INVARIANTS AND COVARIANTS OF CERTAIN FREQUENCY CURVES

By

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Introduction. After the most convenient type of equation $y = f(x, a, b, c \ldots.)$ has been selected and the parameters a, b, c, ..., in the selected equation have been determined so that for a given set of values x_i $(i = 1, 2, \ldots.n)$, the computed values y_i $(i = 1, 2, \ldots n)$ agree as closely as possible or as closely as is consistent with the observed values Y_i $(i = 1, 2, \ldots n)$, it may be desirable to make one or more of the transformations: (1) move the origin, (2) use a different scale (unit of measure), (3) change the total frequency.

This paper discusses certain invariants and covariants of the above transformations which were noted in developing the general theory for the Pearson Curves of frequency.

1. *Change of Origin.* Instead of considering the diff. eq.,

$$\frac{dy}{dx} = \frac{y(x-P)}{b_2 x^2 + b_1 x + b_0} \tag{1}$$

which is the diff. eq. from which the Pearson curves are derived, we take the more general diff. eq.,

$$\frac{dy}{dx} = \frac{y(x-P)}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0}. \tag{2}$$

Equation (1) is a special case of equation (2).

Make the following substitutions:

$$\left\{\begin{array}{ll} x = X + P, & b_n = B_n, \\ & nPb_n + b_{n-1} = B_{n-1}, \\ \frac{n(n-1)}{2!}P^2 b_n + (n-1)P b_{n-1} + & b_{n-2} = B_{n-2}, \\ \multicolumn{2}{c}{\cdots\cdots\cdots\cdots} \\ P^n b_n + P^{n-1} b_{n-1} + \cdots\cdots & + b_0 = B_0, \end{array}\right. \tag{3}$$

and on simplifying we obtain,

$$(4)\qquad \frac{dy}{dx} = \frac{yX}{B_n X^n + B_{n-1} X^{n-1} + \cdots + B_1 X + B_0} = \frac{yX}{F(X)}.$$

If we now write

$$(5)\qquad X = x - P$$

we have:

$$(6)\qquad \frac{dy}{dx} = \frac{y(x-P)}{B_n (x-P)^n + B_{n-1} (x-P)^{n-1} + \cdots + B_1 (x-P) + B_0}.$$

The solutions of equations (4) and (6) can be written in the form

$$(7)\qquad y = G(X) = G(x-P),$$

where P is the mode as will be observed from the diff. eq. In other words the frequency function is a function of $(x-P)$ when it is written in the form of eq. (7). Therefore if we change the origin of x by writing $x' = x - h$ all of the constants of the frequency curve will remain unchanged if at the same time P be subjected to the transformation $P' = P - h$.

2. *Change of Total Frequency.* Let C_0 be the constant of integration when the area under the curve is unity and when the argument is $X = x - P$; K_0 the constant of integration when the argument is $X = x - P$ for an arbitrary area under the curve; and N the total frequency. Now when the total frequency is changed the area under the curve is changed, hence from the above definitions

$$(8)\qquad K_0 = N C_0 .$$

Therefore if the total frequency be N and it is desired to write the equation of the frequency function for a total frequency of $\overline{N}$ then write $\overline{K_0}$ for K_0 where

$$(9)\qquad \overline{K_0} = (K_0 / N) + \overline{N}$$

and leave all of the remaining constants unchanged.

It should be emphasized that in leaving the remaining constants unchanged we assume that the distribution of the new sample or the universe obeys the same law as the old sample. Occasionally one sees the statement in works on probability and statistics in connection with the Theory of Errors that as the number of observations is increased indefinitely, the arithmetic mean tends to the true value of a distribution. This statement is based upon the tacit assumption that an observation less than the true value (most probable) is as likely to occur as an observation greater than the true value. If we make this assumption we will always (if the number of observations be sufficiently large) ultimately obtain a symmetrical frequency curve (the A. M. coincides with the axis of symmetry) and this assumption contradicts the assumption that the distribution of the new sample obeys the same law as the old sample (except the old sample itself be symmetrically distributed).

3. *Change of Scale.* We are now ready to consider the behavior of the constants when the unit of measure is changed. Perhaps it is well to point out here that quite often it is desirable to change the unit from months to years, from feet to yards, from pounds to grams, etc. The behavior of the constants under a change of scale is not as easily arrived at as for the changes of the origin and total frequency.

The behavior of B_n where B_n is the coefficient of the highest power of X in $F(X)$ of the differential equation, $\frac{dy}{dX} = \frac{yX}{F(X)}$, will first be obtained.

Elderton[1] uses moments to determine the constants of a frequency curve. Thorkelsson[2] and Fisher[3] have used Thiele's semi-

[1] W. Palin Elderton, "Frequency Curves and Correlation", Second Edition 1927, London.

[2] Thorkell Thorkelsson, "Frequency Curves Determined by Semi-Invariants" (Visindafelag Islendinga IX) Reykjavik Rikisprentsmidjan Gutenberg.—MCMXXXI.

[3] Arne Fisher, "Frequency Curves", Translated by E. A. Vigfusson American Edition. 1922, The Macmillan Co.

invariants for this purpose. Semi-invariants have an advantage over moments in that the values of the higher semi-invariants do not change when the origin is changed. Moreover Fisher (pp. 12-16, loc. cit.) has pointed out how the semi-invariants behave when the unit is changed, viz:

$$\lambda_1(ax+c) = a\,\lambda_1(x)+c$$

$$\lambda_i(ax+c) = a^i\,\lambda_i(x) \quad \text{for } i>1.$$

Referring to equation (2) let P_0 be the value of P when the origin is at the arithmetic mean, and let b_n', b_{n-1}', ... b_1', and b_0' be the values of b_n, b_{n-1}, ... b_1, and b_0 when the origin is at the arithmetic mean. Now Thorkelsson (loc. cit.) has pointed out that when his method is used for computing the constants of the curve there will be only one equation involving P_0 and only one equation involving b_0'. Moreover the coefficients of the (b')'s and the constant terms of the remaining equations will be of constant weight.

Below is an example of the equations obtained when Thorkelsson's method is used to compute the constants:

$$(10)\quad \begin{cases} -P_0 + b_1' + 3\lambda_2 b_3' = 0 \\ \lambda_2 + b_0' + 3\lambda_2 b_2' + 4\lambda_3 b_3' = 0 \\ \lambda_3 + 2\lambda_2 b_1' + 4\lambda_3 b_2' + (5\lambda_4 + 12\lambda_2^2) b_3' = 0 \\ \lambda_4 + 3\lambda_3 b_1' + (5\lambda_4 + 6\lambda_2^2) b_2' + (6\lambda_5 + 45\lambda_2\lambda_3) b_3' = 0 \\ \lambda_5 + 4\lambda_4 b_1' + (6\lambda_5 + 24\lambda_2\lambda_3) b_2' + (7\lambda_6 + 72\lambda_2\lambda_4 + 54\lambda_3^2 + 24\lambda_2^3) b_3' = 0 \end{cases}$$

Note that only the first of the above equations involves P_0 and only the second involves b_0'.

Since the coefficients are of constant weight they are *invariants*[4] of *index* w where w is the weight of the coefficient when x is subjected to the transformation $x' = ax + c$.

Suppose that we now consider the general case where $F(X)$ is of degree n. Hence, in general, equations (10) will consist of $n+2$ equations in $n+2$ unknowns; the unknowns being P_0, b_0', b_1', ... b_n'. Disregard the two equations which involve P_0 and b_0' then there remain n equations in n unknowns. Observe that the weights of the coefficients of the b_i' form an A.P. whether taken by rows or by columns. Also the weights of the constant terms form the same A.P. as the columns.

We now state the

Lemma: If all of the elements of a determinant are covariants and the weights (indices) of the elements of every row form an A.P. and of every column form an A.P. then when the determinant is expanded every term is of constant weight (index).

Proof: Let the A.P. formed by the weights of the elements of the rows be $w_{ri} = a_r + (i-1)\delta \quad \begin{cases} r = 1, 2, \cdots, n \\ i = 1, 2, \cdots, n. \end{cases}$

Then the weights of the elements can be displayed as follows:

$$\begin{matrix} a_1 & a_1+\delta & a_1+2\delta & a_1+3\delta & \cdots & a_1+(n-1)\delta \\ a_2 & a_2+\delta & a_2+2\delta & a_2+3\delta & \cdots & a_2+(n-1)\delta \\ a_3 & a_3+\delta & a_3+2\delta & a_3+3\delta & \cdots & a_3+(n-1)\delta \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & a_n+\delta & a_n+2\delta & a_n+3\delta & \cdots & a_n+(n-1)\delta \end{matrix}$$

(It should be emphasized that the above is not the determinant mentioned in the statement of the Lemma but the elements of the above array represent the weights of the elements of the determi-

[4] L. E. Dickson. "Modern Algebraic Theories", Benj. H. Sanborn & Co., 1926; Chicago. Chapter I.

nant). Now since by hypothesis the elements of every column have such weights that the weights form an A.P. then a_1, a_2, $a_3, \ldots a_n$ must form an A.P. Let this A.P. be $a_1 + (j-1)\bar{\delta}$. Making use of this notation the weights of the elements of the determinant can be displayed as follows:

$$\begin{array}{llll}
a_1 & a_1+\delta & \cdots & a_1+(n-1)\delta \\
a_1+\bar{\delta} & a_1+\delta+\bar{\delta} & \cdots & a_1+(n-1)\delta+\bar{\delta} \\
\cdots & \cdots & \cdots & \cdots \\
a_1+(n-1)\bar{\delta} & a_1+\delta+(n-1)\bar{\delta} & \cdots & a_1+(n-1)\delta+(n-1)\bar{\delta}
\end{array}$$

Hence the weight of the element in the i^{th} row and the j^{th} column is $a_1+(i-1)\delta+(j-1)\bar{\delta}$. Along the principal diagonal of the determinant $i=j$. Therefore when the determinant is expanded the weight of the term consisting of the elements of the principal diagonal is the sum of the A.P. $w_i = a_1 + (i-1)(\delta+\bar{\delta})$ or

$$\sum_{i=1}^{n} w_i = \frac{n}{2} \cdot \left[2a_1 + (n-1)(\delta+\bar{\delta})\right] = W.$$

Every term in the expansion is of weight W because each term consists of one element from each row and one element from each column and hence the weight is equal to the sum of two series, each being an A.P., plus the weight of the term in the upper left corner.

THEOREM: If all of the coefficients and the "constant" terms of a system of n linear equations in n unknowns are covariants of such respective weights (indices) that the weights (indices) of the elements of every row of the matrix of the system of equations form an A.P. and of the elements of every column of the augmented matrix form an A.P. then the solutions are covariants whose weights (indices) form an A.P. whose common difference is of the same magnitude but of opposite sign to the common difference in the A.P. of the weights (indices) of the elements of the rows.

Proof: By Cramer's rule the solutions are

$$z_i = \frac{D_i}{\Delta} \quad \text{where } \Delta = \begin{vmatrix} K_{11} & \cdots & K_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ K_{n1} & \cdots & K_{nn} \end{vmatrix} \quad \text{and where}$$

D_i is the n-rowed determinant obtained from Δ by replacing the elements of the i^{th} column by the "constant" terms of the system. Let the weight (index) of the element in the i^{th} row and j^{th} column of Δ be $a_1 + (i-1)\delta + (j-1)\bar{\delta}$. Also let the A.P. formed by the weights (indices) of the elements of the r^{th} row of Δ be $w_{ri} = a_r + (i-1)\delta$, hence in particular the A.P. of the weights (indices) of the elements of the first row are $a_1 + (i-1)\delta$. Further let the A.P. formed by the elements of the column of constant terms of the augmented matrix be $w_{ci} = a_c + (i-1)\bar{\delta}$. By the lemma just established we see that when Δ is expanded all of the terms of the expansion will be of the same weight (index) W. Hence Δ is of weight (index) W. Also since the A.P. of the weights (indices) of the column of constant terms is $w_{ci} = a_c + (i-1)\bar{\delta}$ then the weight (index) of each term in the expansion of D_i will be $-[a_1 + (i-1)\delta] + a_c$ different from the weight (index) of each term in the expansion of W. Hence the weight (index) of D_i is $W - [a_1 + (i-1)\delta] + a_c$. Therefore the weight (index) of z_i is $W - [a_1 + (i-1)\delta] + a_c - W = a_c - a_1 + (i-1)(-\delta)$, and the theorem is established.

Applying the above Theorem and observing that $\delta = 1$ in equations (10) we obtain the result:

$$\text{weight of } b_n' = 3 - 2 + (n-1)(-1) = 2 - n.$$

Since $B_n = b_n'$ we have the result that when x is subjected to the transformation $x' = ax + c$ then $B_n' = a^{2-n} B_n$. Or in other words B_n is an *invariant* of *index* $2-n$ under the transformation

$$x' = ax + c.$$

Now we turn to the consideration of P_0. Here we have

$n+1$ equations in $n+1$ unknowns and the augmented matrix has elements of the following weights (* means that an element is lacking):

$$\begin{array}{cccccccc}
P_0 & 0 & * & 2 & 3 \cdots\cdots & n-1 & * \\
* & 2 & 3 & 4 & 5 \cdots\cdots & n+1 & 3 \\
* & 3 & 4 & 5 & 6 \cdots\cdots & n+2 & 4 \\
* & 4 & 5 & 6 & 7 \cdots\cdots & n+3 & 5 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
* & (n+1) & (n+2) & (n+3) & (n+4) \cdots\cdots & 2n & (n+2)
\end{array}$$

Now P_0 is the quotient of two determinants formed from the above matrix and if these two determinants be expanded in terms of the minors of the first column we see that the weight of $(W+1)-W=1$. That is P_0 is of weight 1 regardless of the degree of $F(X)$. Therefore P_0 is an invariant of index 1.

Next considering b_0' the augmented matrix has the same elements as for P_0 except that the first row is now:

$$b_0' \quad * \quad 2 \quad 3 \quad 4 \cdots\cdots \quad n-1 \quad 2$$

Following the same procedure we see that the weight of $b_0' = (W+2)-W = 2$. Therefore b_0' is an invariant of index 2.

We can look upon equations (3) as a transformation. We can reverse this transformation by solving for the b_i in terms of the B_i. Also, by moving the origin to the A.M. equations (3) may be written:

$$(11)\quad \begin{cases}
B_n = b_n' \\
B_{n-1} = {}_nC_n P_0 b_n' + b_{n-1} \\
B_{n-2} = {}_nC_{n-1} P_0^2 b_n' + {}_{n-1}C_{n-1} P_0 b_{n-1}' + b_{n-2} \\
\cdots\cdots\cdots \\
B_{n-r} = {}_nC_{n-r+1} P_0^r b_n' + {}_{n-1}C_{n-r+1} P_0^{r-1} b_{n-1}' + \cdots + b_{n-r}' \\
\cdots\cdots\cdots \\
B_0 = P_0^n b_n' + P_0^{n-1} b_{n-1}' + \cdots\cdots + P_0 b_1' + b_0'.
\end{cases}$$

In equations (11) $P_0', b_n', b_{n-1}', \ldots b_0'$ are the values of P_0, b_n, b_{n-1} $\ldots$, b_0 when the origin is at the A.M.

Note that the right hand numbers of equations (11) are isobaric and that B_0 is of weight 2; B_1 of weight 1; B_2 of weight 0; and in general B_i is of weight $2-i$.

Now let $q_0 = \dfrac{b_0'}{b_n'}$, $q_1 = \dfrac{b_1'}{b_n'}$, in general $q_i = \dfrac{b_i'}{b_n'}$; hence

$q_n = 1$. Therefore when the $q's$ are computed we note that q_0 is of weight n; q_1 is of weight $n-1$ and in general q_i is of weight $(2-i)-(2-n) = n-i$. Since q_0 is the product of all the roots, q_1 the sum of the products taken $(n-1)$ at a time and so on and q_{n-1} is the sum of all the roots (due consideration being taken of the signs) it follows that all the roots of $F(X)$ are invariants of index 1 under the transformation $x' = ax + c$.

Now if equation (4) be solved in the form of equation (7) then it can be seen by actual substitution of the indices of B and the roots of $F(X)$ which are involved in the constants that the exponents K and t of factors of the form $\left(1 - \frac{x-P}{r_1}\right)^K$ and

$$\frac{e^{t_1 t_2 \arctan \frac{x-P+r_3}{r_0}}}{\left[1 + \left(\frac{x-P+r_3}{r_0}\right)^2\right]^{t_1/2}}$$

are invariants of index zero. The factor $\left(1 - \frac{x-P}{r_1}\right)^{K_1}$ occurs for a real root r_1 of $F(X)$ and the factor

$$\frac{e^{t_1 t_2 \arctan \frac{x-P+r_3}{r_0}}}{\left[1 + \left(\frac{x-P+r_3}{r_0}\right)^2\right]^{t_1/2}}$$

occurs for a pair of conjugate complex roots of $F(X)$. The fact that the exponents K are invariants of index zero will be generalized for the case where complex roots do not occur and where no real root is repeated.

If complex roots do not occur the differential equation

$\frac{dy}{dX} = \frac{y\,X}{F(X)}$ can be written $\frac{dy}{y} = \frac{X\,dX}{F(X)} = \frac{1}{B_n}\left[\frac{m_1}{X+\lambda_1} + \cdots + \frac{m_n}{X+\lambda_n}\right] dX$

where in separating $\frac{X}{F(X)}$ into partial fractions and equating coefficients of like powers of X we obtain n equations in n unknowns and since the roots are all of weight 1 the weights of the augmented matrix will be (the unknowns of the system are the m_i):

$$\begin{matrix} n-1 & n-1 & n-1 & \cdots & n-1 & * \\ n-2 & n-2 & n-2 & \cdots & n-2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & * \\ 0 & 0 & 0 & \cdots & 0 & * \end{matrix}$$

Applying the Lemma we see that the m_i are all of the same weight (since $\delta = 0$). Expanding the determinant in the numerator by minors we see that the m_i are of weight $n-2$. Since B_n is of weight $2-n$, we have $\frac{m_i}{B_n} = K_i$ is of weight zero. Therefore the K_i are invariants of index zero under the transformation

$$x' = a\,x + c\,.$$

We have now considered all of the constants of the curves except the constant of integration. Let the solutions be written in the form: $y = C_0 \cdot G(X)$.

Now it is possible to write $G(X)$ in such a form that $G(X)$ is a *covariant* of index zero under the transformation $X' = a\,X$. In the case of real roots this is accomplished by dividing both the numerator and the denominator of each partial fraction by the root involved in the fraction before the integration is performed. Partial fractions which involve complex roots can be similarly treated. This is the way Pearson actually treated his Types I, II, III, IV, and VII curves although he did not deal with his Types V and VI curves in this manner.

After we have our solution in the form which makes $G(X)$ a covariant of index zero then if we write X' for $a\,X$ the total frequency between nX' and $(n+1)X'$ will be the same as the total frequency between naX and $(n+1)aX$. Therefore y is a covariant of index (-1). Hence C_0 is an *invariant* of index (-1).

An example will now be given. Take the equation to which Elderton (loc. cit.) fits a Type I curve on pages 54-59. He has used a unit of 5 years. Suppose we wish to change to a unit of 1 year. Then the constants a_1 and a_2 being the roots of $F(X)$ are invariants of index 1 and are each multiplied by 5 and become 9.98190 and 67.63640 respectively. Since m_1 and m_2 are invariants of index zero they remain unchanged and are as he gives them viz. .409833 and 2.776978. The constant of integration being an invariant of index — 1 it is divided by 5 and becomes 29.892. The equation with a unit of 1 year becomes (See top of page 58):

$$y = 28.892 \left\{1 + \frac{x'}{9.98190}\right\}^{.409833} \cdot \left\{1 - \frac{x'}{67.63640}\right\}^{2.776978}.$$

Suppose that now we wish to move the origin to age 26.75942. Then the above equation becomes:

$$y = 28.892 \left\{1 + \frac{x'' - 26.75942}{9.98190}\right\}^{.409833} \cdot \left\{1 - \frac{x'' - 26.75942}{67.63640}\right\}^{2.776978}.$$

Finally suppose we wish to change to a total frequency of 2000 instead of 1000 as in the given sample. Then the equation becomes:

$$y'' = 59.784 \left\{1 + \frac{x'' - 26.75942}{9.98190}\right\}^{.409833} \cdot \left\{1 - \frac{x'' - 26.75942}{67.63640}\right\}^{2.776978}$$

4. *Conclusion: Benefits of this Information.* If the diff. eq. (2) be written in the form (4) by means of the transformation (3) then the integration is more easily accomplished. That is to say: in general the solution in the form of eq. (7) is more readily obtained from (4) than some equivalent form of solution would be from (2). Thus a solution in the form of eq. (7) is not only

more easily obtained but also lends itself readily to a change of origin.

Each type of Pearson's Curves may be written in a number of ways. The numerical example given above shows the convenience and advantage of writing a solution so that the origin is at the mode, $G(X)$ is a covariant of index zero, y a covariant of index (—1) and the constant of integration an invariant of index (—1).

Regardless of what form is selected for writing a solution the solution will be a covariant and the constants will be invariants, but not necessarily of the indices mentioned above. A knowledge of these invariants will save much labor if it is desired to make a change in the unit of measure.

Similar laws of transformation can be worked out for (1) solutions of the diff. eq. $\frac{dy}{dx} = \frac{y f(x)}{F(x)}$ where both $f(x)$ and $F(x)$ are integral rational functions of x and (2) for the Gram-Charlier Types A and B series. In the first case we obtain the same result as outlined above for the simpler diff. eq. $\frac{dy}{dx} = \frac{y x}{F(x)}$; that is the solution may be written in the form $y = C_0 \cdot G(X)$ where $G(X)$ is a covariant of index zero, y a covariant of index (—1) and C_0 is an invariant of index (—1). In the case of the Type A series the coefficient of each term is an invariant of index zero.

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