a certain excursion with drift have a key role. The connection between additive coalescent and interval partitions is enlightened by an interpretation in

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terms of an aggregating server system. More precisely, the rest of the paper is organized as follows.

The next section focuses on the finite setting (i.e., when at the initial time we start with finitely many clusters). We first introduce an aggregating server system and observe that the evolution of this system can be described in terms of the lengths of the intervals of constancy for the maximal function of a certain excursion with varying drift. This excursion is constructed from some bridge by the following classical path-transformation: one splits the bridge at the location of its infimum and interchanges the resulting two portions of paths. The bridge itself is defined as a piecewise linear function whose jumps represent the initial service rates in the system. When the initial data of the system are suitably randomized, the above random bridge has exchangeable increments. Moreover, the evolution of the server system becomes Markovian and is related to the (discrete) additive coalescent.

This enables us to investigate in the third section the asymptotic behavior of the additive coalescent started with a large number of small clusters, using Kallenberg's results [14] for the convergence of processes with exchangeable increments. We first recast the preceding quantities in Skorohod's space $\mathbf{D}$ of càdlàg paths. Next, we develop some technical results on bridges with exchangeable increments and their associated excursions. Then we derive the main result on the asymptotic behavior of additive coalescents relying on the connections established in Section 2.

Finally, some miscellaneous comments are made in Section 4.

2. An aggregating server system.

2.1. *Mechanism of the server system.* Fix an integer $n \geq 2$ and consider the following system of customers-servers. At the initial time $t = 0$, we have n customers and n servers. Both servers and customers are indexed by $\{1, \dots, n\}$ (we shall often identify this set with $\mathbf{Z}/n\mathbf{Z}$). The ith customer requires a quantity of service $s_i \geq 0$ and is served by the ith server. The service rate (or output rate) of the ith server is $r_i > 0$, in the sense that the quantity of service it delivers on a time duration δ is $r_i\delta$. It is convenient for our future purpose to assume unit total service rate, that is, $r_1 + \cdots + r_n = 1$. At the first time when a server has completely served its customer, this customer leaves the system and the server aggregates with its successor in $\mathbf{Z}/n\mathbf{Z}$ to form a new server whose service rate is just the sum of the two previous service rates.

To give a more formal description, introduce the following notation. If $(x_1, \dots, x_n)$ is a sequence of $n \geq 2$ real numbers and $i \leq n$ a positive integer, we construct a sequence with $(n-1)$ real numbers by

$$(x_1, \dots, x_n)^{\oplus i} = (x_1, \dots, x_{i-1}, x_i + x_{i+1}, x_{i+2}, \dots, x_n) \quad \text{for } i = 1, \dots, n-1$$

and

$$(x_1, \dots, x_n)^{\oplus n} = (x_2, \dots, x_{n-1}, x_n + x_1).$$

Then let $i(1)$ be the index of the first customer completely served and

$$A_1 = \min_{k=1,\dots,n} s_k/r_k = s_{i(1)}/r_{i(1)}$$

the instant when its service is completed. At time A_1 the system evolves into a new system with $(n-1)$ customers and $(n-1)$ servers, both indexed by $\{1,\dots,n-1\}$. In this new system, the sequence of service rates is $(r_1,\dots,r_n)^{\oplus i(1)}$ and the sequence of required services is $(s_1 - r_1A_1,\dots,s_n - r_nA_1)^{\oplus i(1)}$.

The mechanism of evolution after time A_1 is the same as before; for the sake of simplicity, we assume that the initial data are such that there are never two or more customers which are completely served at the same time. Of course, the system stops when all the customers have been completely served.

Let us introduce some notation to describe the state of the system as time evolves. We write $\#(t)$ for the number of customers (or of servers) present at time t,

$$R(t) = \big(R_1(t),\dots,R_{\#(t)}(t)\big)$$

for the sequence of service rates at time t, and

$$R^{\downarrow}(t) = \Big(R_1^{\downarrow}(t),\dots,R_{\#(t)}^{\downarrow}(t)\Big)$$

for the decreasing rearrangement of the sequence $R(t)$. In particular $\#(0)=n$ and $R(0)=(r_1,\dots,r_n)$. We also set $A_0=0$ and for $k=1,\dots,n-1$, we define A_k as the first instant when k customers have been completely served,

$$A_k = \inf\{t\geq 0, \#(t)=n-k\}\ .$$

In other words, for $k=1,\dots,n-1$, A_k is the instant when the kth aggregation happens, and therefore we shall often refer to A_k as the kth aggregation time. It is also convenient to write $A_n = s_1+\cdots+s_n$ for the time when the system stops, where the identity stems from the fact that the total service rate is 1 and the servers are always working.

2.2. *Bridge representation of the server system.* Our purpose here is to show that the ranked state of the aggregating server system can be expressed in terms of the intervals of constancy of a certain family of increasing functions. The latter will be constructed from some bridge that we now introduce (see Figure 1).

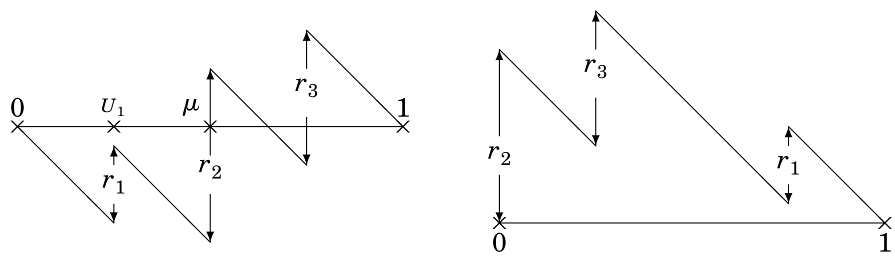
Let U_1 be an arbitrary real number in $[0,1]$. We set

$$U_{i+1} = U_1 + (s_1+\cdots+s_i)/A_n \ [\mathrm{mod}\ 1] \quad \text{for } i=1,\dots,n-1$$

and define

$$b(u) = \sum_{i=1}^{n} r_i\left(\mathbf{1}_{\{u\geq U_i\}} - u\right), \quad 0\leq u\leq 1\,. \tag{2.1}$$

Note that $b(0)=b(1)=0$ whenever $U_i\neq 0[\mathrm{mod}\ 1]$ for $i=1,\dots,n$, and then b can be thought as a bridge on $[0,1]$.



bridge b excursion ε

FIG. 1.

Next, recall our assumption that two different customers are never completely served at the same time, and let $\ell \in \{1, \dots, n\}$ denote the index at the initial time of the customer that will be the last to be completely served. This quantity has a simple interpretation in terms of the bridge. If we set $\mu = U_{\ell+1}$ with the convention that $\ell + 1 = 1$ when $\ell = n$, then we have the following identity.

LEMMA 1. *The bridge b reaches its infimum at a unique location which is μ. More precisely $b(\mu-) = \inf_{0\le u\le 1} b(u) < b(\mu)$ and for every $u \neq \mu$, $b(\mu-) < b(u) \wedge b(u-)$.*

PROOF. For the sake of simplicity, we identify here $\{1, \dots, n\}$ with $\mathbf{Z}/n\mathbf{Z}$, in the sense that additions of indices are taken modulo n. As the customer with label ℓ is the last to be completely served, for every $i = 1, \dots, n-1$, the customers with labels $\ell + 1, \dots, \ell + i$ are served only by servers built from the servers with labels $\ell + 1, \dots, \ell + i$, and they have been completely served before time A_n. In other words, we have

$$(r_{\ell+1} + \cdots + r_{\ell+i})A_n > s_{\ell+1} + \cdots + s_{\ell+i},$$

or equivalently $b(U_{\ell+i+1}-) > b(U_{\ell+1}-)$. This shows that the instant of the infimum of the bridge b must be $\mu = U_{\ell+1}$. $\square$

In order to describe the ranked state of the aggregating server system in terms of the bridge b, we consider the following path transformation $b \to \varepsilon$ which has been introduced by Takács [17] and used by Vervaat [18] to change a Brownian bridge into a normalized Brownian excursion (cf. Figure 1):

$$\varepsilon(x) = b(x + \mu\,[\mathrm{mod}\ 1]) - b(\mu-), \quad x \in [0, 1]. \tag{2.2}$$

Note that ε starts from $b(\mu) - b(\mu-) > 0$, ends at 0, and stays positive on $[0, 1[$. Hence ε can be thought of as an excursion with unit length. We also point out that the excursion ε does not depend on the value of U_1, the arbitrary real number that is used to define the bridge b.

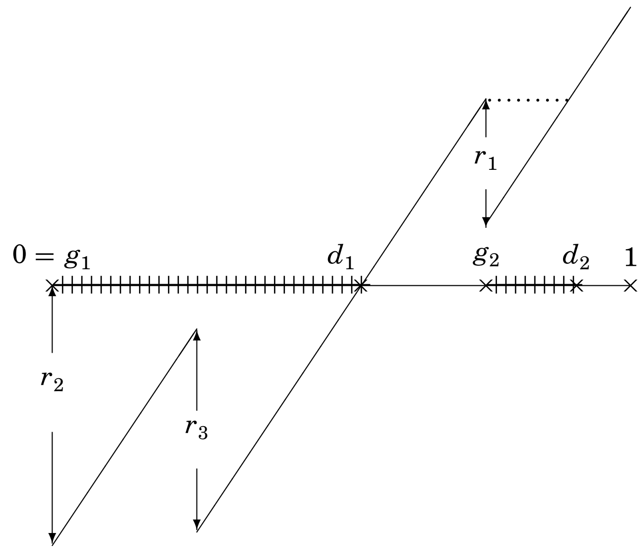


FIG. 2. *Excursion with drift $\varepsilon^{(\tau)}$ and intervals of constancy (hatched) of its supremum.*

Next, we define for every $\tau > 0$ and $u \in [0, 1]$

$$\varepsilon^{(\tau)}(u) = \tau u - \varepsilon(u), \quad \bar{\varepsilon}^{(\tau)}(u) = \sup\left\{0 \vee \bar{\varepsilon}^{(\tau)}(x), 0 \le x \le u\right\}. \tag{2.3}$$

Note that $\bar{\varepsilon}^{(\tau)}(\cdot)$ is a continuous function (because all the jumps of $\varepsilon^{(\tau)}$ are negative), starts from $\bar{\varepsilon}^{(\tau)}(0) = 0$ and ends at $\bar{\varepsilon}^{(\tau)}(1) = \tau$. We call interval of constancy of $\bar{\varepsilon}^{(\tau)}$ any interval component of $]0, 1[\backslash \mathrm{Supp}(d\bar{\varepsilon}^{(\tau)})$, where $\mathrm{Supp}(d\bar{\varepsilon}^{(\tau)})$ denotes the support of the Stieltjes measure $d\bar{\varepsilon}^{(\tau)}$. We write

$$F(\bar{\varepsilon}^{(\tau)}) = \left(F_1(\bar{\varepsilon}^{(\tau)}), \dots\right)$$

for the sequence of the lengths of the intervals of constancy of $\bar{\varepsilon}^{(\tau)}$, ranked in the decreasing order. See Figure 2.

We are now able to state the main result of this section which provides a simple representation of the ranked state of the aggregating server at time t. Recall that $\#(t)$ is the number of servers at time t and that $A_n = s_1 + \cdots + s_n$ is the time at which the system stops.

PROPOSITION 1. *Fix $t \in]0, A_n[$ and set $\tau = t^{-1}A_n - 1$. Then there are exactly $\#(t)$ intervals of constancy of $\bar{\varepsilon}^{(\tau)}$, and the decreasing sequence of their lengths, $F(\bar{\varepsilon}^{(\tau)})$, is $(t/A_n)R^{\downarrow}(t)$.*

PROOF. To ease the notation, we assume that the instant of the infimum of the bridge is $\mu = U_1$, that is (by Lemma 1) that the label at the initial time of the last customer that will be completely served is n. This induces no loss of generality as the mechanism of the server system is invariant by cyclic permutation of indices. For the sake of simplicity, we may also suppose that $U_1 = 0$ (recall that the excursion ε does not depend on U_1), so that $U_{i+1} = (s_1 + \cdots + s_i)/A_n$ and $\varepsilon = b$.

Denote the number of intervals of constancy of $\bar{\varepsilon}^{(\tau)}$ by k, and let $0 = g_1 < d_1 < \cdots < g_k < d_k < 1$ be the sequence of their left and right extremities. The left extremities g_i are jump times of the excursion ε and $d_i = \inf\{x > g_i : \varepsilon^{(\tau)}(x) > \varepsilon^{(\tau)}(g_i-)\}$, where we agree that $\varepsilon^{(\tau)}(0-) = 0$. Because $\varepsilon^{(\tau)}$ is a process with constant drift $\tau + 1 = t^{-1}A_n$ and has only negative jumps of size $-r_j$ at U_j, we have

$$\sum_{j=1}^{n} r_j \mathbf{1}_{\{g_i \le U_j \le d_i\}} = t^{-1} A_n (d_i - g_i), \quad i = 1, \dots, k. \tag{2.4}$$

We claim that the left-hand side of (2.4) is the service rate of the ith server at time t, where the labeling of servers is made according to the rules given in section 2.1. Indeed, consider the first interval $[g_1, d_1]$ and write m for the largest index such that $U_m \le d_1$. Then, whenever $m \ge 2$,

$$\varepsilon^{(\tau)}(U_j-) = t^{-1} A_n U_j - (r_1 + \cdots + r_{j-1}) < 0 \quad \text{for } j = 2, \dots, m$$

and whenever $m < n$,

$$\varepsilon^{(\tau)}(U_{m+1}-) = t^{-1} A_n U_{m+1} - (r_1 + \cdots + r_m) > 0.$$

Equivalently (recall that $A_n = s_1 + \cdots + s_n$), provided that $m \ge 2$,

$$(r_1 + \cdots + r_{j-1})\, t > s_1 + \cdots + s_{j-1} \quad \text{for } j = 2, \dots, m \tag{2.5}$$

and provided that $m < n$,

$$(r_1 + \cdots + r_m)\, t < s_1 + \cdots + s_m. \tag{2.6}$$

The inequality (2.5) implies that the initial first $(m-1)$ customers have been completely served at time t, and therefore, at time t, the initial first $(m-1)$ servers have already aggregated as a single server. Recall the initial nth customer has not been completely served at time $t < A_n$, so (2.6) implies that the initial mth customer has not been completely served either when $m < n$. We conclude that at time t, the first server is built from the servers with initial labels $1, \dots, m$ and no others.

An iteration of this argument completes the proof. □

2.3. *Markovian evolution.* Let us fix a decreasing sequence $r_1^{\downarrow} \ge \cdots \ge r_n^{\downarrow} > 0$ with $r_1^{\downarrow} + \ldots + r_n^{\downarrow} = 1$. We henceforth assume that the data of the server system are random. Specifically, let the customers pick their servers at time $t = 0$ according to the equiprobability, in the sense that the sequence of initial service rates $R(0) = (r_1, \dots, r_n)$ is a random permutation of $(r_1^{\downarrow}, \dots, r_n^{\downarrow})$. We also suppose that the initial quantities of service $s_1, \dots, s_n$ required by the customers are given by an i.i.d. sequence of exponential variables with parameter 1, and are independent of the customers' choice of the servers. This obviously implies that there are never more than one customer which is completely served at any given time.

We first point out that the aggregating server system can be described as a Markov chain.

PROPOSITION 2. (i) *The sequence of aggregation times*, $(A_k, k = 1, \dots, n)$, *has the law of the sequence of the first n jumps times of a Poisson process with intensity* 1, *and is independent of the sequence of the states* $(R(A_k), k = 0, \dots, n-1)$.

(ii) *The ranked sequence of the service rates* $(R^{\downarrow}(A_k), k = 0, \dots, n-1)$ *is a Markov chain with* $(n-1)$-*steps. Its transition probabilities are given as follows. For every* $k = 0, \dots, n-2$ *and* $1 \le i < j \le n-k$, *the conditional probability given* $R^{\downarrow}(A_k)$ *that the servers with respective service rates* $R_i^{\downarrow}(A_k)$ *and* $R_j^{\downarrow}(A_k)$ *aggregate at time* A_{k+1}, *is*

$$\frac{R_i^{\downarrow}(A_k) + R_j^{\downarrow}(A_k)}{n-k-1}.$$

PROOF. We shall first work with the *non-ranked* sequence of service rates $R(\cdot)$. Given the initial service rates $R(0)$, the time that the ith server (i.e., with service rate r_i) would need if it were alone to serve completely the ith customer is $e_i = s_i/r_i$. Plainly $e_1, \dots, e_n$ is a family of n independent exponential variables with parameters $r_1, \dots, r_n$. The first completed service thus occurs at time $A_1 = e_1 \wedge \cdots \wedge e_n$; recall $i(1)$ denotes the index of the first completely served customer, that is, $A_1 = e_{i(1)}$. By a standard property of independent exponential laws, A_1 has an exponential distribution with parameter $\sum_1^n r_i = 1$, and is independent of $i(1)$. Moreover, conditionally on A_1 and $i(1)$, the quantities of services required at time A_1 by the remaining $(n-1)$ customers are i.i.d. exponential variables with parameter 1 and the probability that $i_1 = j$ is $r_j / (\sum_1^n r_i) = r_j$.

Iterating this argument proves the assertion (i), and also that the sequence of service rates $(R(A_k), k = 0, \dots, n-1)$ is a Markov chain. More precisely, its probability transitions can be described as follows. For every $k = 0, \dots, n-2$ and $j = 1, \dots, n-k$, the conditional probability given $R(A_k) = (R_1(A_k), \dots, R_{n-k}(A_k))$ that the jth server [i.e., with service rate $R_j(A_k)$] is the one that will serve completely its customer at time A_{k+1}, is $R_j(A_k)$.

It should also be clear that since $R(0)$ is a random permutation of $R^{\downarrow}(0)$, $R(A_1)$ is a random permutation of $R^{\downarrow}(A_1)$ which is independent of the aggregation time A_1 and the ranked service rates $R^{\downarrow}(A_1)$. By iteration, this ensures that the Markov property of the non-ranked chain $(R(A_k), k = 0, \dots, n-1)$ propagates to the ranked chain $(R^{\downarrow}(A_k), k = 0 \dots, n-1)$. All that we need now is to calculate the transition probabilities. We work conditionally on the ranked state $R^{\downarrow}(A_k)$ of the system at the instant of the kth aggregation. Pick i, j in $\{1, \dots, n-k\}$ with $i \neq j$, and consider the event Λ_{ij} that after time A_k the first server which completely serves its customer is the one with serving rate $R_i^{\downarrow}(A_k)$ and that the server is aggregates with is the one with serving rate $R_j^{\downarrow}(A_k)$. By construction, Λ_{ij} is thus the intersection of two independent events with respective probabilities $R_i^{\downarrow}(A_k)$ (according to the first part of the proof) and $1/(n-k-1)$ (because customers pick their servers at random). In conclusion

$$\mathbf{P}\left(\Lambda_{ij}\right)=\frac{R_i^{\downarrow}(A_k)}{n-k-1}\,.$$

Plainly the events Λ_{ij} and Λ_{ji} are disjoint and their union is precisely the event that the pair of servers with serving rates $\{R_i^{\downarrow}(A_k), R_j^{\downarrow}(A_k)\}$ is the first to merge at time A_{k+1} as a single server in the system. In particular

$$\mathbf{P}\left(\Lambda_{ij}\cup\Lambda_{ji}\right)=\frac{R_i^{\downarrow}(A_k)+R_j^{\downarrow}(A_k)}{n-k-1}\,.$$

This completes the proof of the statement. □

Next we turn our attention to the bridge b defined in the preceding section (recall the notation used there). In this direction, we specify that the number U_1 used to define b is now random; more precisely that U_1 has the uniform distribution on $[0,1]$ and is independent of the preceding data.

LEMMA 2. *The bridge $b=(b(u), 0\le u\le 1)$ has exchangeable increments.*

PROOF. The variables $(s_1+\cdots+s_i)/A_n$, $i=1,\dots,n-1$, are the order statistics of $(n-1)$ independent uniformly distributed random variables. As U_1 is independent of the preceding and also uniformly distributed, we can think of $\{U_1,\dots,U_n\}$ as the set of values taken by n independent uniformly distributed variables on $[0,1]$. Because $R(0)=(r_1,\dots,r_n)$ is a random permutation of $R^{\downarrow}(0)$ and is independent of $U_1,\dots,U_n$, we now see that b has exchangeable increments. □

2.4. *Connection with the additive coalescent* Let us first recall the dynamics of the additive coalescent which has been studied in depth by Aldous, Evans and Pitman [4, 5, 12]. Consider at the initial time $\gamma_0=0$ a decreasing sequence $M(0)=(m_1,\dots,m_n)$ of n positive real numbers with $m_1+\cdots+m_n=1$, which are viewed as the sequence masses of clusters (arranged in the decreasing order) in a system with unit total mass. Suppose that pairs of clusters of masses $\{m_i, m_j\}$ merge into a single cluster of mass m_i+m_j at rate $\kappa(m_i,m_j)=m_i+m_j$, independently of the other pairs. The first coalescence occurs at time γ_1, at which the sequence of masses is re-ranked to form the decreasing sequence $M(\gamma_1)=(M_1(\gamma_1),\dots,M_{n-1}(\gamma_1))$, and the evolution of the system resumes with the same dynamic, and stops when the system reduces to the single unit mass. Let $M(t)$ denote the ranked sequence of masses in the system at time t and for $k=1,\dots,n-1$, γ_k the instant of the kth coalescence. In other words $M(t)=(M_1(\gamma_{k-1}),\dots,M_{n-k}(\gamma_{k-1}))$ for $t\in[\gamma_{k-1},\gamma_k[$.

It is easily checked from the preceding description that the durations between consecutive coalescence times, $\gamma_1, \gamma_2 - \gamma_1, \ldots, \gamma_{n-1} - \gamma_{n-2}$ form a sequence of independent exponential variables with parameters $n-1, n-2, \ldots, 1$, and are independent of the state chain $(M(\gamma_k), k = 0, \ldots, n-1)$. Moreover the latter is Markovian and its transitions can be described as follows. For every $k = 0, \ldots, n-2$, given $M(\gamma_k) = (x_1, \ldots, x_{n-k})$, the probability that $M(\gamma_{k+1})$ is the result of the coalescence of the clusters with masses x_i and x_j, $1 \le i < j \le n-k$, is

$$\frac{x_i + x_j}{n-k-1}.$$

We see from Proposition 2(ii) that the state-chains associated respectively to the additive coalescent and to the aggregating server system, $(M(\gamma_k), k = 0, \ldots, n-1)$ and $(R^{\downarrow}(A_k), k = 0, \ldots, n-1)$, have the same transition probabilities. Comparing the distributions of the jump times in the additive coalescent process and in the aggregating server system enable us connect the two by a simple time-substitution. Recall that $\#(t)$ denotes the number of servers present at time t in the aggregating server system, and that A_{n-1} is the instant when the ultimate aggregation of servers occurs. Set

$$I(t) = \int_0^t \frac{du}{\#(u) - 1}, \quad t \le A_{n-1}$$

and introduce the time change

$$T(t) = u \iff I(u) = t, \quad t < I(A_{n-1}).$$

COROLLARY 1. *If the initial states* $R^{\downarrow}(0)$ *and* $M(0)$ *are the same, then* $(M(t), 0 \le t \le \gamma_{n-1})$ *has the same law as* $(R^{\downarrow}(T(t)), 0 \le t \le I(A_{n-1}))$.

PROOF. Plainly, the sequence of the states of the time-changed process $R^{\downarrow}(T(\cdot))$ is

$$R^{\downarrow}(T(I(A_k))) = R^{\downarrow}(A_k), \quad k = 0, \ldots, n-1,$$

and we know from above that it has the same law as the sequence of the states of the additive coalescent process.

On the other hand, we know from Proposition 2(i) that $(n - \#(t), 0 \le t \le A_{n-1})$ is a Poisson process with unit rate, stopped at its first passage time at $n-1$, and is independent of the sequence of the successive states of the aggregating server system. It follows that the increments of I,

$$I(A_{k+1}) - I(A_k) = \frac{A_{k+1} - A_k}{n-k-1}, \quad k = 0, \ldots, n-2 \tag{2.7}$$

form a sequence of independent exponential variables with parameters $n-1, n-2, \ldots, 1$; thus they have the same law as the sequence of the durations between successive coalescence times in the additive coalescent process. As for both the additive coalescent process and the aggregating server process, the sequence of coalescence (aggregation) times is independent of the chain of successive states, the proof is complete. $\square$

3. Asymptotic regimes. We are concerned here with the asymptotic behavior of various objects considered in the preceding section when the initial number n of clusters (or servers) tends to infinity. To that end, we shall first investigate continuity properties of the functionals of paths appearing in Section 2.2. We will then consider bridges with exchangeable increments in the framework of Kallenberg [14]. Putting the pieces together then readily yields the main result on the asymptotic behavior of additive coalescent processes.

3.1. *Continuity of functionals of bridges and excursions.* We write $\mathbf{D}$ for the space of càdlàg paths $\omega : [0,1] \to \mathbf{R}$, and $\mathbf{B}$ for the subspace of bridges, that is, of paths $\omega \in \mathbf{D}$ with $\omega(0) = \omega(1) = \omega(1-) = 0$. Both $\mathbf{D}$ and $\mathbf{B}$ are endowed with Skorohod's topology, so a sequence $(\omega_n, n \in \mathbf{N})$ converges in $\mathbf{D}$ (or in $\mathbf{B}$) to ω if and only if there exists increasing bijections $\alpha_n : [0,1] \to [0,1]$ such that $\lim_{n\to\infty} \alpha_n(x) = x$ for all $x \in [0,1]$ and $\omega_n \circ \alpha_n$ converges uniformly on $[0,1]$ to ω as $n \to \infty$.

Given a path $\omega \in \mathbf{D}$, call $\mu \in [0,1]$ a location of infimum for ω if

$$\omega(\mu-) \wedge \omega(\mu) = \inf\{\omega(u), u \in [0,1]\}\ .$$

If now $\omega \in \mathbf{B}$ is a bridge, we denote by $\bar\mu$ the largest location of its infimum and set

$$\varepsilon_\omega(u) = \omega\left(u + \bar\mu[\mathrm{mod}\ 1]\right) - \inf_{[0,1]} \omega, \quad 0 \le u \le 1\,.$$

Plainly ε_ω is a path in $\mathbf{D}$ which only takes nonnegative values; it will be referred to as the excursion associated to ω. We leave to the reader the proof of the following elementary lemma.

LEMMA 3. *Let $\omega \in \mathbf{B}$ be a bridge; suppose that ω has a unique location of infimum, $\bar\mu = \mu$, and that ω is continuous at μ. Consider a sequence $(\omega_n, n \in \mathbf{N})$ that converges in $\mathbf{B}$ to ω. Then the sequence $(\varepsilon_{\omega_n}, n \in \mathbf{N})$ of the excursions associated to $(\omega_n, n \in \mathbf{N})$ converges in $\mathbf{D}$ to ε_ω.*

Next, we denote by $\mathbb{S}$ the space of decreasing numerical sequences $x_1 \ge x_2 \ge \cdots \ge 0$ with $\sum_1^\infty x_n < \infty$, and by $\mathbb{S}_1$ the subspace of $\mathbb{S}$ of sequences with $\sum_1^\infty x_n = 1$. Both are endowed with the ℓ^1-norm. Given an increasing path $\mathtt{i} \in \mathbf{D}$, we write

$$F(\mathtt{i}) = (F_1(\mathtt{i}), F_2(\mathtt{i}), \dots) \in \mathbb{S}$$

for the sequence of the lengths of the intervals components of the complement of the support of the Stieltjes measure $d\mathtt{i}$, arranged in the decreasing order. Finally, for any path $\omega \in \mathbf{D}$, we set

$$\bar\omega(u) = \sup\left\{\omega(v)^+, 0 \le v \le u\right\}, \quad u \in [0,1]$$

where x^+ stands for the positive part of the real number x.

LEMMA 4. *Let* $\omega \in \mathbf{D}$ *be such that*

$$\omega(u) \vee \omega(u-) < \bar{\omega}(u) \quad \textit{for every } u \in]a, b[$$

whenever $]a, b[\subseteq [0, 1]$ *is an interval of constancy for* $\bar{\omega}$. *Consider a sequence* $(\omega_n, n \in \mathbf{N})$ *that converges in* $\mathbf{D}$ *to* ω. *Then* $F(\bar{\omega}_n)$ *converges pointwise to* $F(\bar{\omega})$. *If moreover* $F(\bar{\omega}) \in \mathbb{S}_1$, *then* $F(\bar{\omega}_n)$ *converges to* $F(\bar{\omega})$ *in* $\mathbb{S}$ (*i.e., for the* ℓ^1-*norm*).

PROOF. The argument for the proof of the first assertion is easy, and we again leave it to the reader. So suppose that $F(\bar{\omega}) \in \mathbb{S}_1$. We know from the first part that for each $k = 1, 2, \ldots$, $F_k(\bar{\omega}_n)$ converges to $F_k(\bar{\omega})$ as $n \to \infty$, and by Fatou's lemma and our assumption that

$$\limsup_{n\to\infty} \sum_{k=1}^{\infty} F_k(\bar{\omega}_n) \leq 1 = \sum_{k=1}^{\infty} F_k(\bar{\omega}).$$

An application of Scheffé's lemma entails that $F(\bar{\omega}_n)$ converges to $F(\bar{\omega})$ in ℓ^1. □

3.2. *Bridges with exchangeable increments and excursions.* In this section, we consider a decreasing sequence of nonnegative real numbers $\theta = (\theta_1, \ldots)$ with

$$\sum_{j=1}^{\infty} \theta_j^2 \leq 1 \tag{3.8}$$

and set

$$\sigma^2 = 1 - \sum_{j=1}^{\infty} \theta_j^2.$$

We shall further suppose that

$$\text{either } \sigma > 0 \text{ or } \sum_{i=1}^{\infty} \theta_i = \infty. \tag{3.9}$$

Following Kallenberg [14], we introduce b_θ, a bridge with exchangeable increments given by

$$b_\theta(x) = \sigma \mathtt{b}(x) + \sum_{j=1}^{\infty} \theta_j \left(\mathbf{1}_{\{x \geq V_j\}} - x\right), \quad x \in [0, 1] \tag{3.10}$$

where $(\mathtt{b}(x), 0 \leq x \leq 1)$ is a standard Brownian bridge and $V_1, \ldots$ an i.i.d. sequence of uniform random variables on $[0, 1]$ which is independent of the bridge $\mathtt{b}$. More precisely, the series in (3.10) converges uniformly on $[0, 1]$, a.s. The hypothesis (3.9) is a necessary and sufficient condition for b_θ to have unbounded variation a.s.

In the sequel, we shall need the following property which is doubtless known.

LEMMA 5. *For every* $c \in \mathbf{R}$, *we have a.s.*

$$\inf\{u \geq 0 : b_\theta(u) > cu\} = 0.$$

PROOF. Recall from Theorem 2.2 of Kallenberg [15] that under the assumption (3.9), we have a.s.

$$\limsup_{u\to 0+} \frac{|b_\theta(u)|}{u} = \infty .$$

So, if the conclusion of the lemma failed, then we would have with positive probability

$$\limsup_{u\to 0+} \frac{b_\theta(u)}{u} \le c \quad \text{and} \quad \liminf_{u\to 0+} \frac{b_\theta(u)}{u} = -\infty .$$

This is impossible, because it is well-known that the process $(\frac{1}{u} b_\theta(u), 1 \ge u > 0)$ is a backward (i.e., when the time parameter u decreases from 1 to 0) martingale which has obviously no positive jumps, and such martingales cannot converge to $-\infty$. □

Next, we consider for each $n \in \mathbf{N}$ a decreasing sequence $r^{\downarrow}_{n,1} \ge \cdots \ge r^{\downarrow}_{n,n} > 0$ of positive real numbers with $r^{\downarrow}_{n,1} + \cdots + r^{\downarrow}_{n,n} = 1$. We set

$$\sigma_n^2 = \sum_{i=1}^{n} \left(r^{\downarrow}_{n,i}\right)^2 ,$$

and we assume that

$$\lim_{n\to\infty} \sigma_n^2 = 0 \quad \text{and} \quad \lim_{n\to\infty} r^{\downarrow}_{n,i}/\sigma_n = \theta_i \quad \text{for every } i = 1, \ldots . \tag{3.11}$$

We denote by b_n the bridge with exchangeable increments that is distributed as the process b defined by (2.1) for $r^{\downarrow}_j = r^{\downarrow}_{n,j}$, $j = 1, \ldots, n$ (cf. also Lemma 2). Note that b_n coincides in distribution with the bridge b_θ defined by (3.10) when $\theta_j = r^{\downarrow}_{n,j}$ for $j = 1, \ldots, n$ and $\theta_j = 0$ for $j > n$.

The hypothesis (3.11) ensures that as $n \to \infty$,

$$\sigma_n^{-1} b_n \text{ converges in distribution on } \mathbf{D} \text{ to } b_\theta; \tag{3.12}$$

see Kallenberg [14]. To ease the notation, we write ε_θ and ε_n for the excursions associated to b_θ and b_n, respectively.

LEMMA 6. *As $n \to \infty$, $\sigma_n^{-1}\varepsilon_n$ converges in distribution on $\mathbf{D}$ to ε_θ.*

PROOF. We check that the sample paths of the bridge $b_\theta = \omega$ fulfill a.s. the requirements of Lemma 3. It is known that with probability one, b_θ reaches its the overall infimum at a unique location μ_θ; see Theorems 1.3(a) and 1.5 in Knight [16]. On the other hand, b_θ has only positive jumps, so $b_\theta(\mu_\theta) - b_\theta(\mu_\theta-)$ can be viewed as the initial jump (if any) of the post-infimum process obtained by splitting the path b_θ at μ_θ. It follows from a version of the Sparre-Andersen identity for processes with exchangeable increments that the latter quantity has the same law as the first positive jump of b_θ across 0, that is, $b_\theta(\rho) - b_\theta(\rho-)$ where $\rho = \inf\{u \ge 0 : b_\theta(u) > 0\}$. See e.g. Theorem 3.1 and the comment on page 28 in [6]. According to Lemma 5, $\rho = 0$ a.s. and we conclude that b_θ is continuous at μ_θ a.s. Our claim thus follows from (3.12) and Lemma 3. □

Finally, just as in (2.3), introduce for an arbitrary $\tau > 0$

$$\varepsilon_\theta^{(\tau)}(u) = \tau u - \varepsilon_\theta(u), \quad u \in [0,1] \tag{3.13}$$

and recall that $\bar{\varepsilon}_\theta^{(\tau)}$ stands for the continuous supremum process of $\varepsilon_\theta^{(\tau)}$ and $F(\bar{\varepsilon}_\theta^{(\tau)})$ for the sequence of the lengths of the intervals of constancy of $\bar{\varepsilon}_\theta^{(\tau)}$, ranked in the decreasing order. Plainly the intervals of constancy of $\bar{\varepsilon}_\theta^{(\tau)}$ get finer as τ increases, so $(F(\bar{\varepsilon}_\theta^{(\tau)}), \tau \geq 0)$ can be thought of as a fragmentation process.

In order to apply Lemma 4 to the sample paths of $\varepsilon_\theta^{(\tau)}$, we shall need the following technical result.

LEMMA 7. *We have with probability one that*:

(i) $\varepsilon_\theta^{(\tau)}(u) < \bar{\varepsilon}_\theta^{(\tau)}(u)$ *for every* $u \in]a,b[$ *whenever* $]a,b[\subseteq [0,1]$ *is an interval of constancy for* $\bar{\varepsilon}_\theta^{(\tau)}$.

(ii) $F(\bar{\varepsilon}_\theta^{(\tau)}) \in \mathbb{S}_1$.

PROOF. Recall that μ_θ denotes the a.s. unique location of the infimum of b_θ. The argument is based on the fact that

(3.14) μ_θ and ε_θ are independent and μ_θ is uniformly distributed on $[0,1]$.

This is a straightforward extension of results in [16]. More precisely, pick an arbitrary $y \in [0,1]$ and set $b'(x) = b_\theta(x + y\,[\mathrm{mod}\ 1]) - b_\theta(y)$. Then b' has the same law as b_θ and, in the obvious notation, $\mu' = \mu_\theta + y[\mathrm{mod}\ 1]$ and $\varepsilon' = \varepsilon_\theta$. This entails (3.14).

(i) If (i) failed with positive probability, then by (3.14) and the fact that b_θ can be recovered by splitting ε_θ at $1 - \mu_\theta$, we would deduce that with positive probability, the process with exchangeable increments $\tau u - b_\theta(u)$, $u \in [0,1]$, reaches the same local maximum at two distinct locations. We see from Corollary 1.4 and Lemma 1.2 in Knight [16] that this is impossible.

(ii) Fix $u \in]0,1[$ so that by (3.14),

$$\mathbf{P}\left(\varepsilon_\theta^{(\tau)}(u) = \bar{\varepsilon}_\theta^{(\tau)}(u)\right) = \frac{1}{u}\mathbf{P}\left(\varepsilon_\theta^{(\tau)}(u) = \bar{\varepsilon}_\theta^{(\tau)}(u), \mu_\theta > 1-u\right).$$

Again recovering b_θ by splitting ε_θ at $1 - \mu_\theta$, we get

$$\mathbf{P}\left(\varepsilon_\theta^{(\tau)}(u) = \bar{\varepsilon}_\theta^{(\tau)}(u)\right) \leq \frac{1}{u}\int_0^u \mathbf{P}\left(\tau y - b_\theta(y) = \sup_{0\leq x\leq y}(\tau x - b_\theta(x))\right) dy.$$

On the other hand, it is seen by time-reversal that

$$\mathbf{P}\left(\tau y - b_\theta(y) = \sup_{0\leq x\leq y}(\tau x - b_\theta(x))\right) = \mathbf{P}\left(b_\theta(x) - \tau x \leq 0 \text{ for all } x \in [0,y]\right),$$

and we know from Lemma 5 that the right-hand side is zero. It now follows from Fubini theorem that $\{u \in [0,1] : \varepsilon_\theta^{(\tau)}(u) = \bar{\varepsilon}_\theta^{(\tau)}(u)\}$ has Lebesgue measure zero a.s., which concludes the proof. □

3.3. *A limit theorem for additive coalescents.* We are now able to investigate the asymptotic behavior of additive coalescents. Recall that we consider for every integer n a ranked probability distribution $(r_{n,1}^{\downarrow}, \dots, r_{n,n}^{\downarrow})$ which represents the ranked masses of the initial n clusters, and which can also be thought of as the ranked service rates of the n servers at the initial time. We write $M^{(n)} = (M^{(n)}(t), t \geq 0)$ for the additive coalescent process started at time $t = 0$ with n clusters with masses $r_{n,1}^{\downarrow} \geq \cdots \geq r_{n,n}^{\downarrow} > 0$. More precisely $M^{(n)}(t) = (M_1^{(n)}(t), \dots)$ where $M_k^{(n)}(t)$ is the mass of the kth heavier cluster at time t with the convention that $M_k^{(n)}(t) = 0$ when there are less than k clusters at time t.

Just as in Aldous and Pitman [5], we assume that the asymptotic behavior as $n \to \infty$ of these ranked probabilities is given by (3.11) (recall also the conditions (3.8) and (3.9) on the limit θ). It has been proved in [5] that in this situation, the sequence of shifted coalescent processes $(M^{(n)}(-\log \sigma_n - \log t)), t \geq \sigma_n)$ has a limit distribution as $n \to \infty$, which can be described in terms of a certain inhomogeneous random tree cut by an independent Poisson point process on its skeleton. We are now able to state the following variant of this result, which provides an alternative description of the limit process. Recall that $\varepsilon_\theta^{(\tau)}$ is the excursion process with drift τ defined by (3.13), and that $\mathbb{S}_1$ denotes the space of decreasing numerical sequences with sum 1, endowed with the ℓ^1-norm.

THEOREM 1. *Under the preceding assumptions, the $\mathbb{S}_1$-valued process*

$$\left(M^{(n)}(-\log \sigma_n - \log t)), 0 < t < 1/\sigma_n\right)$$

converges in the sense of finite dimensional distributions as $n \to \infty$ toward the fragmentation process

$$\left(F(\bar{\varepsilon}_\theta^{(t)}), t \geq 0\right) .$$

PROOF. To ease the notation, we shall only prove the convergence for the one-dimensional distributions. The argument obviously extends to finite dimensional distributions.

We shall mainly work in the setting of the aggregating server system. For every $n \in \mathbf{N}$, consider the process $R^{(n)\downarrow}$ of ranked service rates defined in Section 2 when at the initial time, the ranked service rates are given by $r_j^{\downarrow} = r_{n,j}^{\downarrow}$, $j = 1, \dots, n$. In the obvious notation, we know from Proposition 2(i) that the durations between successive aggregating times $(A_{k+1}^{(n)} - A_k^{(n)}, k = 0, \dots, n-2)$ form a sequence of $(n-1)$ i.i.d. variables distributed according to the exponential law with parameter 1. Fix $t \geq 0$ and set

$$t(n) := A^{(n)}_{[n-n\sigma_n t]}$$

where $[\cdot]$ denotes the integer part. Recall from (2.7) that for $k = 1, \dots, n$,

$$I^{(n)}(A_k^{(n)}) = \sum_{j=1}^{k} \frac{A_j^{(n)} - A_{j-1}^{(n)}}{n-j}.$$

It is easily seen that if we define

$$o(n,t) := I^{(n)}\left(A^{(n)}_{[n-n\sigma_n t]}\right) + \log \sigma_n + \log t,$$

then

$$\lim_{n\to\infty} o(n,t) = 0 \quad \text{in probability.} \tag{3.15}$$

Recall also that $T^{(n)}$ is the inverse of the functional $I^{(n)}$ that appears in the time substitution of Corollary 1. By definition,

$$T^{(n)}\left(-(\log \sigma_n + \log t) + o(n,t)\right) = t(n),$$

and by Corollary 1,

$$M^{(n)}\left(-(\log \sigma_n + \log t) + o(n,t)\right) \stackrel{d}{=} R^{(n)\downarrow}(t(n)).$$

We study the right-hand side using the representation in terms of lengths of intervals of constancy of the supremum process of an excursion with drift. Recall that b_n denotes the bridge with exchangeable increments that is distributed as the process b defined by (2.1) when $r_j^{\downarrow} = r_{n,j}^{\downarrow}$, $j = 1, \dots, n$, and that ε_n is the excursion associated to b_n by (2.2). We know from Proposition 1 that

$$\frac{t(n)}{A_n^{(n)}} R^{(n)\downarrow}(t(n)) = F\left(\bar{\varepsilon}_n^{(\tau(n))}\right),$$

where

$$\tau(n) = \frac{A_n^{(n)}}{t(n)} - 1 = \frac{A_n^{(n)} - A^{(n)}_{[n-n\sigma_n t]}}{A^{(n)}_{[n-n\sigma_n t]}}.$$

On the one hand, as $A_k^{(n)} = A_1^{(n)} + \cdots + (A_k^{(n)} - A_{k-1}^{(n)})$ is the sum of k i.i.d. exponential variables with unit mean, it is seen from the weak law of large numbers that

$$\lim_{n\to\infty} \frac{t(n)}{A_n^{(n)}} = 1 \quad \text{and} \quad \lim_{n\to\infty} \sigma_n^{-1}\tau(n) = t \quad \text{in probability.} \tag{3.16}$$

On the other hand, introduce the sequence of processes

$$\eta_n(u) = \sigma_n^{-1}\left(\tau(n)u - \varepsilon_n(u)\right), \quad u \in [0,1],$$

and note that $F\left(\bar{\varepsilon}_n^{(\tau(n))}\right) = F\left(\bar{\eta}_n\right)$. Recall from Lemma 6 that $\sigma_n^{-1}\varepsilon_n$ converges in distribution on $\mathbf{D}$ to ε_θ as $n \to \infty$, so by (3.16) η_n converges in distribution on $\mathbf{D}$ to $\varepsilon_\theta^{(t)}$. Thanks to Lemmas 4 and 7, we deduce that

$$\lim_{n\to\infty} F\left(\bar{\varepsilon}_n^{(\tau(n))}\right) = \lim_{n\to\infty} F\left(\bar{\eta}_n\right) = F\left(\bar{\varepsilon}_\theta^{(t)}\right) \quad \text{in distribution on } \mathbb{S}_1,$$

and hence

$$\lim_{n\to\infty} M^{(n)}(-(\log\sigma_n + \log t) + o(n,t)) = F\left(\bar{\varepsilon}_\theta^{(t)}\right) \quad \text{in distribution on } \mathbb{S}_1. \tag{3.17}$$

To complete the proof, we point out that the process $(F(\bar{\varepsilon}_\theta^{(t)}), t \geq 0)$ is continuous in probability at each time t (this follows from Lemmas 4 and 7) and that for every $k = 1, \ldots,$ the processes $M_1^{(n)}(t) + \cdots + M_k^{(n)}(t)$ are monotone decreasing in the time variable t. Theorem 1 now derives from a standard argument of monotonicity from (3.15) and (3.17). $\square$

4. Miscellaneous comments. The dynamics of the aggregating server system bear some striking similarities with hashing with linear probing in computer science. That the latter area is related to the additive coalescent has been observed by Chassaing and Louchard [11]. See also [10] and the references therein.

It should be pointed out that Aldous and Limic [3] (see also [1]) have given representations of the state at a fixed time of eternal *multiplicative* coalescents using as well ladder times of certain Lévy type processes.

When $\theta \equiv 0$, b_0 is the Brownian bridge, and according to Vervaat [18], ε_0 is the normalized Brownian excursion. Hence $(F(\bar{\varepsilon}_0^{(t)}), t \geq 0)$ is the fragmentation process constructed from the Brownian excursion in [8].

We stress that Aldous and Pitman [5] have also shown that conversely, any *extreme* eternal additive coalescent can be obtained as the limit of a sequence of additive coalescent processes fulfilling the assumptions of Theorem 1. In this direction, we focussed here on the situation where the ranked masses of clusters at the initial time are deterministic. Nonetheless, as mixtures of bridges of the type (3.10) are again bridges with exchangeable increments (see Kallenberg [14]), the present approach yields more generally a construction of any (i.e., not necessarily extreme) eternal additive coalescent based on some bridge with exchangeable increments.

The present construction of eternal additive coalescents should have a crucial role to extend the connection between the standard additive coalescent and sticky particle systems with Brownian initial velocity (see [7]) to some other classes of random initial velocities.

Let us discuss the situation when the assumption (3.9) fails, that is, suppose that $\sigma = 0$ and $\Theta := \sum_{i=1}^{\infty} \theta_i < \infty$. The argument as in the proof of Proposition 1 shows that for every $\tau \geq 0$, one has a.s.

$$\sum_{i=1}^{\infty} F_i(\bar{\varepsilon}_\theta^{(\tau)}) = \frac{\Theta}{\Theta + \tau}\,.$$

Informally, this implies that a portion with mass $\tau/(\Theta+\tau)$ of the system has been reduced to dust at time τ by the fragmentation process.

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