consideration which of them is the easiest to employ, and neither this nor the description of the forms of the laws of errors demands any higher qualification than an elementary knowledge of mathematics. But we must take into account also, how far the different forms are calculated to emphasise the important features of the laws of errors, i. e. those which may be transferred from the laws of actual errors to the laws of presumptive errors. On this single point, certainly, a more thorough knowledge of mathematics would be desirable than that which may be expected from the majority of those students who are obliged to occupy themselves with observations. As the definition of the law of presumptive errors presupposes the determination of limiting values to infinitely numerous approximations, some propositions from the differential calculus would, strictly speaking, be necessary.

III. TABULAR ARRANGEMENTS.

§ 9. In stating the results of all the several repetitions we give the law of errors in its simplest form. Identical results will of course be noted by stating the number of the observations which give them.

The table of errors, when arranged, will state all the various results and the frequency of each of them.

The table of errors is certainly improved, when we include in it the *relative frequencies* of the several results, that is, the ratio which each absolute frequency bears to the total number of repetitions. It must be the *relative* frequencies which, according to the law of large numbers, are, as the number of observations is increased, to approach the constant values of the law of presumptive errors. Long usage gives us a special word to denote this transition in our ideas: *probability* is the relative frequency in a law of presumptive errors, the proportion of the number of coincident results to the total number, on the supposition of infinitely numerous repetitions. There can be no objection to considering the *relative frequency* of the law of actual errors as an approximation to the corresponding *probability* of the law of presumptive errors, and the doubt whether the relative frequency itself is the best approximation that can be got from the results of the given repetitions, is rather of theoretical than practical interest. Compare § 73.

It makes some difference in several other respects — as well as in the one just mentioned — if the phenomenon is such that the results of the repetitions show qualitative differences or only differences of magnitude.

§ 10. In the former case, in which no transition occurs, but where there are such abrupt differences that none of the results are more closely connected with one another than with the rest, the tabular form will be the only possible one, in which the law of errors can be given. This case frequently occurs in statistics and in games of chance, and for this reason the theory of probabilities, which is the form of the theory of observations in which these cases are particularly taken into consideration, demands special attention. All previous authors have begun with it, and made it the basis of the other parts of the science of observation. I am of opinion, however, that it is both safer and easier to keep it to the last.

§ 11. If, however, there is such a difference between the results of repetitions, that there is either a continuous transition between them, or that some results are nearer each other than all the rest, there will be ample opportunity to apply mathematical methods; and when the tabular form is retained, we must take care to bring together the results that are near one another. A table of the results of firing at a target may for instance have the following form:

	1 foot to the left	Central	1 foot to the right	Total
1 foot too high	3	17	6	26
Central	13	109	19	141
1 foot too low	4	8	1	13
Total ...	20	134	26	180

If here the heading "1 foot to the left" means that the shot has swerved to the left between half a foot and one foot and a half, this will remind us that we cannot give the exact measures in such tables, but are obliged to give them in round numbers. The number of results then will not correspond to such as were exactly the same, but disregarding small differences, we gather into each column those that approach nearest to one another, and which all fall within arbitrarily chosen limits.

In the simple case, where the result of the observation can be expressed by a single real number, the arranged table not only takes the extremely simple form of a table of functions with a single argument, but, as we shall see in the following chapters, leads us to the representation of the law of errors by means of curves of errors and functional laws of errors.

It is an obvious course to fix the attention on the two extreme results in the table, and not seldom these alone are given, instead of a law of error, as a sort of index of the exactness of the whole series of repetitions, and as the higher and lower limits of the observed phenomenon. This index of exactness, however, must be rejected as itself too inexact for the purpose, for the oftener the observations are repeated, the farther we must expect the extremes to move from one another; and thus the most valuable series of observations will appear to possess the greatest range of discrepancy.

On the other hand, if, in a table arranged according to the magnitude of the values, we select a single middle value, preceded and followed by nearly equal numbers of values, we shall get a quantity which is very well fitted to represent the whole series of repetitions.

If, while we are thus counting the results arranged according to their magnitude, we also take note of these two values with which we respectively (*a*) leave the first sixth part of the total number, and (*b*) enter upon the last sixth part (more exactly we ought to say 16 per ct.), we may consider these two as indicating the limits between great and small deviations. If we state these two values along with the middle one above referred to, we give a serviceable expression for the law of errors, in a way which is very convenient, and although rough, is not to be despised. Why we ought to select just the middle value and the two sixth-part values for this purpose, will appear from the following chapters.

IV. CURVES OF ERRORS.

§ 12. Curves of actual errors of repeated observations, each of which we must be able to express by one real number, are generally constructed as follows. On a straight line as the axis of abscissae, we mark off points corresponding to the observed numerical quantities, and at each of these points we draw an ordinate, proportional to the number of the repetitions which gave the result indicated by the abscissa. We then with a free hand draw the curve of errors through the ends of the ordinates, making it as smooth and regular as possible. For quantities and their corresponding abscissae which, from the nature of the case, *might* have appeared, but do not really appear, among the repetitions, the ordinate will be = 0, or the point of the curve falls on the axis of abscissae. Where this case occurs very frequently, the form of the curves of errors becomes very tortuous, almost discontinuous. If the observation is essentially bound to discontinuous numbers, for instance to integers, this cannot be helped.

§ 13. If the observation is either of necessity or arbitrarily, in spite of some inevitable loss of accuracy, made in round numbers, so that it gives a lower and a higher limit for each observation, a somewhat different construction of the curve of errors ought to be applied, viz. such a one, that the area included between the curve of error, the axis of abscissae, and the ordinates of the limits, is proportional to the frequency of repetitions within these limits. But in this way the curve of errors may depend very much on the degree of accuracy involved in the use of round numbers. This construction of areas can be made by laying down rectangles between the bounding ordinates, or still better, trapezoids with their free sides approximately parallel to the tangents of the curve. If the