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PERTURBATION OF THE EQUILIBRIUM FOR A TOTALLY ASYMMETRIC STICK PROCESS IN ONE DIMENSION[1]

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We study the evolution of a small perturbation of the equilibrium of a totally asymmetric one-dimensional interacting system. The model we take as an example is Hammersley's process as seen from a tagged particle, which can be viewed as a process of interacting positive-valued stick heights on the sites of $\mathbf{Z}$. It is known that under Euler scaling (space and time scale n) the empirical stick profile obeys the Burgers equation. We refine this result in two ways. If the process starts close enough to equilibrium, then over times n^ν for $1 \le \nu < 3$, and up to errors that vanish in hydrodynamic scale, the dynamics merely translates the initial stick configuration. In particular, on the hydrodynamic time scale, diffusive fluctuations are translated rigidly. A time evolution for the perturbation is visible under a particular family of scalings: over times n^ν, $1 < \nu < 3/2$, a perturbation of order $n^{1-\nu}$ from equilibrium follows the inviscid Burgers equation. The results for the stick model are derived from asymptotic results for tagged particles in Hammersley's process.

1. Introduction. A number of recent papers have sought various refinements to the basic hydrodynamic limits of interacting particle systems. One type of refinement is to study the time evolution of a small perturbation of the equilibrium of the process. For asymmetric exclusion in dimensions 3 and higher, Esposito, Marra and Yau (1994) proved that under diffusive scaling (time scale n^2 and space scale n) a perturbation of order n^{-1} follows a conservation law with a diffusion term. The backdrop of this result is the standard hydrodynamic limit of asymmetric processes, which leads to a conservation law *without* diffusion term under *Euler scaling* (time scale and space scale both n). A context for the result of Esposito, Marra and Yau is the search for microscopic interpretations of the Navier–Stokes equations. We refer the reader to pages 185–188 of Kipnis and Landim's (1999) monograph for a description of this program and further references.

Our paper looks at the question of Esposito, Marra and Yau in one dimension. We add a perturbation of order $n^{-\beta}$ to the equilibrium, $\beta > 0$. The perturbation vanishes in the hydrodynamic limit $n \to \infty$, and we study the effect of this perturbation under various time scales $n^\nu t$, $\nu \ge 1$. We have two types of results: (1) for $\beta \in (0, 1/2)$, a hydrodynamic limit in the time scale $n^{1+\beta} t$ shows that the perturbation obeys macroscopically the Burgers equation *without* diffusion term; and (2) for $\nu \in [1, 3)$ and β close enough to 1, we

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show that the dynamics is simply a translation of the initial configuration, up to $o(n)$ error terms.

The most popular models for studies of the hydrodynamics of asymmetric stochastic dynamics are the exclusion and zero-range processes. Instead of these processes, we prove our results for the so-called *stick process*, which can also be regarded as Hammersley's process as seen from a tagged particle. This process has nonnegative variables $\eta(i)$ (stick heights) on the sites of $\mathbf{Z}$ that exchange pieces between each other. The stick process lacks some of the good properties of the exclusion or zero-range process: the state space is not compact, the rates are unbounded, and the amount of material that jumps is also unbounded.

To make up for these complications, the totally asymmetric one-dimensional process has a beautiful combinatorial structure uncovered by Aldous and Diaconis (1995). This structure connects Hammersley's process and the stick model to the increasing sequences problem on planar Poisson points. A key ingredient of our proofs are sharp deviation estimates for the increasing sequences problem from Kim (1996), Seppäläinen (1998b) and Baik, Deift and Johansson (1999).

We believe that the results of our paper hold also for totally asymmetric exclusion and zero-range processes. The basis for this conjecture is that these processes possess particle-level variational formulations that involve a planar growth model, analogous to the increasing sequences connection of Hammersley's process [Seppäläinen (1998a, c)]. Johansson (2000) has shown that the limiting fluctuations for this growth model are the same as for the increasing sequences model. At the moment our proof cannot be carried out for exclusion or zero-range processes because estimate (5.5) has not been derived for these models. See Remark 5.2 in Section 5.

ORGANIZATION OF THE PAPER. In Section 2 we describe the stick model and state the results mentioned previously. Theorem 1 gives the translation, and Theorem 2 the hydrodynamic limit of the perturbation. Theorems 1 and 2 are corollaries of corresponding Theorems 3 and 4 for tagged particles in Hammersley's process. These are stated in Section 3. The translation Theorem 3 for Hammersley's process is compared to a similar result of Ferrari and Fontes (1994) for asymmetric exclusion. Section 4 addresses briefly the rigorous construction of Hammersley's process and the stick process and the connection with increasing sequences. Sections 5–7 contain the proofs. Section 5 contains lemmas, Section 6 the proof of Theorem 3, and Section 7 the proof of Theorem 4. A frequently used notation is $[x] = \max\{n \in \mathbf{Z} : n \le x\}$.

2. The stick model and the results. Here is an informal description of the model. A rigorous construction will follow in Section 4. The state of the process is a configuration $\eta = (\eta(i) : i \in \mathbf{Z})$, where each $\eta(i)$ is a nonnegative real number. Think of $\eta(i)$ as the height of a vertical stick attached to site $i \in \mathbf{Z}$. At exponential rate equal to $\eta(i)$, the following event takes place: pick a random quantity u uniformly distributed on $[0, \eta(i)]$. Break off a piece of length u from the stick at i, and attach this piece to the stick at site $i+1$. Thus, if the neighboring stick lengths before the event were $(\eta(i), \eta(i+1))$, then after the event they would be $(\eta(i)-u, \eta(i+1)+u)$. These events happen at all sites i independently of each other. In the language of generators, this dynamics is expressed as

$$Lf(\eta) = \sum_{i\in\mathbf{Z}} \int_0^{\eta(i)} \big[f(\eta^{u,i,i+1}) - f(\eta)\big]\,du, \tag{2.1}$$

where $\eta^{u,i,i+1}$ is the configuration after the jump,

$$\eta^{u,i,i+1}(j) = \begin{cases} \eta(i) - u, & j = i,\\ \eta(i+1) + u, & j = i+1,\\ \eta(j), & j \neq i, i+1.\end{cases}$$

In Seppäläinen (1996) a Markov process $\eta(t) = (\eta(i,t) : i \in \mathbf{Z})$, $t \geq 0$, is constructed that operates according to the description given previously. The state space of the process is

$$Y = \left\{\eta \in [0,\infty)^{\mathbf{Z}} : \lim_{N\to-\infty} N^{-2} \sum_{i=N}^{-1} \eta(i) = 0\right\} \tag{2.2}$$

and the paths of the process are in the Skorohod space $D([0,\infty), Y)$. Note that Y is not closed in the product topology, but is given a stronger topology with a complete, separable metric. L in (2.1) is the generator of the process, in the sense that

$$E^\eta\big[f\big(\eta(t)\big)\big] - f(\eta) = \int_0^t E^\eta\big[Lf\big(\eta(s)\big)\big]\,ds \tag{2.3}$$

for all bounded continuous cylinder functions f on Y and all initial states $\eta \in Y$. Here E^η stands for the expectation under the path measure of the process started at state η. Furthermore, the process has a one-parameter family of invariant distributions, namely, the i.i.d. exponential distributions on the variables $(\eta(i) : i \in \mathbf{Z})$.

We focus now on the hydrodynamic behavior of this process. The basic result [Seppäläinen (1996)] is that under Euler scaling the empirical stick profile obeys the Burgers equation. Suppose $u(x,t)$, $(x,t) \in \mathbf{R}\times[0,\infty)$, is the entropy solution of the Burgers equation

$$u_t + (u^2)_x = 0, \qquad u(x,0) = u_0(x), \tag{2.4}$$

with nonnegative initial data $u_0 \in L^\infty(\mathbf{R})$. Consider a sequence η^n, $n = 1, 2, 3, \ldots$, of stick processes, and assume that a law of large numbers is satisfied at time $t = 0$: for all $a < b$ in $\mathbf{R}$,

$$\lim_{n\to\infty} n^{-1} \sum_{i=[na]+1}^{[nb]} \eta^n(i,0) = \int_a^b u_0(x)\,dx \quad \text{in probability.} \tag{2.5}$$

The theorem is that the law of large numbers continues to hold at all later times $t > 0$:

$$\lim_{n\to\infty} n^{-1} \sum_{i=[na]+1}^{[nb]} \eta^n(i, nt) = \int_a^b u(x,t)\,dx \quad \text{in probability.} \tag{2.6}$$

Euler scaling refers to the scaling in the preceding limit, where the ratio of macrosopic and microscopic units is the same n for both space and time. A macroscopic space interval $(a, b]$ corresponds to approximately $n(b-a)$ microscopic lattice sites, and macrosopic time t corresponds to microscopic time nt. The derivation of the hydrodynamic limit (2.6) from the hypothesis (2.5) requires some technical assumptions, and the details can be found in Seppäläinen (1996).

A trivial special case of the hydrodynamic limit is of course the case of a process in equilibrium: if the sticks are initially i.i.d. exponentially distributed with common expectation $E[\eta(i,0)] = q$, then this situation persists, and the macroscopic profile is the constant $u(x,t) \equiv q$.

In the present paper we study the evolution of a small perturbation of the equilibrium. The initial macroscopic profile is

$$u_0(x) = q + n^{-\beta} v_0(x), \tag{2.7}$$

where $q > 0$ is the fixed equilibrium density, v_0 is a bounded measurable function on $\mathbf{R}$ and $\beta \in (0, 1]$ is a parameter that we adjust to investigate different scalings. The function v_0 is not assumed to take any particular sign, so to have nonnegative profiles we consider only n large enough to have $q > n^{-\beta}\|v_0\|_\infty$. For each n, the initial stick configuration $(\eta^n(i,0) : i \in \mathbf{Z})$ is assumed to be in *local equilibrium* with macroscopic profile u_0. Precisely speaking, our assumption is this:

(2.8) The variables $(\eta^n(i,0) : i \in \mathbf{Z})$ are mutually independent, exponentially distributed and have expectations

$$E\big[\eta^n(i,0)\big] = q + n^{1-\beta} \int_{(i-1)/n}^{i/n} v_0(x)\,dx.$$

The perturbation of the expected density is taken to be

$$n^{-\beta} \cdot \big\{\text{the average of } v_0 \text{ over the interval } \big((i-1)/n, i/n\big]\big\},$$

instead of $n^{-\beta} \cdot \{\text{the point value } v_0(i/n)\}$ because we make no regularity assumption on v_0. If v_0 is Lipschitz continuous, we can substitute $v_0(i/n)$ for $n\int_{(i-1)/n}^{i/n} v_0(x)\,dx$ in (2.8), and all the results remain true. A standing assumption is also that $q > 0$. In Section 4.1 we explain how the case $q = 0$ is reduced to the basic hydrodynamic limit (2.5)–(2.6) when the time scale is chosen appropriately.

Since $\|u_0 - q\|_\infty \to 0$ as $n \to \infty$, the limit (2.6) is valid again with constant profile $u(x,t) \equiv q$. To escape the regime of (2.6), we subtract the equilibrium density q and speed up time more, beyond the hydrodynamic scale nt.

We introduce a second parameter $\nu \in [1, \infty)$, and look at the evolution of the stick profile over times of order n^{ν}. The space scaling will be the same as in (2.5)–(2.6), so the lattice of sites scales as $n^{-1}\mathbf{Z}$. Our object of study is the empirical perturbation profile

$$\sum_{i \in \mathbf{Z}} \big\{ \eta^n(i, n^{\nu} t) - q \big\} \delta_{i/n}.$$

In other words, we follow either integrals $\sum_{i \in \mathbf{Z}} \big\{ \eta^n(i, n^{\nu} t) - q \big\} \phi(i/n)$ of compactly supported, continuous test functions ϕ, or, equivalently, the total stick mass in macroscopic intervals $(x, y]$, $\sum_{i=[nx]+1}^{[ny]} \eta^n(i, n^{\nu} t) - nq(y - x)$.

Let us derive an easy "benchmark" result against which we can compare later results. It is proved in Seppäläinen (1996) that the stick process is *attractive*. This means that, if η and ζ are two initial states that satisfy $\eta \geq \zeta$ [inequalities are interpreted coordinatewise, $\eta(i) \geq \zeta(i)$ for all $i \in \mathbf{Z}$], then it is possible to construct the processes $\eta(t)$ and $\zeta(t)$ on a common probability space so that the inequality $\eta(t) \geq \zeta(t)$ holds at all times $t \geq 0$, almost surely.

Fix n for the moment. Let ζ^1 and ζ^2 be stick processes in equilibrium, with expectations

$$E\big[\zeta^1(i, t)\big] = q - n^{-\beta} \|v_0\|_{\infty} \quad \text{and} \quad E\big[\zeta^2(i, t)\big] = q + n^{-\beta} \|v_0\|_{\infty}.$$

In other words, for each fixed time t and for $r \in \{1, 2\}$, the stick heights $(\zeta^r(i, t) : i \in \mathbf{Z})$ are exponentially distributed i.i.d. random variables with expectations as above. At time $t = 0$, we can construct the initial configurations of all three processes ζ^1, η^n and ζ^2 on a single probability space so that

$$\zeta^1(i, 0) \leq \eta^n(i, 0) \leq \zeta^2(i, 0) \quad \text{for all } i, \text{ a.s.}$$

To do this, take an i.i.d. sequence of Exp(1) variables $\{X_i\}$ and set

$$\begin{aligned} \zeta^r(i, 0) &= E\big[\zeta^r(i, 0)\big] X_i \quad \text{and} \\ \eta^n(i, 0) &= E\big[\eta^n(i, t)\big] X_i \quad \text{for } r = 1, 2 \text{ and all } i. \end{aligned}$$

We then construct all three processes on a common probability space so that $\zeta^1(i, t) \leq \eta^n(i, t) \leq \zeta^2(i, t)$ for all t and i a.s.

Let $\{K_n\}$ be an arbitrary sequence of integers, to be used as translations on the lattice. The construction gives these inequalities:

$$\begin{aligned} &\sum_{i=[nx]+1}^{[ny]} \big\{ \zeta^1(K_n + i, n^{\nu} t) - E\big[\zeta^1(i, n^{\nu} t)\big] \big\} - \big([ny] - [nx]\big) n^{-\beta} \|v_0\|_{\infty} \\ &\quad \leq \sum_{i=[nx]+1}^{[ny]} \big\{ \eta^n(K_n + i, n^{\nu} t) - q \big\} \\ &\quad \leq \sum_{i=[nx]+1}^{[ny]} \big\{ \zeta^2(K_n + i, n^{\nu} t) - E\big[\zeta^2(i, n^{\nu} t)\big] \big\} + \big([ny] - [nx]\big) n^{-\beta} \|v_0\|_{\infty}. \end{aligned} \tag{2.9}$$

A terminological convention: throughout, we shall say

$$X = Y + o(n^{\alpha}) \quad \text{a.s.} \tag{2.10}$$

as a shorthand for

$$\lim_{n\to\infty} n^{-\alpha}\big|X - Y\big| = 0 \quad \text{a.s.} \tag{2.11}$$

The sums of the ζ-terms in (2.9) are almost surely $o(n^{1/2+\delta})$ for any $\delta > 0$ because the variables are i.i.d. and have sufficient moments.

Thus we get the result

$$\sum_{i=[nx]+1}^{[ny]} \eta^n(K_n + i, n^{\nu}t) = nq(y-x) + o\big(n^{[1/2]\vee[1-\beta]+\delta}\big) \quad \text{a.s.} \tag{2.12}$$

for any ν, $\beta > 0$ and arbitrarily small $\delta > 0$. The translation K_n was included in anticipation of later results. Because we are speeding up time beyond the hydrodynamic scale, a certain translation will appear naturally. The goal of the paper is to improve on (2.12), by obtaining results that reveal how the perturbation evolves in time or that have a smaller error term.

As the last preparatory step, we construct the solution of the Burgers equation (2.4) by the Hopf–Lax formula. The perturbation $v_0(x)$ is now the initial data. Define $V_0(x)$ by

$$V_0(0) = 0 \quad \text{and} \quad V_0(y) - V_0(x) = \int_x^y v_0(z)\,dz \quad \text{for all } x < y.$$

V_0 is a Lipschitz function with a bounded derivative a.e. For $(x, t) \in \mathbf{R}\times[0,\infty)$, define $V(x, 0) = V_0(x)$ and, for $t > 0$,

$$V(x,t) = \inf_{y\in\mathbf{R}}\left\{V_0(y) + \frac{1}{4t}(x-y)^2\right\}. \tag{2.13}$$

Then V is the unique *viscosity solution* of the Hamilton–Jacobi equation

$$V_t + (V_x)^2 = 0, \qquad V(x,0) = V_0(x). \tag{2.14}$$

For each fixed t, $V(\cdot, t)$ is again a Lipschitz function, so it has a.e. an x-derivative $v = V_x$. This function $v(x, t)$ is the unique *entropy solution* of (2.4) with initial data v_0. The reader can find a development of these p.d.e. results in Evans (1998).

Now the results for the stick process. The most general result, valid for all scalings, does not identify any time evolution, only a translation of the initial sticks.

THEOREM 1. *Assume that $\beta > 0$ and $\nu \geq 1$. Let $\eta^n(t)$ denote the stick process started from the initial configuration* (2.8). *Fix $x < y$ in $\mathbf{R}$ and $t > 0$. Then, for any $\delta > 0$, the following asymptotic equality is valid almost surely as $n \to \infty$:*

$$\sum_{i=[nx]+1}^{[ny]} \eta^n\big([2n^{\nu}qt] + i, n^{\nu}t\big) = \sum_{i=[nx]+1}^{[ny]} \eta^n(i, 0) + o(n^{[\nu-2\beta]\vee[\nu/3]+\delta}). \tag{2.15}$$

Why the translation $[2n^\nu qt]$ appears naturally is explained in Section 4.2. The error exponent in the statement (2.15) satisfies

$$[\nu - 2\beta] \vee [\nu/3] = \begin{cases} \nu - 2\beta, & \nu > 3\beta, \\ \nu/3, & \nu \leq 3\beta. \end{cases} \tag{2.16}$$

If $\nu < 3$ and $\beta > (\nu - 1)/2$ the error in (2.15) is $o(n)$ and so vanishes in the standard hydrodynamic scaling of (2.6). If $\nu = 1$ and $\beta > 1/4$ the error is $o(\sqrt{n})$, and we see that fluctuations in the central limit scale are rigidly translated by the dynamics.

We find one family of scalings where the perturbation evolves according to the Burgers equation. For this to happen the perturbation has to be larger than $n^{-1/2}$.

THEOREM 2. *Suppose $\beta \in (0, 1/2)$ and set $\nu = 1 + \beta$. Let $\eta^n(t)$ denote the stick process started from the initial configuration* (2.8). *Let ϕ be a compactly supported, continuous test function on $\mathbf{R}$. Then almost surely*

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n^{1-\beta}} \sum_{i\in\mathbf{Z}} \big\{\eta^n([2n^{1+\beta}qt] + i, n^{1+\beta}t) - q\big\}\phi(i/n) \\ &\quad = \int_{\mathbf{R}} \phi(x)v(x, t)\,dx. \end{aligned} \tag{2.17}$$

This result compares directly with Corollary 2.3 in Esposito, Marra and Yau (1994), where the corresponding result is proved in dimensions $d \geq 3$ for an exclusion process. The deterministic limit (2.17) cannot be valid for $\beta = 1/2$ because in equilibrium this would be the central limit theorem scaling.

REMARK ABOUT CONSTRUCTION. The almost sure results of Theorems 1 and 2 are proved for a special construction explained in Section 4. In this construction the processes η^n are defined on one common probability space, and the variables $\eta^n(i, t)$ are realized as interparticle distances of Hammersley's process. This is *not* the construction used in (2.9) that makes η attractive. Both theorems are proved by Borel–Cantelli arguments, and the probability estimates for the arguments are derived with the help of the special construction. But once derived, the estimates are valid in all constructions because they are statements about the distributions of the processes. Hence Theorems 1 and 2 are valid for any construction of the stick process.

To compare (2.17) directly with (2.12), we can write it in the form

$$\begin{aligned} &\sum_{i=[nx]+1}^{[ny]} \eta^n\big([2n^{1+\beta}qt] + i, n^{1+\beta}t\big) \\ &\quad = nq(y - x) + n^{1-\beta}\int_x^y v(z, t)\,dz + o\big(n^{1-\beta}\big). \end{aligned} \tag{2.18}$$

A comparison of the errors in (2.12), (2.15) and (2.18) reveals that for $\nu > 1+\beta$ the easy result (2.12) in fact has the smallest error. For $\nu = 1 + \beta$ (2.18) has the smallest error, while for $\nu < 1+\beta$ it can be either one of (2.12) and (2.15), depending on the exact relation between β and ν.

3. Asymptotics for a tagged particle in Hammersley's process. In Hammersley's process a countable collection of point particles evolves on $\mathbf{R}$ according to the following rule: if $x < y$ are two locations of neighboring particles, then with rate equal to the distance $y - x$ the particle at y jumps to a randomly (uniformly) chosen location in the interval (x, y). All particles execute jumps independently of each other.

This evolution can be graphically constructed with a rate one, homogeneous Poisson point process on the space–time plane $\mathbf{R} \times (0, \infty)$: Suppose (x, t) is a point of the Poisson process. Then at time t the leftmost particle in $[x, \infty)$ jumps to x. If the leftmost particle were already at x, or if there were no leftmost particle in $[x, \infty)$, no jump would take place. This latter case can happen if there are infinitely many particles in some bounded interval.

There is an obvious connection between Hammersley's process and our stick process. We assume that we can label the particles by integers in an order-preserving way. Let $z(i, t)$ denote the position of particle i at time t. The assumption is

$$z(i-1, t) \le z(i, t) \quad \text{for all } i \text{ and } t. \tag{3.1}$$

Suppose we have constructed the process $z(t) = (z(i, t) : i \in \mathbf{Z})$ that operates according to the description given previously. Define

$$\eta(i, t) = z(i, t) - z(i-1, t) \quad \text{for } i \in \mathbf{Z}. \tag{3.2}$$

Then it is clear that $\eta(t)$ evolves as our stick process. When particle $z(i)$ jumps to the left, stick $\eta(i)$ donates a piece to stick $\eta(i+1)$. In particle system jargon, the stick process is Hammersley's process as seen from a tagged particle. What this means is that knowing $\eta(t)$ and the evolution of one particle $z(j, t)$ is equivalent to knowing the process $z(t)$. The simultaneous construction of Hammersley's process and the stick process is discussed in Section 4.

Assume that the initial sticks $\eta^n(0) = (\eta^n(i, 0) : i \in \mathbf{Z})$ that satisfy (2.8) have been defined on some probability space. Initial particle configurations $z^n(0) = (z^n(i, 0) : i \in \mathbf{Z})$ are defined on this same probability space by

$$\begin{gathered} z^n(0,0) = 0, \qquad z^n(i, 0) = \sum_{j=1}^{i} \eta^n(j, 0) \quad \text{for } i > 0, \\ z^n(i, 0) = -\sum_{j=i+1}^{0} \eta^n(j, 0) \quad \text{for } i < 0. \end{gathered} \tag{3.3}$$

Thus $z^n(i, 0)$ is a sum of independent exponential random variables with uniformly bounded expectations, and

$$E[z^n(i, 0)] = qi + n^{1-\beta} V_0(i/n). \tag{3.4}$$

The processes $\{z^n(t)\}$ are then constructed together on one probability space where the initial configurations $\{z^n(0)\}$ and the space–time Poisson points are independent. All processes $z^n(t)$ use the same realization of the Poisson points to construct the dynamics. This is not really necessary because our a.s. results come from Borel–Cantelli arguments. But in the proof it is convenient to work with a single Poisson process and the family $\{z^n(0)\}$ of initial configurations, instead of giving each process $z^n(t)$ its own space–time Poisson process.

THEOREM 3. *Assume that $\beta > 0$ and $\nu \geq 1$. Let $z^n(i,t)$ denote Hammersley's process started from the initial configuration described in* (3.3) *and* (2.8). *Fix $x \in \mathbf{R}$ and $t > 0$. Then, for any $\delta > 0$, we have the following asymptotic equality almost surely as $n \to \infty$*:

$$z^n\big([nx]+[2n^\nu qt], n^\nu t\big) = n^\nu tq^2 + z^n([nx],0) + o\big(n^{[\nu-2\beta]\vee[\nu/3]+\delta}\big). \tag{3.5}$$

Theorem 1 is an immediate consequence of (3.2) and Theorem 3.

Ferrari and Fontes (1994) proved a translation result of this type for the exclusion process. Suppose for the moment that the $\eta(i)$'s are occupation variables of totally asymmetric one-dimensional simple exclusion in equilibrium at density ρ. Then the jumps of the z-variables correspond to the current of particles. Statement (1.5) in Theorem 1 of Ferrari and Fontes (1994) implies that, in the L^2 sense as $n \to \infty$,

$$z(0,nt) = nt\rho^2 + z([nth(\rho)],0) + o(n^{1/2}), \tag{3.6}$$

where $h(\rho) = 2\rho - 1$. This can be compared with our result for Hammersley's process: with $\nu = 1$ and $\beta \geq 1/3$, (3.5) implies that

$$z^n(0,nt) = ntq^2 + z^n\big(-[2tqn],0\big) + o(n^{1/3+\delta}). \tag{3.7}$$

The error is smaller in (3.7) than in (3.6), but the Ferrari–Fontes result is valid for more general asymmetric exclusions, not only for totally asymmetric.

Next we give a result with explicit time evolution. In one of the cases treated by the next theorem, we will assume that $V_0(x)$ has asymptotic slopes in the sense that these limits exist:

$$v_0(-\infty) = \lim_{x\to-\infty} \frac{V_0(x)}{x} \quad \text{and} \quad v_0(+\infty) = \lim_{x\to+\infty} \frac{V_0(x)}{x}. \tag{3.8}$$

When this is the case, we define the piecewise linear "asymptotic profile"

$$V_\infty(x,0) = \begin{cases} v_0(-\infty)x, & x < 0, \\ 0, & x = 0, \\ v_0(+\infty)x, & x > 0, \end{cases} \tag{3.9}$$

and its evolution for $t > 0$ by

$$V_\infty(x,t) = \inf_{y\in\mathbf{R}} \left\{ V_\infty(y,0) + \frac{1}{4t}(x-y)^2 \right\}. \tag{3.10}$$

THEOREM 4. *Assume that* $\nu > 3\beta$, *in addition to the basic assumption* $\beta > 0$ *and* $\nu \geq 1$. *Let* $z^n(i,t)$ *denote Hammersley's process started from the initial configuration* $z^n(i,0)$ *described in* (3.3) *and* (2.8). *Fix* $x \in \mathbf{R}$ *and* $t > 0$. *Then we have the following asymptotic equalities, each statement valid almost surely as* $n \to \infty$.

Case 1. $\nu > 1+\beta$. *Assume that the limits in* (3.8) *exist. Then*

$$\begin{aligned} z^n\big([nx]+[2n^\nu qt], n^\nu t\big) \\ = n^\nu tq^2 + nxq + n^{\nu-2\beta} V_\infty(0,t) + o(n^{\nu-2\beta}). \end{aligned} \tag{3.11}$$

Case 2. $\nu = 1+\beta$. *Then*

$$\begin{aligned} z^n\big([nx]+[2n^\nu qt], n^\nu t\big) \\ = n^\nu tq^2 + nxq + n^{1-\beta} V(x,t) + o(n^{1-\beta}). \end{aligned} \tag{3.12}$$

Case 3. $1 \leq \nu < 1+\beta$. *Then, for any* $\delta > 0$,

$$\begin{aligned} z^n\big([nx]+[2n^\nu qt], n^\nu t\big) \\ = n^\nu tq^2 + nxq + n^{1-\beta} V_0(x) + o\big(n^{[1/2]\vee[\nu-2\beta]+\delta}\big). \end{aligned} \tag{3.13}$$

REMARKS. Recall again (2.10)–(2.11) for the precise meaning of the almost sure $o(n^\alpha)$ error terms. In Case 1, the term nxq in (3.11) may or may not be included in the error $o(n^{\nu-2\beta})$, depending on whether $\nu > 1+2\beta$ or not. The statement (3.11) for Case 1 does not improve (2.12) because the error $n^{\nu-2\beta}$ is strictly larger than $n^{1-\beta}$ in this case. Theorem 2 follows from Case 2.

The remark about construction at end of Section 2 applies here, too. The proofs of Theorems 3 and 4 are Borel–Cantelli arguments that depend on estimates of the distributions of the processes, and hence are valid in all constructions.

The three cases reveal the effect of the time scale on the evolution of the perturbation: For fast times $\nu > 1+\beta$ we only see the asymptotic effect $V_\infty(0,t)$ which is independent of the reference point x. For slow times $\nu < 1+\beta$ we only see the initial perturbation $V_0(x)$. And exactly at $\nu = 1+\beta$, we see the perturbation evolve according to the Burgers equation.

It remains to prove Theorems 3 and 4. This proof uses a special construction of Hammersley's process in terms of increasing sequences of space–time Poisson points.

4. Graphical construction and increasing sequences. Consider a planar, rate one, homogeneous Poisson point process. A sequence (x_1, t_1), (x_2, t_2), ..., (x_m, t_m) of Poisson points is *increasing* if

$$x_1 < x_2 < \cdots < x_m \quad \text{and} \quad t_1 < t_2 < \cdots < t_m.$$

For arbitrary (a,s), (b,t) on the plane, define the random variable $\mathbf{L}((a,s), (b,t))$ as the maximal number of Poisson points on an increasing sequence contained in the rectangle $(a, b] \times (s, t]$. Abbreviate $\mathbf{L}(b, t) = \mathbf{L}((0, 0), (b, t))$ for the case where the lower left corner is the origin.

An inverse to $\mathbf{L}$ is defined by

$$\mathbf{\Gamma}\big((a, s), m, \tau\big) = \inf\big\{h > 0 : \mathbf{L}\big((a, s), (a + h, s + \tau)\big) \geq m\big\}. \tag{4.1}$$

In words: $\mathbf{\Gamma}((a, s), m, \tau)$ is the minimal horizontal distance h for which the rectangle $(a, a + h] \times (s, s + \tau]$ contains an increasing sequence of m points. Again abbreviate $\mathbf{\Gamma}(m, \tau) = \mathbf{\Gamma}((0, 0), m, \tau)$.

These random variables satisfy laws of large numbers:

$$\lim_{s\to\infty} \frac{1}{s}\mathbf{L}(sb, st) = 2\sqrt{bt} \quad \text{and} \quad \lim_{s\to\infty} \frac{1}{s}\mathbf{\Gamma}\big([sa], st\big) = \frac{a^2}{4t} \quad \text{a.s.} \tag{4.2}$$

The existence of the limits follows from the subadditive ergodic theorem. The exact values were first calculated by Vershik and Kerov (1977).

In the previous section we suggested how to construct Hammersley's process with a rate one space–time Poisson point process. The rule was that the Poisson point (x, t) pulls the leftmost particle in $[x, \infty)$ to the location x at time t. As usual in particle system contexts, constructing the process rigorously from this description, on the infinite real line, needs a proof.

We can take an elegant way out with the help of the increasing paths. Assume given an initial configuration $(z(i, 0) : i \in \mathbf{Z})$ that satisfies the ordering convention (3.1) for $t = 0$. Given a realization of the Poisson points, *define*

$$z(k, t) = \inf_{i: i\leq k} \big\{z(i, 0) + \mathbf{\Gamma}\big((z(i, 0), 0), k - i, t\big)\big\} \tag{4.3}$$

for all $k \in \mathbf{Z}$ and $t > 0$. In words: the potential locations of $z(k, t)$ are all points x such that the rectangle $(z(i, 0), x] \times (0, t]$ contains an increasing sequence of $k - i$ Poisson points. Of these potential locations $z(k, t)$ chooses the leftmost.

If we permit $-\infty$ as a value for $z(k, t)$, (4.3) defines a process $z(t) = (z(k, t) : k \in \mathbf{Z})$ that satisfies (3.1). To rule out the possibility of jumping to $-\infty$ in finite time, define the state space

$$Z = \big\{z = \big(z(i)\big) \in \mathbf{R}^{\mathbf{Z}} : z(i - 1) \leq z(i) \text{ for all } i, \text{ and } \lim_{i\to-\infty} i^{-2}z(i) = 0\big\}. \tag{4.4}$$

One can check that if $(z(i, 0)) \in Z$, then almost surely the infimum in (4.3) is always attained at some finite i and $z(t) \in Z$ for all t. Homogeneity of the space–time Poisson point process then implies that (4.3) defines a time-homogeneous Markov process $z(t)$ with state space Z.

Definitions (2.2) and (4.4) show that Y maps injectively into Z through (3.3), and Z back onto Y through (3.2). So given an initial stick configuration $(\eta(i, 0) : i \in \mathbf{Z})$ in Y, we define an initial particle configuration $(z(i, 0)) \in Z$ as in (3.3), then define the process $z(t)$ by (4.3), and finally use (3.2) to define the stick process $\eta(t)$.

This is convenient as a rigorous definition of the processes $\eta(t)$ and $z(t)$, but it is not so obvious that the resulting dynamics follows our earlier descriptions.

One can prove that when $\eta(t)$ is defined this way, (2.3) is satisfied so the generator of $\eta(t)$ is L. All the facts mentioned here can be found in Sections 3–5 in Seppäläinen (1996).

We can also argue from definition (4.3) that if (x, t) is a space–time Poisson point, then at time t the leftmost particle in $[x, \infty)$ is at x, if such a particle exists. Suppose not, so that for some k, $z(k-1, t) < x < z(k, t)$. Pick $i \le k-1$ so that

$$z(k-1, t) = z(i, 0) + \mathbf{\Gamma}\big((z(i, 0), 0), k-1-i, t\big).$$

Barring the null event that space–time Poisson points can lie on the same horizontal line, there must be an increasing sequence of $k-1-i$ Poisson points from $(z(i, 0), 0)$ to a point (y, s) such that $y = z(k-1, t) < x$ and $s < t$. (In the extreme case $i = k-1$, this sequence is empty, and $s = 0$.) Consequently, we can append the new point (x, t) to this increasing sequence to produce a sequence of $k-i$ points from $(z(i, 0), 0)$ to (x, t). Then

$$z(k, t) \le z(i, 0) + \mathbf{\Gamma}\big((z(i, 0), 0), k-i, t\big) \le x,$$

contradicting $x < z(k, t)$.

The remainder of the paper proves Theorems 3 and 4 through definition (4.3). The construction of the family of processes $\{z^n(t)\}$ is the following. There is a single probability space $(\Omega, \mathscr{F}, P)$ on which are defined the initial locations $\{z^n(0)\}$ and, independently of them, the space–time Poisson point process. On this probability space define the random variables

$$\Gamma^n(i, m, t) = \mathbf{\Gamma}\big((z^n(i, 0), 0), m, t\big). \tag{4.5}$$

Then, following (4.3), the processes $\{z^n(t)\}$ are defined by

$$z^n(k, t) = \inf_{i: i \le k} \Big\{ z^n(i, 0) + \Gamma^n\big(i, k-i, t\big) \Big\}. \tag{4.6}$$

Our arguments use distributional bounds on the initial locations $z^n(i, 0)$ and the variables $\Gamma^n(i, m, t)$. In distribution $\Gamma^n(i, m, t)$ is equal to $\mathbf{\Gamma}(m, t)$, so we can ignore the indices n and i and switch to $\mathbf{\Gamma}(m, t)$ as soon as only distributional properties are studied.

We close this section with two comments about matters that came up in Section 2.

4.1. *The case $q = 0$.* Theorems 1–4 are proved for the case where the fixed equilibrium density q is strictly positive. Here we show how the case $q = 0$ reduces to the standard hydrodynamic setting, through a space–time scaling of the graphical picture.

Suppose $q = 0$, and let the initial configurations $\eta^n(0)$ and $z^n(0)$ be as in (2.8) and (3.3). Define another initial particle configuration by $\tilde{z}^n(0) = n^\beta z^n(0)$. Construct the two processes $z^n(t)$ and $\tilde{z}^n(t)$ by formula (4.3), with these Poisson processes: for $z^n(t)$ take a realization Π of the rate one, space–time Poisson points, and for $\tilde{z}^n(t)$ use the space–time points $\widetilde{\Pi}$ obtained by mapping the points of Π by $(x,t) \mapsto (n^{\beta}x, n^{-\beta}t)$. By the scaling properties of Poisson processes, $\widetilde{\Pi}$ is again a rate one, homogeneous Poisson point process. In the graphical construction the difference between the evolutions $z^n(t)$ and $\tilde{z}^n(t)$ is merely a stretching and shrinking of the space and time axes: $z^n(i,t) = n^{-\beta}\tilde{z}^n(i, n^{-\beta}t)$. By the standard hydrodynamic result $n^{-1}\tilde{z}^n([nx], nt) \to V(x,t)$, so we get that $n^{\beta-1}z^n([nx], n^{1+\beta}t) \to V(x,t)$. This is Case 2 of Theorem 4. And similarly, Theorem 2 holds for the stick model.

4.2. *The translation* $[2n^{\nu}qt]$. We advance here some explanation for the spatial translation $[2n^{\nu}qt]$ that appears in the theorems.

First, consider the variational formula (4.3). Suppose that the process is in equilibrium with $E[\eta(i,t)] = q$. Equations (4.2) and (4.3) give

$$z(0, n^{\nu}t) = \inf_{y\le 0}\left\{qy + \frac{y^2}{4n^{\nu}t} + [\text{fluctuations}]\right\}. \tag{4.7}$$

Neglecting fluctuations, the infimum is attained at $y = -2n^{\nu}qt$. So roughly speaking,

$$\begin{aligned} z(0, n^{\nu}t) = {} & z(-[2n^{\nu}qt], 0) + \mathbf{\Gamma}\big((z(-[2n^{\nu}qt], 0), 0), [2n^{\nu}qt], n^{\nu}t\big) \\ & + [\text{fluctuations}]. \end{aligned} \tag{4.8}$$

As a sum of independents the term $z(-[2n^{\nu}qt], 0)$ has fluctuations of order $n^{\nu/2}$, while the $\mathbf{\Gamma}$-term has fluctuations of order $n^{\nu/3}$ (Lemma 5.2). It is advantageous to move the translation $[2n^{\nu}qt]$ to the left-hand side of (4.8), so that we study the dynamics of $z([2n^{\nu}qt], n^{\nu}t)$. Then the minimizer in (4.7) is $y = 0$, and we get smaller fluctuations on the right-hand side of (4.8).

Alternatively, we can look at the macroscopic equation to find the right scaling and translation for nontrivial dynamics. Suppose first that $u(x,t) = q + n^{-\beta}\rho(x,t)$ satisfies the Burgers equation (2.4). Then $\rho(x,t)$ satisfies the equation

$$\rho_t + 2q\rho_x + n^{-\beta}(\rho^2)_x = 0.$$

In the limit $n \to \infty$ this gives $\rho_t + 2q\rho_x = 0$ which is solved by a spatial translation $\rho(x,t) = \rho_0(x - 2qt)$. To get nontrivial dynamics, we speed up time and set $w(x,t) = \rho(x, n^{\beta}t)$ that satisfies

$$w_t + 2n^{\beta}qw_x + (w^2)_x = 0.$$

To eliminate the n^{β}-term, let $v(x,t) = w(x + 2n^{\beta}qt, t)$. The function v then solves Burgers equation $v_t + (v^2)_x = 0$ again. Working backwards, $v(x,t) = \rho(x + 2n^{\beta}qt, n^{\beta}t)$.

To see how $v(x,t)$ should arise microscopically, start with the "hydrodynamic heuristic" $u(x,t) \approx (2n\varepsilon)^{-1}\sum_{|i|\le n\varepsilon}\eta([nx]+i, nt)$. From this,

$$\rho(x,t) \approx \frac{1}{2\varepsilon}\cdot\frac{1}{n^{1-\beta}}\sum_{|i|\le n\varepsilon}\big\{\eta\big([nx]+i, nt\big) - q\big\},$$

and then

$$v(x,t) \approx \frac{1}{2\varepsilon}\cdot\frac{1}{n^{1-\beta}}\sum_{|i|\le n\varepsilon}\bigl\{\eta\bigl([nx]+[2n^{1+\beta}qt]+i,\,n^{1+\beta}t\bigr)-q\bigr\}.$$

This is exactly what Theorem 2 states, with the right translation $[2n^{1+\beta}qt]$ again.

5. Auxiliary lemmas. Throughout the proofs, $C, C_1, C_2, \ldots$ stand for constants independent of the important indices of the proof (such as m, n, i or j). The values of C, $C_1, C_2, \ldots$ may change freely from one inequality to the next.

We start with an inequality for bounding the initial locations $z^n(i,0)$.

LEMMA 5.1. *Suppose $\{X_i\}$ are independent exponentially distributed random variables with expectations $E[X_i] = q_i \in [0,b]$, where b is a finite constant. Then for all $\varepsilon \in (0,1/2)$ there is a finite constant $C = C(b,\varepsilon) > 0$ such that, for large enough $m \in \mathbf{N}$,*

$$P\left\{\left|\sum_{i=1}^m X_i - \sum_{i=1}^m q_i\right| \ge \varepsilon m^{1/2+\varepsilon}\right\} \le 2\exp(-Cm^{2\varepsilon})\,. \tag{5.1}$$

PROOF. The standard exponential Chebyshev argument. Let $t \in (0, 1/b)$. Then

$$\begin{aligned}
&P\left\{\sum_{i=1}^m X_i \ge \sum_{i=1}^m q_i + \varepsilon m^{1/2+\varepsilon}\right\}\\
&\quad\le \exp\left\{-t\sum_{i=1}^m q_i - t\varepsilon m^{1/2+\varepsilon}\right\}\prod_{i=1}^m E\bigl[e^{tX_i}\bigr]\\
&\quad= \exp\left\{-t\sum_{i=1}^m q_i - t\varepsilon m^{1/2+\varepsilon} - \sum_{i=1}^m \log(1-tq_i)\right\}\\
&\quad= \exp\left\{-t\varepsilon m^{1/2+\varepsilon} + \sum_{i=1}^m\left(\frac{q_i^2t^2}{2} + O(t^3)\right)\right\}\\
&\quad\le \exp\left\{-t\varepsilon m^{1/2+\varepsilon} + \frac{mb^2t^2}{2} + O(t^3m)\right\} \qquad [\text{choose } t = b^{-2}\varepsilon m^{\varepsilon-1/2}]\\
&\quad= \exp\left\{\frac{-\varepsilon^2b^{-2}m^{2\varepsilon}}{2} + O\bigl(m^{3\varepsilon-1/2}\bigr)\right\}\\
&\quad\le \exp(-Cm^{2\varepsilon}).
\end{aligned}$$

The last step is valid for $\varepsilon < 1/2$. In the second equality we expanded $\log(1 - tq_i) = -tq_i - q_i^2t^2/2 + O(t^3)$, where the O-term is uniform over i because of the uniform bound $q_i \le b$. The expansion is valid because $t = b^{-2}\varepsilon m^{\varepsilon-1/2}$ can be made arbitrarily small by restricting m to be large. The same argument with some sign changes also proves the other inequality. □

Next we give bounds on the fluctuations of the increasing paths.

LEMMA 5.2. *Suppose a, s and h are positive real numbers.*

(a) *For $x \ge 2$, define*

$$I(x) = 2x\cosh^{-1}(x/2) - 2\sqrt{x^2-4}. \tag{5.2}$$

Then, for $a \le hs < a^2/4$,

$$P\left\{\mathbf{\Gamma}\big([a], s\big) \le \frac{a^2}{4s} - h\right\} \le \exp\left\{-\frac{1}{2}\sqrt{a^2-4hs}\; I\left(2+\frac{hs}{a^2}\right)\right\}. \tag{5.3}$$

When $x = hs/a^2$ is small, we can use the expansion

$$I(2+x) \ge Cx^{3/2}. \tag{5.4}$$

(b) *There are fixed positive constants B_0, B_1, d_0, C_0 and C_1 such that if $a \ge B_0$ and $B_1 a^{4/3} \le hs \le d_0 a^2$, then*

$$P\left\{\mathbf{\Gamma}\big([a], s\big) > \frac{a^2}{4s} + h\right\} \le C_0 \exp\left\{-C_1\frac{s^3h^3}{a^4}\right\}. \tag{5.5}$$

REMARK 5.1. In our typical application of Lemma 5.2, a and s are of the same large order m, and h is of the order $m^{1/3+\varepsilon}$. Then the bound in (5.3) is $C_1\exp(-C_2 m^{3\varepsilon/2})$ and in (5.5) $C_1\exp(-C_2 m^{3\varepsilon})$.

REMARK 5.2. For the growth model associated with the exclusion and zero-range processes, the lower tail estimate (5.3) is available in Seppäläinen (1998c) and Johansson (2000). But the upper tail estimate (5.5) has not been derived at the time of writing this paper.

PROOF OF LEMMA 5.2. Part (a). The random variables $\mathbf{L}(s,s)$ are superadditive in the sense that, for any $0 < s < t$,

$$\mathbf{L}\big((0,0),(s,s)\big) + \mathbf{L}\big((s,s),(t,t)\big) \le \mathbf{L}\big((0,0),(t,t)\big) \qquad \text{a.s.}$$

It follows that there exists a function $I(x)$ such that

$$\sup_{s>0}\frac{1}{s}\log P\big\{\mathbf{L}(s,s)\ge sx\big\} = \lim_{s\to\infty}\frac{1}{s}\log P\big\{\mathbf{L}(s,s)\ge sx\big\} = I(x). \tag{5.6}$$

Since $s^{-1}\mathbf{L}(s,s)\to 2$ as $s\to\infty$, $I(x) = 0$ for $x < 2$. Kim (1996) proved that $I(x)$ is bounded below by the expression in (5.2), and Seppäläinen (1998b) showed that this expression equals $I(x)$. By (5.6) and the observation that $\mathbf{L}(a,b) \stackrel{d}{=} \mathbf{L}(\sqrt{ab}, \sqrt{ab}\,)$,

$$P\left\{\boldsymbol{\Gamma}\big([a], s\big) \le \frac{a^2}{4s} - h\right\} = P\left\{\mathbf{L}\left(\frac{a^2}{4s} - h, s\right) \ge [a]\right\}$$
$$\le \exp\left\{-\frac{1}{2}\sqrt{a^2 - 4hs}\; I\left(\frac{2[a]}{\sqrt{a^2-4hs}}\right)\right\}.$$

The argument of $I(\cdot)$ is estimated below by

$$\frac{2[a]}{\sqrt{a^2-4hs}} \ge \frac{2(1-1/a)}{\sqrt{1-4hsa^{-2}}} \ge 2\left(1-\frac{1}{a}\right)\left(1+\frac{2hs}{a^2}\right)$$
$$\ge\; 2 + \frac{hs}{a^2},$$

provided $hs \ge a$.

Part (b) is a consequence of case 4 of Lemma 7.1 in Baik, Deift and Johansson (1999). We check the assumptions of that lemma. First, we express the probability (5.5) in terms of $\mathbf{L}$, then convert it to the $\phi_n(\lambda)$-notation of Baik, Deift and Johansson (1999):

$$P\left\{\boldsymbol{\Gamma}\big([a], s\big) > \frac{a^2}{4s} + h\right\} = P\left\{\mathbf{L}\left(\frac{a^2}{4s} + h, s\right) < [a]\right\}$$
$$= P\left\{\mathbf{L}\left(\frac{a^2}{4s} + h, s\right) \le [a]-1\right\} = \phi_{[a]-1}\left(\frac{a^2}{4} + hs\right).$$

According to case 4 of Lemma 7.1 in Baik, Deift and Johansson (1999), we can bound

$$\phi_{[a]-1}\left(\frac{a^2}{4} + hs\right) \le C_0 \exp(C_1 t^3), \tag{5.7}$$

with t defined by the equation

$$1 - \frac{t}{2^{1/3}[a]^{2/3}} = \frac{\sqrt{a^2+4hs}}{[a]}, \tag{5.8}$$

provided

$$1 + \frac{M_7}{2^{1/3}[a]^{2/3}} \le \frac{\sqrt{a^2+4hs}}{[a]} \le 1 + \delta_6, \tag{5.9}$$

where M_7 and δ_6 are certain positive constants that appear in the development of Baik, Deift and Johansson.

The first inequality in (5.9) is equivalent to

$$M_7 \le 2^{1/3}\left(\frac{\sqrt{a^2+4hs}}{[a]^{1/3}} - [a]^{2/3}\right). \tag{5.10}$$

Provided $hs \le 2a^2$, the right-hand side of (5.10) is bounded below by

$$2^{1/3}\left(\frac{a + hs/a}{a^{1/3}} - a^{2/3}\right) = 2^{1/3}\frac{hs}{a^{4/3}}\,. \tag{5.11}$$

Thus the first inequality in (5.9) is satisfied if $B_1 a^{4/3} \le hs \le 2a^2$ for a large enough constant B_1.

For the second inequality in (5.9) observe that

$$\frac{\sqrt{a^2+4hs}}{[a]} \le \frac{a}{[a]}\left(1+\frac{4hs}{a^2}\right),$$

which is less than or equal to $1+\delta_6$, provided a is large enough and $hs \le d_0 a^2$ for a small enough d_0.

We have verified the conditions of case 4 of Lemma 7.1 in Baik, Deift and Johansson. Again because (5.11) is below the right-hand side of (5.10), we see that t defined by (5.8) satisfies $-t \ge 2^{1/3} hsa^{-4/3}$, so inequality (5.7) becomes (5.5). □

LEMMA 5.3. *For $a > 0$, $\beta > 0$, $\nu \ge 1$ and $\varepsilon \in (0, 1/2)$, define a deterministic quantity R_n by*

$$R_n = n^\nu t q^2 + nxq + n^{1-\beta}\|v_0\|_\infty |x| + a(n^{\nu/3+\varepsilon} + n^{1/2+\varepsilon}). \tag{5.12}$$

Then there are finite constants $C_i > 0$ such that

$$P\{z^n([nx]+[2n^\nu qt], n^\nu t) > R_n\} \le C_1 \exp(-C_2 n^{2\varepsilon}) \tag{5.13}$$

for all n.

PROOF. By the variational formula (4.6) and Lemmas 5.1 and 5.2,

$$\begin{aligned} &P\{z^n([nx]+[2n^\nu qt], n^\nu t) > R_n\} \\ &\quad\le P\{z^n([nx],0) \ge nxq + n^{1-\beta}\|v_0\|_\infty|x| + an^{1/2+\varepsilon}\} \\ &\qquad + P\{\Gamma^n([nx],[2n^\nu qt], n^\nu t) \ge n^\nu tq^2 + an^{\nu/3+\varepsilon}\} \\ &\quad\le C_1\exp(-C_2 n^{2\varepsilon}) + C_1 \exp(-C_2 n^{3\varepsilon}). \end{aligned}$$

□

The main lemma of this section reduces the range of indices that need to be considered in the variational formula (4.6).

LEMMA 5.4. *Suppose $\beta > 0$ and $\nu \ge 1$, and let ξ be any number that satisfies $\xi \ge \nu - \beta$ and $\xi > 2\nu/3$. Then if $b > 0$ is large enough, the following holds with probability 1: for large enough n,*

$$\begin{aligned} &z^n([nx]+[2n^\nu qt], n^\nu t) \\ &\quad = \min\{z^n(i,0) + \Gamma^n(i, [nx]+[2n^\nu qt]-i, n^\nu t) : |i-[nx]| \le [bn^\xi]\}. \end{aligned} \tag{5.14}$$

Furthermore,

$$\sum_n P\{(5.14)\ \textit{fails for}\ n\} < \infty. \tag{5.15}$$

REMARK 5.3. It is not hard to understand heuristically why the thresholds $\nu - \beta$ and $2\nu/3$ are the correct ones. Take $x = 0$. To leading order the quantity minimized in the variational formula (4.6) for $z^n([2n^\nu qt], n^\nu t)$ is

$$z^n(i,0) + \Gamma^n\big(i,\ [2n^\nu qt] - i,\ n^\nu t\big) = n^\nu q^2 t + i^2 n^{-\nu}/(4t) + O(in^{-\beta}) + \text{[fluctuations]}.$$

The fluctuations of $z^n(i,0)$ are of order $|i|^{1/2}$, and those of $\Gamma^n(i,\ [2n^\nu qt] - i,\ n^\nu t)$ of order $n^{\nu/3} + |i|^{1/3}$. Which values of i can be safely ignored? Clearly, if $i^2 n^{-\nu}$ dominates both $in^{-\beta}$ and the fluctuations, this value of i cannot be a minimizer. Comparison of $i^2 n^{-\nu}$ with these terms suggests the correct thresholds.

REMARK 5.4. The distinction between the weak inequality $\xi \geq \nu - \beta$ and the strict inequality $\xi > 2\nu/3$ in the hypothesis is significant. It arises in the proof of Lemma 5.4 and influences the error terms of our theorems.

PROOF OF LEMMA 5.4. We shall prove (5.14). The reader can accumulate the estimates as we proceed and observe that (5.15) is also true.

By Lemma 5.3 and Borel–Cantelli, we may suppose that

$$z^n([nx] + [2n^\nu qt], n^\nu t) \leq R_n,$$

at the expense of discarding an event of probability 0 and by taking n large enough. Fix $\delta_0 \in (0,1)$ and take ε small enough in the definition (5.12) of R_n so that $\nu/3 + \varepsilon < \nu$ and $1/2 + \varepsilon < 1$. Then, since $|z^n([(1+\delta_0)n^\nu tq] + [nx], 0)|$ is a sum of $|[(1+\delta_0)n^\nu tq] + [nx]|$ independent exponential random variables with expectations in the range $q \pm n^{-\beta}\|v_0\|_\infty$, basic large-deviation estimates show that

$$\sum_n P\big\{z^n([(1+\delta_0)n^\nu tq] + [nx], 0) \leq R_n\big\} < \infty. \tag{5.16}$$

Consequently, we may also assume that

$$z^n\left(\big[(1+\delta_0)n^\nu tq\big] + [nx], 0\right) > R_n. \tag{5.17}$$

It then follows from the variational formula (4.6) that, almost surely,

$$\begin{aligned} &z^n\big([nx] + [2n^\nu qt], n^\nu t\big) \\ &\quad = \inf\big\{z^n(j,0) + \Gamma^n(j,\ [nx] + [2n^\nu qt] - j,\ n^\nu t) : \\ &\qquad\qquad j < \big[(1+\delta_0)n^\nu tq\big] + [nx]\big\} \end{aligned} \tag{5.18}$$

for large enough n.

To conclude the proof, we shall show that indices j outside the range $|j - [nx]| \leq [bn^\xi]$ cannot give the infimum in (5.18). This will follow by showing that, almost surely,

$$\begin{aligned} &z^n([nx]+[bn^\xi]+i,0)+\Gamma^n([nx]+[bn^\xi]+i,[2n^\nu qt]-[bn^\xi]-i,n^\nu t) \\ &\quad \geq z^n([nx],0)+\Gamma^n([nx],[2n^\nu qt],n^\nu t) \end{aligned} \tag{5.19}$$

for all $0 \le i \le [(1+\delta_0)n^\nu tq]$, and that

$$z^n\big([nx]-[bn^\xi]-i,0\big)+\Gamma^n\big([nx]-[bn^\xi]-i,[2n^\nu qt]+[bn^\xi]+i,n^\nu t\big) \ge z^n\big([nx],0\big)+\Gamma^n\big([nx],[2n^\nu qt],n^\nu t\big) \tag{5.20}$$

for all $i \ge 0$. The upper bound $j < [(1+\delta_0)n^\nu tq]+[nx]$ in (5.18) permitted us to restrict the range of i to $i \le [(1+\delta_0)n^\nu tq]$ in (5.19). The benefit is that the argument $[2n^\nu qt]-[bn^\xi]-i$ of Γ^n in (5.19) is of order n^ν throughout the range of i, which makes the estimation easier because there is no need for separate arguments for values of smaller order.

We estimate various terms separately in three sublemmas.

SUBLEMMA 5.1. *For any fixed $b>0$ and $\delta>0$, the following statements hold almost surely*: *for all large enough n*,

$$\Gamma^n\big([nx]+[bn^\xi]+i,\ [2n^\nu qt]-[bn^\xi]-i,\ n^\nu t\big) > \frac{1}{4n^\nu t}\big(2n^\nu qt - bn^\xi - i\big)^2 - \delta n^{\nu/3+\delta\nu} \tag{5.21}$$

for all $0 \le i \le [(1+\delta_0)n^\nu tq]$, and

$$\Gamma^n\big([nx]-[bn^\xi]-i,\ [2n^\nu qt]+[bn^\xi]+i,\ n^\nu t\big) > \frac{1}{4n^\nu t}\big(2n^\nu qt + bn^\xi + i\big)^2 - \delta\big(n^{\nu/3+\delta\nu} + i^{1/3+\delta}\big) \tag{5.22}$$

for all $i \ge 0$.

PROOF. We shall prove (5.22) and leave (5.21) to the reader. Their difference is that in (5.22), due to the unbounded range of i, we need i explicitly in the estimates and sum over $i \ge 0$ in the end. Equation (5.21) is easier because one estimate uniformly over i is sufficient.

Let A_n denote the event that (5.22) holds for all $i \ge 0$. Our goal is to prove

$$\sum_n P(A_n^c) < \infty \tag{5.23}$$

so that by Borel–Cantelli A_n happens for all large enough n a.s. For fixed n and i, the event that (5.22) fails has the same probability as the event

$$\mathbf{\Gamma}\big([2n^\nu qt]+[bn^\xi]+i,\ n^\nu t\big) \le \frac{1}{4n^\nu t}\big(2n^\nu qt + bn^\xi + i\big)^2 - \delta\big(n^{\nu/3+\delta\nu} + i^{1/3+\delta}\big). \tag{5.24}$$

By shrinking b slightly we can discard the integer parts. We bound the probability of the event (5.24) by (5.3). Now

$$\begin{aligned}\sqrt{a^2-4hs} &= \Big[\big(2n^\nu qt + bn^\xi + i\big)^2 - 4\delta t\big(n^{4\nu/3+\delta\nu} + i^{1/3+\delta}n^\nu\big)\Big]^{1/2}\\ &\ge C_1(n^\nu + i)\end{aligned}$$

and

$$\frac{hs}{a^2} \geq C_2 \frac{n^{4\nu/3+\delta\nu} + i^{1/3+\delta} n^{\nu}}{n^{\nu} + i} \geq C_3 n^{-2\nu/3+\delta\nu},$$

uniformly over $i \geq 0$ for all large enough n. Thus the probability of the event (5.24) is at most

$$\exp\bigl[-C(n^{\nu} + i) I(2 + C_1 n^{-2\nu/3+\delta\nu})\bigr] \leq \exp\bigl(-C n^{3\delta\nu/2} - C i n^{-\nu+3\delta\nu/2}\bigr),$$

where we applied the expansion (5.4). Summing this over $i \geq 0$, we get

$$\begin{aligned} P(A_n^c) &\leq \sum_{i \geq 0} \exp\bigl(-C n^{3\delta\nu/2} - C i n^{-\nu+3\delta\nu/2}\bigr) \\ &\leq C_1 n^{\nu(1-3\delta/2)} \exp\bigl(-C n^{3\delta\nu/2}\bigr). \end{aligned}$$

This last expression is summable over n, so (5.23) happens. □

SUBLEMMA 5.2. *For any fixed* $\delta > 0$, *this holds almost surely*: *for all large enough* n,

$$\Gamma^n\bigl([nx], [2n^{\nu} q t], n^{\nu} t\bigr) \leq n^{\nu} t q^2 + \delta n^{\nu/3+\delta\nu}. \tag{5.25}$$

PROOF. Deviation bound (5.5) and Borel–Cantelli. □

SUBLEMMA 5.3. *For any fixed* $b > 0$ *and* $\delta > 0$, *the following statements hold almost surely*: *for all large enough* n,

$$\begin{aligned} &z^n([nx], 0) - z^n\bigl([nx] + [bn^{\xi}] + i, 0\bigr) \\ &\quad \leq -\bigl(bn^{\xi} + i\bigr)\bigl(q - n^{-\beta}\|v_0\|_{\infty}\bigr) + \delta\bigl(n^{\xi/2+\delta} + i^{1/2+\delta}\bigr) \end{aligned} \tag{5.26}$$

for all $0 \leq i \leq [(1 + \delta_0) n^{\nu} t q]$, *and*

$$\begin{aligned} &z^n([nx], 0) - z^n\bigl([nx] - [bn^{\xi}] - i, 0\bigr) \\ &\quad \leq \bigl(bn^{\xi} + i\bigr)\bigl(q + n^{-\beta}\|v_0\|_{\infty}\bigr) + \delta\bigl(n^{\xi/2+\delta} + i^{1/2+\delta}\bigr) \end{aligned} \tag{5.27}$$

for all $i \geq 0$.

PROOF. This lemma is a consequence of Borel–Cantelli and the distribution of the increments $z^n(j+1, 0) - z^n(j, 0)$. We prove (5.27). The argument for (5.26) is the same.

We can realize the initial locations $z^n(j, 0)$ so that the inequalities

$$(q - n^{-\beta}\|v_0\|_{\infty}) X_j \leq z^n(j, 0) - z^n(j-1, 0) \leq (q + n^{-\beta}\|v_0\|_{\infty}) X_j \tag{5.28}$$

are valid, where $\{X_j\}$ are i.i.d. exponential random variables with expectation $E[X_j] = 1$. For fixed n and i, the probability that (5.27) fails is bounded above by

$$P\Bigg\{\sum_{j=1}^{[bn^{\xi}]+i}(q+n^{-\beta}\|v_0\|_\infty)X_j > \big(bn^{\xi}+i\big)\big(q+n^{-\beta}\|v_0\|_\infty\big) + \delta\big(n^{\xi/2+\delta}+i^{1/2+\delta}\big)\Bigg\} \le P\Bigg\{\sum_{j=1}^{[bn^{\xi}]+i}X_j > \big(bn^{\xi}+i\big)+\delta\frac{n^{\xi/2+\delta}+i^{1/2+\delta}}{q+n^{-\beta}\|v_0\|_\infty}\Bigg\}. \tag{5.29}$$

This probability is bounded above by

$$\exp\Bigg\{-\big([bn^{\xi}]+i\big)\kappa\bigg(1+\frac{\delta(n^{\xi/2+\delta}+i^{1/2+\delta})}{(q+n^{-\beta}\|v_0\|_\infty)(bn^{\xi}+i)}\bigg)\Bigg\}, \tag{5.30}$$

where $\kappa(x) = x-1-\log x$ is the Cramér rate function for the Exp(1)-distribution. In case the reader is used to thinking of large deviation rate functions only asymptotically valid, note that the inequality $P(\sum_1^m X_j \ge ma) \le \exp\{-m\kappa(a)\}$ for $a > 1$, and its lower tail counterpart, are valid for finite m due to the supermultiplicativity $P(\sum_1^{l+m} X_j \ge (l+m)a) \ge P(\sum_1^{l} X_j \ge la)\cdot P(\sum_1^m X_j \ge ma)$.

For small x we have the quadratic lower bound $\kappa(1+x) \ge Cx^2$, so (5.30) is further bounded above by

$$\exp\bigg\{-C\frac{n^{\xi+2\delta}+i^{1+2\delta}}{bn^{\xi}+i}\bigg\}. \tag{5.31}$$

Summing the quantities in (5.31) over $i \ge 0$, we bound the probability that, for a fixed n, (5.27) fails for *some* $i \ge 0$, by

$$\sum_{i\ge 0}\exp\bigg\{-C\frac{n^{\xi+2\delta}+i^{1+2\delta}}{bn^{\xi}+i}\bigg\} \le \sum_{0\le i\le n^{\xi}}\exp(-Cn^{2\delta}) + \sum_{i>n^{\xi}}\exp(-Ci^{2\delta}).$$

The last line is summable over n. Hence, by Borel–Cantelli, it is almost surely true that, for large enough n, (5.27) holds for all $i \ge 0$. □

We return to complete the proof of Lemma 5.4. By (5.21), (5.25) and (5.26), inequality (5.19) will hold for large enough n if we can show that

$$\frac{1}{4n^{\nu}t}\big(2n^{\nu}qt - bn^{\xi} - i\big)^2 - n^{\nu}tq^2 - 2\delta n^{\nu/3+\delta\nu} \ge -\big(bn^{\xi}+i\big)\big(q-n^{-\beta}\|v_0\|_\infty\big) + \delta\big(n^{\xi/2+\delta}+i^{1/2+\delta}\big) \tag{5.32}$$

holds for all $0 \le i \le [(1+\delta_0)n^{\nu}tq]$. Equation (5.32) simplifies to

$$\frac{b^2}{4t}n^{2\xi-\nu} + \frac{b}{2t}in^{\xi-\nu} + \frac{i^2}{4n^{\nu}t} - 2\delta n^{\nu/3+\delta\nu} \ge b\|v_0\|_\infty n^{\xi-\beta} + \|v_0\|_\infty in^{-\beta} + \delta\big(n^{\xi/2+\delta}+i^{1/2+\delta}\big). \tag{5.33}$$

Now suppose $\xi \geq \nu - \beta$ so that $n^{2\xi-\nu} \geq n^{\xi-\beta}$ and $n^{\xi-\nu} \geq n^{-\beta}$. Then (5.33) follows from

$$\left(\frac{b}{4t} - \|v_0\|_\infty\right) b n^{2\xi-\nu} + \left(\frac{b}{2t} - \|v_0\|_\infty\right) i n^{\xi-\nu} + \frac{i^2}{4n^\nu t} \geq \delta\left(2n^{\nu/3+\delta\nu} + n^{\xi/2+\delta} + i^{1/2+\delta}\right).$$

Now choose $b > 4t\|v_0\|_\infty$ so that the two coefficients in parentheses on the left-hand side are positive and large. The condition $\xi > 2\nu/3$ is exactly what is needed to have $n^{2\xi-\nu} > n^{\nu/3+\delta\nu} + n^{\xi/2+\delta}$ for all large enough n, if $\delta > 0$ is small enough. The i-term on the right-hand side is controlled by the observation

$$\delta i^{1/2+\delta} \leq C\left(n^{2\xi-\nu} + i n^{\xi-\nu}\right)$$

for all $i \geq 0$, provided n is large enough.

The argument for (5.20) goes exactly the same way. This completes the proof of Lemma 5.4. □

6. Proof of Theorem 3. Let ξ satisfy $\xi > 2\nu/3$ and $\nu - \beta \leq \xi \leq \nu - \varepsilon$ for some small $\varepsilon > 0$. Let $M < \infty$ be a large finite constant, to be chosen later. Define the events

$$A_n = \left\{\left|z^n\left([nx] + [2n^\nu qt], n^\nu t\right) - z^n\left([nx], 0\right) - n^\nu t q^2\right| > 2Mn^{2\xi-\nu}\right\}.$$

By Borel–Cantelli, Theorem 3 will follow by proving the summability

$$\sum_n P(A_n) < \infty. \tag{6.1}$$

To see this, compare $2\xi - \nu$ with the exponent in the error of (3.5): in the case $\nu \leq 3\beta$, take $\xi = 2\nu/3 + \delta/3$, so that $2\xi - \nu < \nu/3 + \delta$. In the case $\nu > 3\beta$, we can take $\xi = \nu - \beta$ so that $2\xi - \nu = \nu - 2\beta < \nu - 2\beta + \delta$.

Now we shall prove (6.1). By Lemma 5.4, for large enough n it is the case that

$$\begin{aligned} &z^n\left([nx] + [2n^\nu qt], n^\nu t\right) - z^n([nx], 0) - n^\nu t q^2 \\ &\quad = \inf_{|i| \leq bn^\xi}\left\{z^n\left([nx] - i, 0\right) - z^n\left([nx], 0\right) + qi \right. \\ &\qquad \left. + \Gamma^n\left([nx] - i, [2n^\nu qt] + i, n^\nu t\right) - n^\nu t q^2 - qi\right\}. \end{aligned} \tag{6.2}$$

The probability $P(A_n)$ is bounded above by

$$\begin{aligned} &\sum_{|i| \leq bn^\xi} P\left\{\left|z^n\left([nx] - i, 0\right) - z^n\left([nx], 0\right) + qi\right| \geq Mn^{2\xi-\nu}\right\} \\ &\quad + \sum_{|i| \leq bn^\xi} P\left\{\left|\mathbf{\Gamma}\left([2n^\nu qt] + i, n^\nu t\right) - n^\nu t q^2 - qi\right| \geq Mn^{2\xi-\nu}\right\} \\ &\quad + P\left\{(6.2) \text{ fails for } n\right\}. \end{aligned} \tag{6.3}$$

To bound the first probability in (6.3), apply (5.28) to get, for each i,

$$\begin{aligned} P\{|z^n([nx]-i,0)-z^n([nx],0)+qi| \geq Mn^{2\xi-\nu}\} \\ \leq P\left\{\sum_{j=1}^{|i|}(q-n^{-\beta}\|v_0\|_\infty)X_j \leq q|i| - Mn^{2\xi-\nu}\right\} \\ + P\left\{\sum_{j=1}^{|i|}(q+n^{-\beta}\|v_0\|_\infty)X_j \geq q|i| + Mn^{2\xi-\nu}\right\} \\ \leq P\left\{\sum_{j=1}^{|i|} X_j \leq |i| + C_1 n^{-\beta}|i| - M_1 n^{2\xi-\nu}\right\} \\ + P\left\{\sum_{j=1}^{|i|} X_j \geq |i| - C_1 n^{-\beta}|i| + M_1 n^{2\xi-\nu}\right\}. \end{aligned}$$

In the second inequality, C_1 is a new constant, and $M_1 = M/(q+1)$ accounts for the effect of dividing M by $(q \pm n^{-\beta}\|v_0\|_\infty)$ when n is large enough.

Since $n^{-\beta}|i| \leq bn^{\xi-\beta}$ and $\xi \geq \nu - \beta \Rightarrow 2\xi - \nu \geq \xi - \beta$, by choosing M large enough at the outset we have a further constant $M_2 > 0$ such that $M_1 n^{2\xi-\nu} - C_1 n^{-\beta}|i| \geq M_2 n^{2\xi-\nu}$. Next, apply the large-deviation rate function κ for Exp(1)-variables as in (5.29)–(5.30). The new upper bound becomes

$$\begin{aligned} P\left\{\sum_{j=1}^{|i|} X_j \leq |i| - M_2 n^{2\xi-\nu}\right\} + P\left\{\sum_{j=1}^{|i|} X_j \geq |i| + M_2 n^{2\xi-\nu}\right\} \\ \leq \exp\{-|i|\kappa(1 - M_2 n^{2\xi-\nu}|i|^{-1})\} \\ + \exp\{-|i|\kappa(1 + M_2 n^{2\xi-\nu}|i|^{-1})\} \\ \leq \exp\{M_2 n^{2\xi-\nu} + |i|\log(1 - M_2 n^{2\xi-\nu}|i|^{-1})\} \\ + \exp\{-M_2 n^{2\xi-\nu} + |i|\log(1 + M_2 n^{2\xi-\nu}|i|^{-1})\}. \end{aligned} \tag{6.4}$$

Check that the functions $(1/x)\log(1 \pm x)$ are maximized by taking $x > 0$ as small as possible. Thus we get an upper bound for (6.4) by replacing $|i|$ by its upper bound bn^ξ. Expanding the log then gives the upper bound $2\exp(-Cn^{3\xi-2\nu})$ for (6.4) (here the assumption $\xi \leq \nu$ becomes useful). Tracing backwards, we conclude that

$$[\text{the first sum in (6.3)}] \leq C_1 n^\xi \exp(-Cn^{3\xi-2\nu}). \tag{6.5}$$

Now we shall bound the second probability in (6.3). Again because $|i| = O(n^\xi)$,

$$\begin{aligned} P\{|\mathbf{\Gamma}([2n^\nu qt]+i, n^\nu t) - n^\nu tq^2 - qi| \geq Mn^{2\xi-\nu}\} \\ \leq P\left\{\left|\mathbf{\Gamma}([2n^\nu qt]+i, n^\nu t) - \frac{1}{4n^\nu t}(2n^\nu qt + i)^2\right| > M_1 n^{2\xi-\nu}\right\} \end{aligned} \tag{6.6}$$

for a constant $M_1 > 0$, provided M was chosen large enough. Since $|i| \le Cn^{\xi} \le Cn^{\nu-\varepsilon}$, Lemma 5.2 implies that the probability in (6.6) is at most $C_1 \exp(-C_2 n^{3\xi-2\nu})$. It follows that the bound in (6.5) works also for the second sum in (6.3). Thus the sum in (6.1) has the following bound:

$$\sum_n P(A_n) \le \sum_n C_1 n^{\xi} \exp(-Cn^{3\xi-2\nu}) + \sum_n P\{(6.2) \text{ fails for } n\} < \infty.$$

The summability is a consequence of the assumption $\xi > 2\nu/3$ and (5.15). Equation (6.1) holds, and we have proved Theorem 3.

7. Proof of Theorem 4. The proofs for the different cases are Borel–Cantelli estimates for the distribution of the random variable $z^n([nx]+[2n^{\nu}qt], n^{\nu}t)$. For the sake of readability, we do not formulate explicit probability estimates and instead write statements of the type (2.10)–(2.11). Behind each a.s. error estimate is a summable deviation probability, as the reader can verify from the arguments.

Case 3 can be proved quickly from Theorem 3: in (3.5) replace the term $z^n([nx], 0)$ by its expectation (3.4) plus fluctuation $o(n^{1/2+\delta})$. We concentrate on proving Cases 1 and 2.

PROOF OF THEOREM 4, CASE 1. Assuming $\nu > 1 + \beta$, the goal is to show that

$$\lim_{n\to\infty} n^{2\beta-\nu}\{z^n([nx] + [2n^{\nu}qt], n^{\nu}t) - n^{\nu}tq^2 - nxq\} = V_{\infty}(0, t). \tag{7.1}$$

Let y be a number that achieves the infimum in (3.10) for $x = 0$. Set $i = [n^{\nu-\beta}y]$ in the expression inside the braces on the right-hand side of (5.14). For large n, we get the upper bound

$$\begin{aligned} &z^n([nx] + [2n^{\nu}qt], n^{\nu}t) \\ &\quad\le z^n([n^{\nu-\beta}y], 0) + \Gamma^n([n^{\nu-\beta}y], [nx] + [2n^{\nu}qt] - [n^{\nu-\beta}y], n^{\nu}t) \\ &\quad\le n^{\nu-\beta}yq + n^{1-\beta}V_0(n^{-1}[n^{\nu-\beta}y]) \\ &\qquad + n^{\nu}tq^2 + \frac{y^2}{4t}n^{\nu-2\beta} + nxq - n^{\nu-\beta}yq + o(n^{\nu-2\beta}) \\ &\quad\le n^{\nu}tq^2 + nxq + n^{\nu-2\beta}\left\{n^{1-\nu+\beta}V_0(n^{\nu-\beta-1}y) + \frac{y^2}{4t}\right\} + o(n^{\nu-2\beta}). \end{aligned} \tag{7.2}$$

The following steps were taken previously: for small $\varepsilon > 0$, it is almost surely true that, for large enough n,

$$\begin{aligned} &z^n([n^{\nu-\beta}y], 0) \\ &\quad\le E\{z^n([n^{\nu-\beta}y], 0)\} + \varepsilon n^{(\nu-\beta)/2+\varepsilon} \\ &\quad= [n^{\nu-\beta}y]q + n^{1-\beta}V_0(n^{-1}[n^{\nu-\beta}y]) + \varepsilon n^{(\nu-\beta)/2+\varepsilon} \\ &\quad\le n^{\nu-\beta}yq + n^{1-\beta}V_0(n^{\nu-\beta-1}y) + o(n^{\nu-2\beta}). \end{aligned} \tag{7.3}$$

The first step above is by Lemma 5.1. The assumption $\nu > 3\beta$ guarantees that $n^{(\nu-\beta)/2+\varepsilon} = o(n^{\nu-2\beta})$ if $\varepsilon > 0$ is small enough.

Similarly by (5.5), for large enough n,

$$\begin{aligned} &\Gamma^n\big([n^{\nu-\beta}y], [nx]+[2n^\nu qt]-[n^{\nu-\beta}y], n^\nu t\big) \\ &\quad\le \frac{1}{4n^\nu t}\big([nx]+[2n^\nu qt]-[n^{\nu-\beta}y]\big)^2 + \varepsilon n^{\nu/3+\varepsilon} \\ &\quad\le n^\nu tq^2 + \frac{y^2}{4t}n^{\nu-2\beta} + nxq - n^{\nu-\beta}yq + C_1 + C_2 n^{1-\beta} + \varepsilon n^{\nu/3+\varepsilon}. \end{aligned} \tag{7.4}$$

The term $C_1 + C_2 n^{1-\beta}$ accounts for terms left out after expanding the square and for removal of integer parts $[\cdot]$. The assumptions $\nu > 1+\beta$ and $\nu > 3\beta$ guarantee that $C_1 + C_2 n^{1-\beta} + \varepsilon n^{\nu/3+\varepsilon} = o(n^{\nu-2\beta})$ if $\varepsilon > 0$ is small enough.

Equations (7.3) and (7.4) justify the validity of (7.2) for large enough n, almost surely. Now we can prove one half of (7.1):

$$\begin{aligned} &\limsup_{n\to\infty} n^{2\beta-\nu}\big\{z^n\big([nx]+[2n^\nu qt], n^\nu t\big) - n^\nu tq^2 - nxq\big\} \\ &\quad\le \limsup_{n\to\infty}\left\{n^{1-\nu+\beta}V_0(n^{\nu-\beta-1}y) + \frac{y^2}{4t}\right\} = V_\infty(y,0) + \frac{y^2}{4t} = V_\infty(0,t). \end{aligned} \tag{7.5}$$

The second last equality follows from

$$\lim_{m\to\infty} m^{-1}V_0(my) = V_\infty(y,0). \tag{7.6}$$

It remains to bound the lim inf in (7.1) from below. By the assumptions $\nu > 3\beta$ and $\nu > 1+\beta$, we can choose a number ϱ that satisfies

$$1-\beta < \varrho < \nu - 2\beta, \qquad \varrho > \nu/3 > \beta, \qquad \varrho > (\nu-\beta)/2. \tag{7.7}$$

Define a sequence of deterministic numbers by

$$r_n = n^\nu tq^2 + nxq + \min_{|i|\le bn^{\nu-\beta}}\left\{n^{1-\beta}V_0\left(\frac{i}{n}\right) + \frac{i^2}{4n^\nu t}\right\} - 2n^\varrho.$$

LEMMA 7.1. *Almost surely, the inequality* $z^n([nx]+[2n^\nu qt], n^\nu t) \ge r_n$ *holds for large enough* n.

Before proving the lemma, let us use it to finish the proof of Case 1 of Theorem 4:

$$\begin{aligned} &\liminf_{n\to\infty} n^{2\beta-\nu}\big\{z^n\big([nx]+[2n^\nu qt], n^\nu t\big) - n^\nu tq^2 - nxq\big\} \\ &\quad\ge \liminf_{n\to\infty}\min_{|i|\le bn^{\nu-\beta}}\left\{n^{1-\nu+\beta}V_0\left(\frac{i}{n}\right) + \frac{i^2}{4n^{2\nu-2\beta}t}\right\} \\ &\qquad\qquad [\text{change of variable } i = n^{\nu-\beta}y] \\ &\quad\ge \liminf_{n\to\infty}\inf_{|y|\le b}\left\{n^{1-\nu+\beta}V_0(n^{\nu-1-\beta}y) + \frac{y^2}{4t}\right\} \\ &\quad\ge V_\infty(0,t). \end{aligned} \tag{7.8}$$

To check the last inequality, let n_j be a subsequence along which the $\liminf_{n\to\infty}$ is realized. For each j pick y_{n_j} that realizes the infimum, pass to a further convergent subsequence $y_{n_j} \to y$, and now consider different cases: if y_{n_j} stays bounded away from 0, it follows from (7.6) that

$$\lim_{j\to\infty}\left\{n_j^{1-\nu+\beta}V_0(n_j^{\nu-1-\beta}y_{n_j})+\frac{y_{n_j}^2}{4t}\right\}=V_\infty(y,0)+\frac{y^2}{4t}\geq V_\infty(0,t). \tag{7.9}$$

And if $y=0$, Lipschitz continuity of V_0 gives

$$\left|n_j^{1-\nu+\beta}V_0(n_j^{\nu-1-\beta}y_{n_j})\right|\leq\|v_0\|_\infty|y_{n_j}|\longrightarrow 0,$$

so in this case, too, the limit in (7.9) is $V_\infty(y,0)+y^2/4t=0$.

Equations (7.5) and (7.8) together imply (7.1), and thereby prove Case 1 of Theorem 4. Before moving on to Case 2, we check Lemma 7.1.

PROOF OF LEMMA 7.1. Abbreviate temporarily

$$Z_n=\min\left\{z^n(i,0)+\Gamma^n(i,[nx]+[2n^\nu qt]-i,n^\nu t):|i|\leq bn^{\nu-\beta}\right\}. \tag{7.10}$$

The assumption $\nu>3\beta$ permits us to set $\xi=\nu-\beta$ in Lemma 5.4, so $z^n([nx]+[2n^\nu qt],n^\nu t)=Z_n$ for large enough n. The difference between $|i|\leq bn^{\nu-\beta}$ in (7.10) and $|i-[nx]|\leq bn^{\nu-\beta}$ in (5.14) is irrelevant now because $\nu-\beta>1$ and we can always increase b. To prove Lemma 7.1, we show that

$$Z_n\geq r_n \text{ holds for large enough } n. \tag{7.11}$$

The complementary probability $P\{Z_n<r_n\}$ is bounded above by the sum

$$\begin{aligned}&\sum_{|i|\leq bn^{\nu-\beta}}P\left\{z^n(i,0)<qi+n^{1-\beta}V_0\left(\frac{i}{n}\right)-n^\varrho\right\}\\&\quad+\sum_{|i|\leq bn^{\nu-\beta}}P\left\{\Gamma^n\big(i,[nx]+[2n^\nu qt]-i,n^\nu t\big)\right.\\&\qquad\qquad\left.<n^\nu tq^2+nxq-qi+\frac{i^2}{4n^\nu t}-n^\varrho\right\}.\end{aligned} \tag{7.12}$$

In the first sum in (7.12) the term for $i=0$ vanishes because by construction $z^n(0,0)=V_0(0)=0$ with probability 1. We bound the sum over $1\leq i\leq bn^{\nu-\beta}$ and leave the matching argument for negative i's to the reader. Let $\{X_j\}$ be as in (5.28). First, split the sum:

$$\begin{aligned}&\sum_{1\leq i\leq bn^{\nu-\beta}}P\left\{z^n(i,0)<qi+n^{1-\beta}V_0\left(\frac{i}{n}\right)-n^\varrho\right\}\\&\quad\leq\sum_{1\leq i\leq\varepsilon_1 n^{\varrho+\beta}}P\left\{\sum_{j=0}^{i-1}(q-n^{-\beta}\|v_0\|_\infty)X_j<qi+n^{1-\beta}V_0\left(\frac{i}{n}\right)-n^\varrho\right\}\\&\qquad+\sum_{\varepsilon_1 n^{\varrho+\beta}<i\leq bn^{\nu-\beta}}P\left\{z^n(i,0)<qi+n^{1-\beta}V_0\left(\frac{i}{n}\right)-n^\varrho\right\}.\end{aligned} \tag{7.13}$$

To the first sum in (7.13) apply a large-deviation argument as in (5.29)–(5.30). Pick $\varepsilon_1, \varepsilon_2 > 0$ small enough and take n large enough so that

$$(q - n^{-\beta}\|v_0\|_\infty)^{-1}\big(qi + n^{1-\beta}V_0(i/n) - n^{\varrho}\big) \le i - \varepsilon_2 n^{\varrho}$$

for all $1 \le i \le \varepsilon_1 n^{\varrho+\beta}$. Then

$$\begin{aligned} &P\left\{\sum_{j=0}^{i-1}(q - n^{-\beta}\|v_0\|_\infty)X_j < qi + n^{1-\beta}V_0\left(\frac{i}{n}\right) - n^{\varrho}\right\} \\ &\quad\le P\left\{\sum_{j=0}^{i-1} X_j < i - \varepsilon_2 n^{\varrho}\right\} \le \exp\{-i\kappa(1 - \varepsilon_2 n^{\varrho} i^{-1})\} \\ &\quad\le \exp\big(-Cn^{2\varrho}i^{-1}\big) \le \exp\big(-C_1 n^{\varrho-\beta}\big). \end{aligned}$$

To the last sum in (7.13) we apply Lemma 5.1. Pick $0 < \varepsilon < \varrho/(\nu-\beta) - 1/2$ so that $\varepsilon \in (0, 1/2)$. Then, for this range of i's,

$$\begin{aligned} &P\{z^n(i,0) < qi + n^{1-\beta}V_0(i/n) - n^{\varrho}\} \\ &\quad\le P\{z^n(i,0) < qi + n^{1-\beta}V_0(i/n) - \varepsilon i^{1/2+\varepsilon}\} \\ &\quad\le C_2 \exp\big(-C_3 i^{2\varepsilon}\big) \le C_2 \exp\big(-C_3 n^{2\varepsilon(\varrho+\beta)}\big). \end{aligned}$$

Combining the estimates gives

$$\sum_{1\le i\le bn^{\nu-\beta}} P\{z^n(i,0) < qi + n^{1-\beta}V_0(i/n) - n^{\varrho}\} \le C_1 n^{\nu-\beta}\exp(-C_2 n^{\gamma}),$$

where $\gamma > 0$ is a new exponent that depends on the earlier constants. The same bound is valid for the entire first sum in (7.12).

By Lemma 5.2, the probability in the second sum in (7.12) is at most

$$\begin{aligned} &P\left\{\mathbf{\Gamma}([nx] + [2n^{\nu}qt] - i, n^{\nu}t) < \frac{1}{4n^{\nu}t}\big(nx + 2n^{\nu}qt - i\big)^2 - C_3 n^{\varrho}\right\} \\ &\quad\le \exp(-C_4 n^{(3/2)(\varrho-\nu/3)}). \end{aligned} \tag{7.14}$$

The constant $C_3 \in (0,1)$ appeared in front of n^{ϱ} to subsume the difference between $n^{\nu}tq^2 + nxq - qi + i^2/(4n^{\nu}t)$ in (7.12) and $(nx + 2n^{\nu}qt - i)^2/(4n^{\nu}t)$ in (7.14). Combining the estimates, we get

$$\sum_n P\{Z_n < r_n\} < \infty.$$

Borel–Cantelli now gives (7.11) and completes the proof of Lemma 7.1. □

PROOF OF THEOREM 4, CASE 2. Assuming $\nu = 1 + \beta$, the goal is now

$$\lim_{n\to\infty} n^{\beta-1}\big\{z^n\big([nx] + [2n^{\nu}qt], n^{\nu}t\big) - n^{\nu}tq^2 - nxq\big\} = V(x,t). \tag{7.15}$$

Let y be a number that achieves the infimum in (2.13) so that $V(x,t) = V_0(y) + (x-y)^2/4t$. Set $i = [ny]$ in the expression inside the braces on the right-hand side of (5.14). Repeat the calculation in (7.2) to get an upper bound for large n:

$$
\begin{aligned}
&z^n\big([nx]+[2n^\nu qt], n^\nu t\big)\\
&\quad\le z^n([ny],0)+\Gamma^n\big([ny],[nx]-[ny]+[2n^\nu qt], n^\nu t\big)\\
&\quad\le nyq+n^{1-\beta}V_0(y)+n^{1/2+\varepsilon}\\
&\qquad+n^\nu tq^2+\frac{(x-y)^2}{4t}n^{1-\beta}+nxq-nyq+n^{(1+\beta)/3+\varepsilon}\\
&\quad\le n^\nu tq^2+nxq+n^{1-\beta}V(x,t)+o(n^{1-\beta}).
\end{aligned}
\tag{7.16}
$$

The preceding steps are justified by Lemmas 5.1 and 5.2, as was done in (7.3)–(7.4). Again $\varepsilon>0$ needs to be small enough. The estimate $n^{1/2+\varepsilon}+n^{(1+\beta)/3+\varepsilon}=o(n^{1-\beta})$ follows from $\beta<1/2$, which itself is a consequence of the assumptions $\nu=1+\beta$ and $\nu>3\beta$. This gives one half of the goal (7.15).

For the other half of the proof we can also follow the argument of Case 1. Since $\beta<1/2$, we can choose ϱ so that

$$\frac{1+\beta}{3}<\frac{1}{2}<\varrho<1-\beta.$$

By the variational formula (2.13), for all i,

$$
\begin{aligned}
&n^\nu tq^2+nxq+n^{1-\beta}V(x,t)-2n^\varrho\\
&\quad\le n^\nu tq^2+nxq+n^{1-\beta}V_0\Big(\frac{i}{n}\Big)+n^{1-\beta}\frac{(x-i/n)^2}{4t}-2n^\varrho\\
&\quad=\Big[qi+n^{1-\beta}V_0\Big(\frac{i}{n}\Big)-n^\varrho\Big]+\Big[\frac{1}{4n^\nu t}\big(nx+2n^\nu qt-i\big)^2-n^\varrho\Big].
\end{aligned}
$$

Now the argument of Lemma 7.1 can be repeated to conclude that almost surely, for large enough n,

$$z^n\big([nx]+[2n^\nu qt], n^\nu t\big)\ge n^\nu tq^2+nxq+n^{1-\beta}V(x,t)-2n^\varrho,$$

which together with (7.16) implies (7.15). Case 2 is proved, and thereby Theorem 4. □

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