ON SYMMETRIC FUNCTIONS OF MORE THAN ONE VARIABLE AND OF FREQUENCY FUNCTIONS

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In a paper published in this journal[1] the writer has developed a simple differential operator method for expressing any symmetric function of the n variates $x_1, x_2, \cdots\cdots, x_n$ as a rational, integral, algebraic function of the power sums $s_1, s_2, \cdots, s_w$ where w is the weight of the symmetric function and

$$s_k = \Sigma x_i^k = x_1^k + x_2^k + \cdots\cdots + x_n^k.$$

The transformation to moments is then simply a matter of recognizing that $n\mu'_k = s_k$ if μ'_k is the kth moment of the n variates with respect to the origin from which they are measured. If the origin is at the arithmetic mean of the n variates the prime may be dropped and then $n\mu_k = s_k$.

In the above mentioned paper the variates x_i are of the serial distribution type, but, of course, not necessarily integers. The extensions to the case of more than one set of variates and to frequency functions now suggest themselves. It is the purpose of this note to discuss these problems simultaneously.

Suppose that two sets, of n variates each, $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ are given and that x_i, y_i $(i = 1,2, \ldots\ldots, n)$ are corresponding pairs. Modifying the partition notation used in the previous paper the symmetric function to be considered may be written in the form $(a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots\cdots b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots)$ i.e. the sum of all such terms as

$$x_1^{a_1} x_2^{a_1} \cdots x_{n_1}^{a_1} x_{n_1+1}^{a_2} \cdots x_{n_1+n_2}^{a_2} \cdots\cdots y_1^{b_1} y_2^{b_1}$$

$$\cdots y_{m_1}^{b_1} y_{m_1+1}^{b_2} \cdots y_{m_1+m_2}^{b_2} \cdots$$

[1] Symmetric Functions and Symmetric Functions of Symmetric Functions, Vol. II. No. 2 (May, 1931), pp. 102-149.

where

$$a_1, a_2, a_3, \cdots, b_1, b_2, b_3, \cdots\ n_1, n_2, n_3, \cdots\ m_1, m_2, m_3, \cdots$$

are positive integers and where

$$a_1 > a_2 > a_3 > \cdots > 0.$$

$$n_1 + n_2 + n_3 + \cdots = m_1 + m_2 + m_3 + \cdots.$$

e.g. $(3^2 2.451) = (332.451) = \sum x_i^3 x_j^3 x_k^2 y_i^4 y_j^5 y_k,\ i \neq j \neq k$. Obviously the order of the parts in the bipartite notation will be important, corresponding parts being equidistant from the left of their respective sections of the partition. The double partition will be said to be of weight $w_1 = a_1 n_1 + a_2 n_2 + a_3 n_3 + \cdots$ in x and of weight $w_2 = b_1 m_1 + b_2 m_2 + b_3 m_3 + \cdots$ in y. There will be no loss of generality if it is assumed that $w_1 \geq w_2$.

By a procedure similar to that employed in the first chapter of the paper already referred to, it may be shown that any symmetric function of the type defined above can be expressed as rational, integral, algebraic function of the symmetric functions

$$s_{11},\ s_{12},\ s_{21}, \cdots\cdots, s_{w_1 w_2}$$

where

$$s_{hk} = \sum_{i=1}^{n} x_i^h y_i^k,\quad h = 1, 2, \cdots, w_1,\quad k = 1, 2, \cdots, w_2.$$

Moreover, the terms of this function will be isobaric of weight w_1 in x, isobaric of weight w_2 in y, and hence isobaric of weight $w_1 + w_2$ in x and y together.

e. g. Multiply s_{21} by itself:

$$\begin{aligned} s_{21}^2 &= (x_1^2 y_1 + x_2^2 y_2 + \cdots + x_n^2 y_n)^2 \\ &= (x_1^4 y_1^2 + x_2^4 y_2^2 + \cdots + x_n^4 y_n^2) + 2(x_1^2 x_2^2 y_1 y_2 + x_1^2 x_3^2 y_1 y_3 + \cdots) \\ &= s_{42} + 2(2^2 \cdot 1^2). \end{aligned}$$

$$(2^2 \cdot 1^2) = (s_{21}^2 - s_{42})/2,$$

each term being of weight 4 in x, 2 in y, and 6 in x and y together. It is possible then to write

$$(a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots) \equiv$$

$$f(s_{11}, s_{12}, \cdots, s_{ij}, \cdots, s_{w_1 w_2}) \equiv f$$

where f stands for a rational, integral, algebraic function of the sums (or product moments) s_{ij}, $i = 1, \cdots, w_1$, $j = 1, \cdots, w_2$, isobaric as explained above. Suppose that a new pair of variates $x_{n+1} = \alpha$, $y_{n+1} = \beta$ are introduced. Obviously s_{ij} becomes $s_{ij} + \alpha^i \beta^j$. Hence applying Taylor's Theorem f becomes

$$f + (\alpha\beta d_{11} + \alpha\beta^2 d_{12} + \cdots + \alpha^i\beta^j d_{ij} + \cdots + \alpha^{w_1}\beta^{w_2} d_{w_1 w_2}) f$$

$$+ (\alpha\beta d_{11} + \alpha\beta^2 d_{12} + \cdots + \alpha^i\beta^j d_{ij} + \cdots + \alpha^{w_1}\beta^{w_2} d_{w_1 w_2})^2 f/2!$$

$$+ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$$

$$+ (\alpha\beta d_{11} + \alpha\beta^2 d_{12} + \cdots + \alpha^i\beta^j d_{ij} + \cdots + \alpha^{w_1}\beta^{w_2} d_{w_1 w_2})^{w_2} f/w_2!$$

all other terms being identically zero, where $d_{ij} = \partial/\partial_{ij}$ and $d_{ij}^k = \partial^k/\partial s_{ij}^k$, $i = 1, \cdots, w_1$, $j = 1, \cdots, w_2$.

Using the multinomial theorem and collecting coefficients of $\alpha^i\beta^j$ the above expression may be written in the form

$$(1 + \alpha\beta D_{11} + \alpha\beta^2 D_{12} + \cdots + \alpha^i\beta^j D_{ij} + \cdots) f$$

where

$$(1) \quad \left[\begin{array}{l} [\, D_{11} = d_{11}, \\ \left[\begin{array}{l} D_{12} = d_{12}, \\ D_{21} = d_{21} \end{array}\right. \\ \left[\begin{array}{l} D_{13} = d_{13} \\ D_{22} = d_{22} + d_{11}^2/2, \\ D_{31} = d_{31}, \end{array}\right. \\ \left[\begin{array}{l} D_{14} = d_{14}, \\ D_{23} = d_{23} + d_{11} d_{12}, \\ D_{32} = d_{32} + d_{11} d_{21}, \\ D_{41} = d_{41}, \end{array}\right. \end{array}\right.$$

$$\left\{\begin{array}{l} D_{15} = d_{15}, \\ D_{24} = d_{24} + d_{11} d_{13} + d_{12}^2/2, \\ D_{33} = d_{33} + d_{11} d_{22} + d_{12} d_{21} + d_{11}^3/6, \\ D_{42} = d_{42} + d_{11} d_{31} + d_{21}^2/2, \\ D_{51} = d_{51} \end{array}\right.$$

$$D_{ij} = \sum \frac{d_{i_1 j_1}^{k_1} \, d_{i_2 j_2}^{k_2} \, d_{i_3 j_3}^{k_3} \cdots\cdots}{k_1! \; k_2! \; k_3! \cdots\cdots}, \; i=1, \cdots, w_1, \; j=1, \cdots, w_2,$$

where $k_1 i_1 + k_2 i_2 + k_3 i_3 + \cdots\cdots = i,$
$k_1 j_1 + k_2 j_2 + k_3 j_3 + \cdots\cdots = j,$

$i_1, i_2, i_3, \cdots\cdots, j_1, j_2, j_3, \cdots, k_1, k_2, k_3, \cdots$ being positive integers. The inverse relation is given bv

$$d_{ij} = \sum \frac{(-1)^{v+1}(v-1)! D_{i_1 j_1}^{k_1} \, D_{i_2 j_2}^{k_2} \, D_{i_3 j_3}^{k_3} \cdots\cdots}{k_1! \; k_2! \; k_3! \cdots\cdots},$$

where $i = 1, \cdots\cdots, w_1, \quad j = 1, \cdots\cdots, w_2,$

$k_1 i_1 + k_2 i_2 + k_3 i_3 + \cdots\cdots = i,$

$k_1 j_1 + k_2 j_2 + k_3 j_3 + \cdots\cdots = j,$

$k_1 + k_2 + k_3 + \cdots\cdots\cdots = v,$

$i_1, i_2, i_3, \cdots, j_1, j_2, j_3, \cdots, k_1, k_2, k_3, \cdots$ being positive integers.

The effect of the new variates α, β on

$$(a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots\cdots b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots)$$

is to replace this symmetric function by

$$(a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots \; b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots)$$
$$+ \alpha^{a_1} \beta^{b_1} (a_1^{n_1-1} a_2^{n_2} a_3^{n_3} \cdots \; b_1^{m_1-1} b_2^{m_2} b_3^{m_3} \cdots)$$
$$+ \alpha^{a_2} \beta^{b_2} (a_1^{n_1} a_2^{n_2-1} a_3^{n_3} \cdots \; b_1^{m_1} b_2^{m_2-1} b_3^{m_3} \cdots) + \cdots.$$

Hence replacing f by $(a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots)$, then

$$(1 + \alpha\beta D_{11} + \alpha\beta^2 D_{12} + \alpha^2\beta D_{21} + \cdots$$
$$+ \alpha^i \beta^j D_{ij} + \cdots)(a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots)$$

$$=(a_1^{n_1}a_2^{n_2}a_3^{n_3}\cdots b_1^{m_1}b_2^{m_2}b_3^{m_3}\cdots)$$

$$+\alpha^{a_1}\beta^{b_1}(a_1^{n_1-1}a_2^{n_2}a_3^{n_3}\cdots b_1^{m_1-1}b_2^{m_2}b_3^{m_3}\cdots)$$

$$+\alpha^{a_2}\beta^{b_2}(a_1^{n_1}a_2^{n_2-1}a_3^{n_3}\cdots b_1^{m_1}b_2^{m_2-1}b_3^{m_3}\cdots)+\cdots.$$

Equating coefficients of like terms in α and β it is seen that

(2)
$$D_{a_1b_1}(a_1^{n_1}a_2^{n_2}a_3^{n_3}\cdots b_1^{m_1}b_2^{m_2}b_3^{m_3}\cdots)$$
$$=(a_1^{n_1-1}a_2^{n_2}a_3^{n_3}\cdots b_1^{m_1-1}b_2^{m_2}b_3^{m_3}\cdots),$$

$$D_{a_2b_2}(a_1^{n_1}a_2^{n_2}a_3^{n_3}\cdots b_1^{m_1}b_2^{m_2}b_3^{m_3}\cdots)$$
$$=(a_1^{n_1}a_2^{n_2-1}a_3^{n_3}\cdots b_1^{m_1}b_2^{m_2-1}b_3^{m_3}\cdots),$$

etc. and

$D_{hk}(a_1^{n_1}a_2^{n_2}a_3^{n_3}\cdots b_1^{m_1}b_2^{m_2}b_3^{m_3}\cdots)=0$ if both h and k are not among $a_1, a_2, a_3, \cdots$ and $b_1, b_2, b_3, \cdots$ respectively. Hence also

$$D_{a_1b_1}^{t_1}D_{a_2b_2}^{t_2}D_{a_3b_3}^{t_3}\cdots D_{a_kb_k}^{t_k}(a_1^{n_1}a_2^{n_2}a_3^{n_3}\cdots b_1^{m_1}b_2^{m_2}b_3^{m_3}\cdots)=0$$

if $t_1, t_2, t_3, \cdots, t_k$ are not all respectively less than or equal to $n_1, n_2, n_3, \cdots, n_k$ and also respectively less than or equal to $m_1, m_2, m_3, \cdots; m_k$ where $t_1, t_2, t_3, \cdots, t_k$ are positive integers or zeros; that is, if even one of the values of t is greater than the corresponding n or m then the effect of the operation is to give zero. This last property of the D operators is important not only in minimizing the work of computation as will be seen in the illustrations given below, but will be of fundamental importance in the theorem to be stated in the closing paragraph of this paper. It should be noted here that the multiplication of the operators is commutative.

To illustrate the application of the operators consider:

a) $(21.1^2)=c_1s_{21}s_{11}+c_2s_{32}$ where c_1 and c_2 are constants to be determined. These are the only terms which satisfy the weight conditions. Operating on the left with D_{21} and on the right with

d_{21} gives $(1.1) = c_1 s_{11}$, i.e. $s_{11} = c_1 s_{11}$, hence $c_1 = 1$.
Operating on the left with D_{32} and on the right with $d_{32} + d_{11} d_{21}$ gives $0 = c_1 + c_2$, hence $c_2 = -1$ and thus $(21.1^2) = s_{21} s_{11} - s_{32}$.

b) $$(321.1^3) = c_1 s_{63} + c_2 s_{52} s_{11} + c_3 s_{51} s_{12} + c_4 s_{42} s_{21} + c_5 s_{41} s_{22} + c_6 s_{41} s_{11}^2 + c_7 s_{32} s_{31} + c_8 s_{31} s_{21} s_{11} + c_9 s_{21}^3.$$

These are the only terms which satisfy the weight conditions. If the expressions for all the symmetric functions of lower weights are known then to determine the constants it is sufficient to operate on the left with

$D_{11}, D_{21}, D_{31}, D_{41}, D_{51}, D_{63}$ and on the right with their respective equivalents in terms of d. However if the expressions for the symmetric functions of lower weights are not known then it is perhaps simpler to operate with

$$d_{12},\ d_{11}^2,\ d_{21}^2,\quad d_{31} d_{21} d_{11},\ d_{31} d_{32},\ d_{41} d_{22},\ d_{42} d_{21},$$

$d_{11} d_{52}$, d_{63} on the right and with their respective equivalents in terms of D on the left. d_{12} and D_{12} give $0 = c_3 s_{51}$ and hence $c_3 = 0$. d_{11}^2 and D_{11}^2 give $0 = 2c_6 s_{41}$ and $c_6 = 0$. d_{21}^2 and D_{21}^2 give $0 = 6 c_9 s_{21}$ and $c_9 = 0$. Similarly $d_{31} d_{21} d_{11}$ and $D_{31} D_{21} D_{11}$ give $c_8 = 1$. $d_{31} d_{32}$ and $D_{31}(D_{32} - D_{21} D_{11})$ give $c_7 = -1$. $d_{41} d_{22}$ and $D_{41}(D_{22} - D_{11}^2/2)$ give $c_5 = 0$. $d_{21} d_{42}$ and $D_{21}(D_{42} - D_{31} D_{11} - D_{21}^2/2)$ give $C_4 = -1$. d_{11} d_{52} and $D_{11}(D_{52} - D_{41} D_{11} - D_{31} D_{21})$ give $c_2 = -1$.
d_{63} and $D_{63} - D_{52} D_{11} - D_{51} D_{12} - D_{42} D_{21} - D_{41} D_{22} + D_{41} D_{11}^2 - D_{32} D_{31} + 2 D_{31} D_{21} D_{11} + D_{21}^3/2$ give $c_1 = 2$. Hence $(321.1^3) = 2 s_{63} - s_{52} s_{11} - s_{42} s_{21} - s_{32} s_{31} + s_{31} s_{21} s_{11}$.

Suppose that $y_k = 1$, $k = 1, 2, \cdots, n$. Then s_{ij} is simply s_i and D_{ij}, d_{ij} have no meaning except when $i = j$ and then D_{ii} and d_{ii} become the operators D_i and d_i respectively, of the earlier paper.

The operator relations for any number of sets of corresponding variates are now obvious. For instance, in the case of 3 sets

x_i, y_i, z_i the result is

$$D_{ijk} = \sum \frac{d_{i_1 j_1 k_1}^{h_1} \, d_{i_2 j_2 k_2}^{h_2} \, d_{i_3 j_3 k_3}^{h_3} \cdots\cdots}{h_1! \quad h_2! \quad h_3! \quad \cdots\cdots},$$

$$i = 1, 2, \cdots\cdots, w_1,$$

$$j = 1, 2, \cdots\cdots, w_2,$$

$$k = 1, 2, \cdots\cdots, w_3,$$

$$h_1 i_1 + h_2 i_2 + h_3 i_3 + \cdots\cdots = i,$$

$$h_1 j_1 + h_2 j_2 + h_3 j_3 + \cdots\cdots = j,$$

$$h_1 k_1 + h_2 k_2 + h_3 k_3 + \cdots\cdots = k,$$

$$i_1, i_2, i_3, \cdots, j_1, j_2, j_3, \cdots, k_1, k_2, k_3, \cdots, h_1, h_2, h_3, \cdots$$

being positive integers , w_1, w_2, w_3 the weights in xyz respectively of the symmetric function ,

$$d_{ijk} = \partial / \partial s_{ijk} \, , \quad s_{ijk} = \sum_{t=1}^{n} x_t^i y_t^j z_t^k \, .$$

Returning now to the case of two variates x and y, suppose that x_i takes on only integral values for $i = 1, 2, \ldots\ldots, n$ and that $y_i = f(x_i)$ where $f(x_i)$ is thought of as the frequency corresponding to $x = x_i$ If, further, $b_1 = b_2 = b_3 = \cdots\cdots = 1$ then the operators developed in this note give the expressions for $(a^\alpha b^\beta c^\gamma \cdots)$ of the earlier paper when each serial x_i^k is replaced by $x_i^k f(x_i)$. More generally, let $y_i = f(x_i)\Delta x_i$ represent the frequency of x in the interval Δx_i. If x takes on only integral values then of course $\Delta x_i = 1$, $i = 1, 2, \ldots\ldots, n$.
Consider

$$(32.11) = \sum_{\substack{i=1\\j=1}}^{n} x_i^3 x_j^2 f(x_i) f(x_j) \Delta x_i \Delta x_j \, , \; i \neq j \, ,$$

$$= \sum_{i=1}^{n} x_i^3 f(x_i) \Delta x_i \cdot \sum_{j=1}^{n} x_j^2 f(x_j) \Delta x_j - \sum_{i=1}^{n} x_i^5 f^2(x_i) \overline{\Delta x}_i^2 .$$

If the lower and upper bounds for x are respectively a and b then in the limit as n becomes infinite and the maximum Δx_i, $i = 1, 2, \cdots, n$, approaches zero, the last summation on the right approaches zero, $f(x)$ being an ordinary frequency function. Thus in this limiting case

$$(32.11) = \int_a^b \int_a^b x_i^3 x_j^2 f(x_i) f(x_j) dx_i \, dx_j, \; i \neq j,$$

$$= \int_a^b x_i^3 f(x_i) dx_i \cdot \int_a^b x_j^2 f(x_j)\, dx_j .$$

In general, under these limiting conditions, any summation $\sum_{i=1}^{n} x_i^k f^r(x_i)\overline{\Delta x_i}^r$ approaches zero, if r is greater than 1, k and r being positive integers. For let $\mathcal{E}_n$ be the maximum Δx_i for specified n. Now $x_i^k f^r(x_i) \leqq M$ for $i = 1,2, \ldots ,n$. Hence

$$\sum_{i=1}^{n} x_i^k f^r(x_i)\overline{\Delta x}_i^r \leqq M \sum_{i=1}^{n} \overline{\Delta x}_i^r \leqq M\mathcal{E}_n^{r-1} \sum_{i=1}^{n} \Delta x_i = M\mathcal{E}_n^{r-1}(b-a)$$

which approaches zero with $\mathcal{E}_n$. This establishes the well known statement that, the values of x being independent,

$$\int_a^b \int_a^b \cdots \int_a^b x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k} f(x_{i_1}) f(x_{i_2}) \cdots f(x_{i_k}) dx_{i_1} dx_{i_2} \cdots dx_{i_k}$$

$$= \int_a^b x_{i_1}^{a_1} f(x_{i_1}) dx_{i_1} \cdot \int_a^b x_{i_2}^{a_2} f(x_{i_2}) dx_{i_2} \cdots \int_a^b x_{i_k}^{a_k} f(x_{i_k}) dx_{i_k}.$$

For, under the above limiting conditions, all those terms which contain a sum $s_{hk} = \sum x_i^h f^k(x_i) \overline{\Delta x_i}^k$ with k greater than 1 must vanish. By the last property of the D operators given in (2) it is seen that there is always only one term which does not contain such an s_{hk}; and from this term arises the product of the definite integrals.

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