A SHORT METHOD FOR SOLVING FOR A CO-EFFICIENT OF MULTIPLE CORRELATION

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The method which we present presupposes a familiarity with the Doolittle method [1] for solving normal equations. We start with the determinant

$$(1) \qquad R = \begin{vmatrix} 1 & r_{12} & \cdots & r_{1n} \\ r_{12} & 1 & \cdots & r_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{1n} & r_{2n} & \cdots & 1 \end{vmatrix}$$

where the elements are zero order coefficients of correlation.

Now the adjoint determinant of (1) may be written

$$(2) \qquad r = \begin{vmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{12} & R_{22} & \cdots & R_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1n} & R_{2n} & \cdots & R_{nn} \end{vmatrix}$$

where the elements are the cofactors of the elements in (1).

From the elementary theory of determinants [2] we know that

$$(3) \qquad r = R^{n-1}$$

The adjoint determinant of r may be designated by KR where

$$(4) \qquad KR = r^{n-1}$$

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[1] Mills, F. C., Statistical Methods, p. 577.
[2] Bôcher, Maxime, Introduction to Higher Algebra, p. 33.

From (3) and (4) we have

$$KR = R^{(n-1)^2}$$

or

$$K = R^{n(n-2)} \tag{5}$$

Hence, the adjoint or r is obtained by multiplying each element of R by R^{n-2}. And if we write

$$KR = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{12} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{vmatrix} \tag{6}$$

then

$$\frac{A_{ij}}{R^{n-2}} = r_{ij} \tag{7}$$

so that (1) may be rewritten

$$R = \begin{vmatrix} \frac{A_{11}}{R^{n-2}} & \frac{A_{12}}{R^{n-2}} & \cdots & \frac{A_{1n}}{R^{n-2}} \\ \frac{A_{12}}{R^{n-2}} & \frac{A_{22}}{R^{n-2}} & \cdots & \frac{A_{2n}}{R^{n-2}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{A_{1n}}{R^{n-2}} & \frac{A_{2n}}{R^{n-2}} & \cdots & \frac{A_{nn}}{R^{n-2}} \end{vmatrix} \tag{8}$$

The numerators of the elements in (8) are the cofactors of the elements in (2).

Let us now consider (8) as the coefficients of a set of normal equations whose constant terms are zero, and let us follow through literally the Doolittle elimination process.

For simplicity we outline the reduction of a 4-variable problem as follows.

Reciprocal	1	2	3	4	α	β	γ	δ
$-\frac{R^2}{A_{11}}$	$\frac{A_{11}}{R^2}$ -1	$\frac{A_{12}}{R^2}$ $\frac{-A_{12}}{A_{11}}$	$\frac{A_{13}}{R^2}$ $-\frac{A_{13}}{A_{11}}$	$\frac{A_{14}}{R^2}$ $\frac{-A_{14}}{A_{11}}$	α_1		γ_1	δ
		$\frac{A_{22}}{R^2}$ $-\frac{A_{12}^2}{R^2A_{11}}$	$\frac{A_{23}}{R^2}$ $-\frac{A_{12}A_{13}}{R^2A_{11}}$	$\frac{A_{24}}{R^2}$ $-\frac{A_{12}A_{14}}{R^2A_{11}}$	α_2	β_{22}		
$-\frac{R^2A_{11}}{AA_{1122}}$		$\frac{AA_{1122}}{R^2A_{11}}$ -1	$\frac{AA_{1123}}{R^2A_{11}}$ $-\frac{A_{1123}}{A_{1122}}$	$\frac{AA_{1124}}{R^2A_{11}}$ $-\frac{A_{1124}}{A_{1122}}$			γ_2	δ_2
			$\frac{A_{33}}{R^2}$ $-\frac{A_{13}^2}{R^2A_{11}}$ $-\frac{AA_{1123}^2}{R^2A_{11}A_{1122}}$	$\frac{A'_{34}}{R^2}$ $-\frac{A_{13}A_{14}}{R^2A_{11}}$ $-\frac{AA_{1123}A_{1124}}{R^2A_{11}A_{1122}}$	α_3	β_{23} β_{33}		
$-\frac{R^2A_{1122}}{AA_{112233}}$			$\frac{AA_{112233}}{R^2A_{1122}}$ -1	$\frac{AA_{112234}}{R^2A_{1122}}$ $\frac{A_{112234}}{A_{112233}}$			γ_3	δ_3
			$-\sum_2^n \beta_i n =$	$\frac{A'_{44}}{R^2}$ $-\frac{A_{14}^2}{R^2A_{11}}$ $-\frac{AA_{1124}^2}{R^2A_{11}A_{1122}}$ $-\frac{AA_{112234}^2}{R^2A_{1122}A_{112233}}$	α_4	β_{24} β_{34} β_{44}		
$-\frac{R^2A_{112233}}{AA_{11223344}}$				$\frac{AA_{11223344}}{R^2A_{112233}}$ -1			γ_4	δ_4

The α-equations are the original equations with the coefficients to the left of the diagonal omitted. The β-equations are the product equations which are subtracted from the α-equations. The γ-equations are the reduced equations which may be represented symbolically by

$$\gamma = \alpha - \Sigma\beta \tag{9}$$

The δ-equations are the γ-equations divided by the negatives of their respective leading coefficients. That (9) is true may be readily proved from the theorem[8]

$$\begin{vmatrix} A_{ij} & A_{il} \\ A_{kj} & A_{kl} \end{vmatrix} = AA_{ijkl} \tag{10}$$

Where the notation indicates cofactors rather than minors. The proof is made more obvious if (9) is written

$$\gamma = \left\{ \left[(\alpha - \beta_1) - \beta_2 \right] - \beta_3 \right\} - \beta_4 \ \text{etc.}$$

for each successive subtraction reduces the determinant by one.

If we indicate the leading coefficients of the γ-equations by γ_{ii} we may prove that

$$R = \prod_1^n \gamma_{ii} \tag{11}$$

We have in the case of four variables

$$\prod_1^4 \gamma_{ii} = \frac{A_{11}}{R^2} \cdot \frac{rA_{1122}}{R^2 A_{11}} \cdot \frac{rA_{112233}}{R^2 A_{1122}} \cdot \frac{rA_{11223344}}{R^2 A_{112233}} = \frac{r^3}{R^8} \tag{12}$$

$$A \equiv r$$

or from [8]

$$\prod_1^4 \gamma_{ii} = R$$

In the general case we have

$$\prod_1^n \gamma_{ii} = \frac{r^{n-1}}{R^{(n-2)n}} = \frac{R^{(n-1)^2}}{R^{n(n-2)}} = R$$

[8] Bôcher, Maxime, Introduction to Higher Algebra, p. 33.

Now let us consider Kelley's equation for the coefficient of multiple correlation[4] with a slight change in notation to be consistent with the above,

$$R_{n.12--(n-1)} = \sqrt{1 - \frac{R}{R_{nn}}} \tag{13}$$

Obviously

$$R_{nn} = \prod_{1}^{n-1} \gamma_{ii} \tag{14}$$

So that (13) becomes simply

$$R_{n.12--(n-1)} = \sqrt{1 - \gamma_{nn}} \tag{15}$$

But from (9) we have

$$\gamma_{nn} = \alpha_{nn} - \sum_{2}^{n} \beta_{in} \tag{16}$$

hence $R_{n.12--(n-1)} = \sqrt{1 - (\alpha_{nn} - \sum_{2}^{n} \beta_{in})}$

But $\alpha_{nn} = 1$ therefore we get

$$R_{n.12--(n-1)} = \sqrt{\sum_{2}^{n} \beta_{in}} \tag{17}$$

In other words $R_{n.12--}$ is simply the square root of the last product summation.

From (17) it is obvious that the solution for the coefficient of multiple correlation is considerably shorter than the standard Doolittle solution for regression coefficients. All of the back solution work is eliminated, as is also the calculation of the last reciprocal.

The only caution needed with respect to the order of the variables is that the dependent variable shall be the nth variable.

The usual summation check method may be employed exactly as in the solution for regression coefficients.

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[4] Kelley, T. L., Statistical Method, p. 301, eq. 275.

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