A POSTULATE FOR OBSERVATIONS

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When measurements are made by a given observer using a particular instrument, if the mean or expected result of the measurements is not supposed to be equal to the true value of the quantity measured the difference is considered to be an error, personal or instrumental or both. A correction is therefore applied to any such measurement in order to remove the discrepancy. In other words a given combination of observer and instrument is not considered to give correct or balanced measurements until this discrepancy is removed. Also as between two instruments or observers, both giving balanced measurements after the application of known corrections, preference is given to the one which shows the smaller variations between different measurements of the same quantity.

In selecting the formula to be used to determine, from the results of a series of measurements involving certain unknown quantities, the best measures of those quantities we are in a position similar to that of an observer desiring to make a certain measurement and selecting the best available instrument for the purpose. Such an observer would, in the first place, require that the instrument should give balanced measurements and would, in the second place, among a number of such instruments select the one showing the smallest standard deviation. This suggests the following definition and postulate.

Definition A balanced measure of a quantity is one of which the mean or expected value is equal to the true value of the quantity measured.

Postulate Of two or more balanced measures of a quantity the best measure is the one which has the smallest standard deviation.

Repeated Measurements

When we have a number of different results

$$a_1,\ a_2,\ a_3 \cdot\cdot\cdot\cdot\cdot a_n$$

of balanced measurements of the same quantity a, then any function of the form

$$\frac{\ell_1 a_1 + \ell_2 a_2 + \cdots + \ell_n a_n}{\ell_1 + \ell_2 + \cdots \ell_n}$$

will also be a balanced measure of the quantity. If the standard deviations of the individual measures are respectively

$$\sigma_1,\ \sigma_2 \cdot\cdot\cdot\cdot \sigma_n,$$

the square of the standard deviation of the function will be

$$\frac{\ell_1^2\sigma_1^2 + \ell_2^2\sigma_2^2 + \cdots\cdot \ell_n^2\sigma_n^2}{(\ell_1 + \ell_2 + \cdots \ell_n)^2}$$

This is a minimum when $\ell_1\sigma_1^2 = \ell_2\sigma_2^2 = \cdots\cdot = \ell_n\sigma_n^2$ so that the best measure is the average of the individual measures each weighted inversely as the square of its standard deviation. If the individual measures have the same standard deviation this reduces to the ordinary arithmetical average.

Combinations of Observations.

In applying this postulate to the theory of combination of observations, suppose that there are n unknown quantities

$x_i\ (i = 1, 2, 3 \cdots n$ and that we have balanced measures of

$m\ (m > n)$ linear functions of these unknowns of the form

$$y_h = \sum_{i=1}^{i=n} a_{hi}\, x_i \quad (h = 1, 2, 3 \cdots m) \qquad (1)$$

For simplicity we shall assume that the functions have been so taken that the standard deviations of these measures are all equal. Then a linear function of these measures of the form.

$$\sum_{h=1}^{h=m} b_{hj} y_h = \sum_{h=1}^{h=m} \sum_{i=1}^{i=n} b_{hj} a_{hi} x_i \tag{2}$$

will be a balanced measure of x_j if

$$\sum_{h=1}^{h=m} b_{hj} a_{hi} = \delta_j^2 \tag{3}$$

where $\delta_j^2 = 1$ if $i = j$ and $\delta_j^2 = 0$ if $i \neq j$

It will be the best measure of that type if $\sum_{h=1}^{h=m} b_{hj}^2$ is a minimum subject to those conditions.

By the method of indeterminate coefficients we find that this occurs when we can write

$$b_{hj} = \sum_{k=1}^{k=n} t_{jk} a_{hk} \tag{4}$$

and the values of the n^2 coefficients t_{jk} are determined from the n^2 conditions (3). Then

$$x_j = \sum_{k=1}^{k=n} \sum_{h=1}^{h=m} t_{jk} a_{hk} y_h = \sum_{k=1}^{k=n} t_{jk} y_k \tag{5}$$

it we write y_k for $\sum_{h=1}^{h=m} a_{hk} y_h$ The value of each of the n unknowns x_j is thus expressed in terms of the n functions y_k or, in other words, may be determined from the n equations expressing y_k in terms of the n unknowns x_j These equations take the form, if we write

$$A_{jk} \text{ for} \sum_{h=1}^{h=m} a_{hj} a_{hk}$$

$$\sum_{j=1}^{j=n} A_{jk} x_j = y_k \tag{6}$$

It will be noted that although no assumption regarding the law of error, other than that of balance, has been made the equations deduced are the same as those derived, in the ordinary theory of least squares, from the assumption of the normal exponential law.

Measurement of Probabilities.

Where the quantity measured is a probability and the measure is to be determined from the observed result of a finite number of trials we know that, if the probability is p the number of trials n and the number of occurrences of the particular result r then the expected value of r is np. Consequently r/n is a balanced measure of p . The measure usually associated with Bayes' theorem, namely $\frac{r+1}{n+2}$ is not a balanced measure. Its mean value is $p + \frac{1-2p}{n+2}$ which is not equal to p unless p happens to be equal to ½.

For this case a different postulate might consistently with the general methods of science, have been proposed as follows. *That hypothesis is to be adopted which makes the compound probability of the hypothesis and the observed facts a maximum.* If then we considered one value of the probability as likely as another this would mean selecting the value of p which would make $p^r(1-p)^{n-r}$ a maximum. This would have given r/n as before.

Frequency Distributions

The notation on the subject of moments is so unsettled that it appears to be necessary for each writer to specify the notation adopted. In this paper the rth moment about the origin in a finite sample will be written m_r and the corresponding moment about the mean value will be designated by μ_r. The moments in the population from which the sample is drawn will be written $\bar{m}_r$ and $\bar{\mu}_r$ respectively.

In this connection an important consideration arises from the fact that balanced measures are not always consistent under ordinary mathematical transformations. This happens because if y is a balanced measure of x then $f(y)$ is not necessarily a balanced measure of $f(x)$ Let $y = x + h$ and let mean values be indicated by prefixing $\bar{m}_1$ so that

$$\bar{m}_1(h) = \bar{m}_1(y) - x = 0.$$

Then since

$$f(y) = f(h) + hf'(x) + \frac{h^2}{2} f''(x) + etc$$

we have

$$\bar{m}_1\{f(y)\} = f(x) + \frac{\bar{m}_1(h^2)}{2} f''(x) + c = f(x) + \frac{\bar{\mu}_2}{2} f''(x) + etc$$

Ordinarily therefore unless $f(x)$ is a linear function of x or, if not, y is an exact measure of x, $\bar{m}_1 \{ f(y) \}$ will not be equal to $f(x)$.

A simple illustration of this fact arises in connection with the determination, from a sample, of a measure for $\bar{\mu}_2$. By ordinary transformations we have the well known formula $\bar{\mu}_2 = \bar{m}_2 - \bar{m}_1^2$ Also m_2 and m_1 are balanced measure of $\bar{m}_2$ and $\bar{m}_1$ respectively but m_1^2 is not a balanced measure of $\bar{m}_1^2$. We have in fact. $\bar{m}_1(m_1^2) = \bar{m}_1^2 + \frac{1}{n}\bar{\mu}_2$. Therefore,

$$\bar{m}_1(\mu_2) = \bar{m}_1(m_2 - m_1^2) = \bar{m}_2 - \bar{m}_1^2 - \frac{1}{n}\bar{\mu}_2 = (1 - \frac{1}{n})\bar{\mu}_2.$$

The balanced measure of $\bar{\mu}_2$ would therefore be $\frac{n}{n-1}\mu_2$ which is not formally consistent with the balanced measures of

$\bar{m}_1$ and $\bar{m}_2$ in the light of the equation $\bar{\mu}_2 = \bar{m}_2 - \bar{m}_1^2$. By a similar line of reasoning as shown by Thiele, we obtain $\frac{n^2}{(n-1)(n-2)}\mu_3$ as a balanced measure of $\bar{\mu}_3$ and by Tschuprow's modification of Thiele's analysis

$$\frac{n}{(n-1)(n-2)(n-3)}\left\{(n^2-2n+3)\,\mu_4 - 3(2u-3)\mu_2^2\right\}$$

as a balanced measure of $\bar{\mu}_4$. Here however we are faced with the further difficulty that while this is a balanced measure its standard deviation for small values of n is so great that possible values of μ_4 and μ_2 would result in negative values of $\bar{\mu}_4$ whereas in any real frequency distribution not only must $\bar{\mu}_4$ be positive but $(\bar{\mu}_2\,\bar{\mu}_4 - \bar{\mu}_3^2 - \bar{\mu}_2^3)$ which is the mean value of $\frac{(x_1-x_2)^2(x_2-x_3)^2(x_3-x_1)^2}{6}$ must also be positive.

If, therefore, we wish to derive a value of $\bar{\mu}_4$ certainly satisfying this condition we must determine the average value of

$\frac{(x_1-x_2)^2(x_2-x_3)^2(x_3-x_1)^2}{6}$ for all combinations of three values from the sample of n and use it as a balanced measure of

$(\bar{\mu}_2\bar{\mu}_4 - \bar{\mu}_3^2 - \bar{\mu}_2^3)$ This average value is found to be $\frac{n^2}{(n-1)(n-2)}(\mu_2\,\mu_4 - \mu_3^2 - \mu_2^3)$.

The analysis is as follows.

$$\bar{m}_1\left\{\frac{(x_1-x_2)^2(x_2-x_3)^2(x_3-x_1)^2}{6}\right\}$$

$$= \bar{m}_1\{x_1^4x_2^2 + 2\,x_1^3x_2^2\,x_3 - x_1^4x_2\,x_3 - x_1^3x_2^3 - x_1^2x_2^2x_3^2\}$$

$$= \bar{m}_4\,\bar{m}_2 + 2\,\bar{m}_3\bar{m}_2\,\bar{m}_1 - \bar{m}_4\,\bar{m}_1^2 - \bar{m}_3^2 - \bar{m}_2^3$$

$$= \bar{\mu}_2\,\bar{\mu}_4 - \bar{\mu}_3^2 - \bar{\mu}_2^3$$

If repetitions were allowed in the finite sample the average value would be the corresponding expression in moments of the sample, namely, $\mu_2\mu_4 - \mu_3^2 - \mu_2^3$ But since the expression vanishes if two or more of the values of x involved are equal the exclusion of repetitions reduces the total number of permutations three at a time from n^3 to $n(n-1)(n-2)$ without reducing the sum of the values. The average is there-increased to

$$\frac{n^2}{(n-1)(n-2)}\left(\mu_2\mu_4 - \mu_3^2 - \mu_2^3\right)$$

The second moment $\bar{\mu}_2$ might have been similarly derived as the mean value of $\frac{(x_1 - x_2)^2}{2}$ and the third moment $\bar{\mu}_3$ as the mean value of

$$\frac{(2x_1 - x_2 - x_3)(2x_2 - x_3 - x_1)(2x_3 - x_1 - x_2)}{6}$$

In this latter case the expressions averaged do not vanish when two only of the value of x are equal but they cancel one another in pairs so that their sum vanishes.

We have thus as working approximations

$$\bar{\mu}_2 = \frac{n}{n-1}\mu_2$$

$$\bar{\mu}_3 = \frac{n^2}{(n-1)(n-2)}\mu_3$$

$$\bar{\mu}_2\bar{\mu}_4 - \bar{\mu}_3^2 - \bar{\mu}_2^3 = \frac{n^2}{(n-1)(n-2)}(\mu_2\mu_4 - \mu_3^2 - \mu_2^3)$$

The net result of this investigation of the application of balanced measures as presumptive values of moments in frequency distributions seems to be that, in view of the formal inconsistencies involved, it is necessary to carefully select the functions to which such measures are applied. The functions considered above are suggested as well adopted for this purpose and as probably sufficient for all practical purposes. If they are adopted as fundamental the resulting approximations for the Pearson constants $\beta_1 = \mu_3^2/\mu_2^3$ and $\beta_2 = \mu_4/\mu_2^2$ are

$$\bar{\beta}_1 = \bar{\mu}_3^2/\bar{\mu}_2^3 = \frac{n(n-1)}{(n-2)^2}\,\mu_3^2/\mu_2^3 = \frac{n(n-1)}{(n-2)^2}\,\beta_1 \qquad \text{and}$$

$$\bar{\beta}_2 - \bar{\beta}_1 - 1 = \frac{\bar{\mu}_2\bar{\mu}_4 - \bar{\mu}_3^2 - \bar{\mu}_2^3}{\bar{\mu}_2^3} = \frac{(n-1)^2}{n(n-2)}\,\frac{\mu_2\mu_4 - \mu_3^2 - \mu_2^3}{\mu_2^3}$$

$$= \frac{(n-1)^2}{n(n-2)}\,(\beta_2 - \beta_1 - 1).$$

It will be noted that the coefficient in the latter equation is very nearly unity for even moderate values of n.