THE ADEQUACY OF "STUDENT'S" CRITERION OF DEVIATIONS IN SMALL SAMPLE MEANS*

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INTRODUCTION

The origin of the movement toward precise evaluation of probabilities based on the statistics of small samples would generally be located by practical statisticians in the work of "Student" (1908). The problem he considered is of such importance, not only from the historical aspect, but also from a consideration of the elements of statistical interpretation, that we wish to return to an analysis of the adequacy of his solution. "Student" was concerned with the problem of determining the significance to be attached to the deviation of the mean, $\bar{x}$, of a small sample from a probable (or possible) supply† mean, m, when the dispersal of variates in the supply is unknown. The solution he suggested was based upon derivation of the probability integral of the quantity

$$(1) \qquad z = \frac{\bar{x} - m}{s},$$

where s is the standard deviation of the sample. He found the distribution of z to be given by the equation,

$$(2) \qquad df = k_z \left(1 + z^2\right)^{-\frac{N}{2}} dz.$$

In 1915, Fisher indicated that "Student's" partly intuitive derivation was sound, and in 1925 he returned to a more complete exposition of the accuracy of the solution, at the same time widely

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† Following Wicksell (e.g. Biometrika 25, p. 121), we shall use the term "supply" in place of "population."

extending its application. Fisher at that time changed the variable to $t = z\sqrt{n}$, where n is the number of "degrees of freedom" involved in estimating σ (the supply standard deviation) from s. "Student" (1925) cooperated in this extension by preparing tables of the probability integral of t, using n in place of N as the parameter. Since the integrals are of essentially identical curves, and z will prove somewhat more adaptable in the present study, we will conduct the discussion of the problem in terms of z. All conclusions reached will apply with equal validity, of course, when t is used in place of z.

"Student" illustrated the usefulness of his z distribution by considering the x values as a set of differences (between experimental and control pairs, say), thus logically making m equal to zero. He then found the probability that the resulting z would be exceeded solely through random sampling errors. Although it is not by any means clear from "Student's" original memoir that he so intended, the custom has grown of considering this probability as that which might be expected for the deviation of $\bar{x}$ from m if a knowledge of σ were available. Is such a transfer of the probability really acceptable? The usefulness of the z (or t) test depends entirely on the answer to this question.

SIGNIFICANT DEVIATIONS

In a supply of variates, x, whose frequency distribution accords with the "normal" curve and whose total frequency approaches infinity, let the mean be m and the standard deviation σ. Assume a large number of samples, each of total frequency N, to be drawn independently and at random from this supply. Let the mean and standard deviation of each sample be designated as $\bar{x}$ and s respectively. Then the probability that values of $\bar{x}$ will deviate from m by more than a certain amount may be determined exactly from the "normal" integral. Letting

(3) $$y = \frac{\bar{x} - m}{\sigma},$$

the distribution of y will be given by the equation

$$(4) \qquad df = k_y e^{-\frac{N}{2}y^2} dy,$$

a "normal" curve with mean at zero and standard deviation of $N^{-\frac{1}{2}}$. Values of y exceeding $1.96/\sqrt{N}$ will arise but 5 times in 100, and this value would be known therefore as the "5% level of significance." For N equal to 5, this level is .8765.

Let a single sample of 5 individuals, not known to be drawn from the above supply, be made available. It may be desired to test whether the mean, $\bar{x}'$, of this sample differs sufficiently from m to warrant the assumption, on the basis of the mean value alone, that the sample has not been drawn from the above supply. If $(\bar{x}' - m)/\sigma$ should exceed .8765, those depending on a 5% "level of significance" would decide that the sample is significantly different in the respect tested. However, y will exceed this level 5 times in 100. It must therefore be expected that up to 5% of samples like that designated by the prime above which are investigated by this procedure will be erroneously segregated as "differing significantly."

This maximum error of 5% is acceptable to most workers for two reasons:

(*i*) Some such error must be accepted in order to have a basis for differentiation, and 5% or less (generally less) erroneous segregation is sufficiently small to be regarded by many as an acceptable proportion of error;

(*ii*) The cases erroneously segregated in this manner are the most rational ones to be subjected to the error, since they deviate from m by the greatest amount.

In practical statistical problems wherein the significance of the deviation of a mean is to be tested, it is usually impossible to apply the above reasoning because of lack of precise knowledge of the value of σ. "Student's" test aimed to meet this deficiency by finding the integral of z already defined (equations 1 and 2).

Applying the probability integral of this variable, he reached his conclusions about the significance of z in the same way as has been indicated for the variable y.

THE CORRELATION BETWEEN $\bar{x}$ AND s.

In analyzing the adequacy of the procedure suggested by "Student," it seems fruitful to consider the correlation of $\bar{x}$ and s. Defining the latter in its original sense,

$$s = \sqrt{\Sigma (x - \bar{x})^2 / N}, \tag{5}$$

"Student" (unknowingly justifying Helmert's previous work) concluded the distribution of s is given by

$$df = k_s \, e^{-\frac{N}{2}\left(\frac{s}{\sigma}\right)^2} s^{N-2} \, ds. \tag{6}$$

This most important equation has not received the discussion it deserves. Tables of the probability integral of v, where

$$v = s/\sigma \tag{7}$$

would also be most helpful in small sample analysis, if for no other reason than to show the wide variation which must be expected in s for small values of N. An appreciation of this variation is much more pertinent to the adequate solution of the problem analyzed by "Student" than appears to have been realized. We accordingly include here the 2½% points* in v for a few values of N small.

2½% points for v

N	lower	upper
2	.02	1.58
3	.13	1.57
4	.22	1.53
5	.31	1.49

* By 2½% points we mean those points at which the ordinate truncates a tail whose area is 2½% of the total area of the curve.

It will be seen from these figures that, for N equal to 5, s will vary over the relatively very wide range of $.31\sigma$ to 1.49σ even when only the central 95% of cases are considered. Inasmuch as there is no correlation between $(\bar{x} - m)$ and s when sampling is made from a "normal" supply, the values to be expected for z in those samples where $(\bar{x} - m)$ is the same must vary widely solely through the influence of variation in s.

Expressing $(\bar{x} - m)$ and s in terms of σ as the unit of measurement, the simultaneous distribution we wish to analyze will become that of y and v. Since these variables are wholly independent (see Fisher, 1925), their simultaneous distribution will be given by the product of their separate probabilities, yielding

$$df = k_{y,v}\, e^{-\frac{N}{2}y^2}\, e^{-\frac{N}{2}v^2}\, v^{N-2}\, dy\, dv. \tag{8}$$

This surface is graphically portrayed in Figure 1 for the case when N equals 5. The few contours given are sufficient to indicate the general character of the distribution of frequency. Projection of the frequencies onto the two margins gives the univariate distributions drawn in the Figure.

If B and B' be taken as the 2½% points for the y distribution, then lines through them drawn perpendicular to the y axis will cut off in the extreme zones of the surface and in the tails of the y distribution those samples whose means deviate sufficiently from m to permit their segregation according to a "5% level of significance."

Since $$z = y/v,$$
the samples segregated by the 5% level in applying the z test must be bounded on one side (in each direction) by radial lines traversing this surface and passing through the point $(y = 0,\ v = 0)$. Let b be the value of the 2½% point for the z distribution. Then the cotangent of the angle of incidence to the y axis will in each case equal b, i.e. 1.3882 when N equals 5.

All samples given by points in the shaded areas, E and F (Figure 1), would be considered significantly deviating with respect to $\bar{x}$ according to customary interpretation of the z test. Those samples in the shaded areas, F and G, would be segregated by the y test. Only those samples in the cross-shaded regions, F, would be selected by both tests. For the situation under discussion, wherein the sampling is actually made from the one supply, no samples really deviate in $\bar{x}$ from m by an amount not logically to be ascribed to random sampling effects. For reasons given earlier in this discussion, however, the y segregates are all rationally made. Only the z segregates in the double-shaded area F may be designated as rational on the grounds given. Those in the single-shaded area E are irrationally selected; the segregation has been made because s is small, not because $(\bar{x} - m)$ is large.

THE CORRELATION BETWEEN $\bar{x}$ AND z.

An analagous geometric view may be presented by considering the correlation surface for y and z. To obtain the simultaneous distribution of these variables, the substitutions

$$v = y/z\,,$$
$$dv = -\frac{y}{z^2}\,dz\,,$$

may be made in equation (8), yielding:

$$df = k_{y,z}\, e^{-\frac{N}{2}y^2}\, y^{N-1}\, e^{-\frac{N}{2}\left(\frac{y}{z}\right)^2}\, z^{-N}\, dy \cdot dz \tag{9}$$

In slightly different form, Pearson (1931*a*) has given this expression and derived from it the equations for the correlation, regression and scedasticity of the surface in terms of N. He demonstrated that, although regression is rectilinear and $r_{\bar{x}z}$ is very high, the distribution of z for constant $\bar{x}$ is characterized by "excessive leptotosis and extreme skewness" for N small, with gradual approach to "normality" as N increases. Also, there is marked heteroscedasticity of these arrays.

It is a simple matter to truncate the (y, z) surface into volumes of frequency corresponding to the probability of occurrence of given deviates in y or z. This is graphically portrayed in Figure 2, where the surface is approximately represented for $N = 5$ and the planes of truncation, BCD and bCd, correspond to the 2.5% points, B and b respectively, for each variable. Since the frequency surface is radially symmetrical about the point $(y = 0,\ z = 0)$, only one quadrant need be lettered. 2.5% of the area of the "normal" y distribution lies in the minor segment bounded by the ordinate AB, and 2.5% of the "leptokurtic" z distribution lies in the minor segment bounded by the ordinate ab. Also, 2.5% of the total frequency of the correlation surface lies in the two minor volumes truncated by the vertical planes passing through AB and ab respectively. Only that proportion of frequency lying beyond *both* planes, *i.e.* in the area bCd, exceeds the given level for both variables simultaneously.

The corresponding frequency volumes in Figures 1 and 2 representing segregations by the y and z tests are as follows:

Figure 1	Figure 2
Zone E	Zone dCD
Zone F	Zone bCD
Zone G	Zone BCb

That the corresponding zones should not have the same relative areas in the two figures is in accordance with expectation, since the densities of frequency must vary widely within the zones and in different manners from one zone to another. Interpretation of the degrees of rational and irrational segregation by the z test must depend upon evaluation of the integrals defining the respective frequency volumes.

EVALUATION OF INTEGRALS

For the (y, v) surface, the frequency over each double-shaded zone E will be given by the expression

$$(10) \qquad \Delta f_1 = k_{y,v} \int_B^\infty e^{-\frac{N}{2}y^2} dy \int_0^{by} e^{-\frac{N}{2}v^2} v^{N-2} dv.$$

For the (y, z) surface, the corresponding frequency over the area, bCD, will be given by the expression

$$(11) \qquad \Delta f_2 = k_{y,z} \int_B^\infty e^{-\frac{N}{2}y^2} y^{N-1} dy \int_b^\infty e^{-\frac{N}{2}\left(\frac{y}{z}\right)^2} z^{-N} dz.$$

The constants, $k_{y,v}$ and $k_{y,z}$, prove to be identical in magnitude, and we shall therefore give the evaluation of the latter only.

Integrating from zero to infinity in both directions, one secures half the total frequency since the distribution appears equally and solely in the two quadrants of positive product.

$$\frac{1}{2k_{y,z}} = \int_0^\infty e^{-\frac{N}{2}y^2} y^{N-1} dy \int_0^\infty e^{-\frac{N}{2}\left(\frac{y}{z}\right)^2} \cdot z^{-N} dz$$

But

$$\int_0^\infty e^{-\frac{N}{2}\left(\frac{y}{z}\right)^2} z^{-N} dz = \frac{2^{\frac{N-3}{2}}}{N^{\frac{N-1}{2}} y^{N-1}} \int_0^\infty u^{\frac{N-3}{2}} e^{-u} du$$

$$= \frac{2^{\frac{N-3}{2}} \Gamma\left(\frac{N-1}{2}\right)}{N^{\frac{N-1}{2}} y^{N-1}}.$$

Therefore

$$\frac{1}{2k_{y,z}} = \frac{2^{\frac{N-3}{2}} \Gamma\left(\frac{N-1}{2}\right)}{N^{\frac{N-1}{2}}} \int_0^\infty e^{-\frac{N}{2}y^2} dy$$

$$= \frac{2^{\frac{N-4}{2}}\,\Gamma\left(\frac{N-1}{2}\right)\,\pi^{1/2}}{N^{\frac{N}{2}}}$$

and

(12) $$k_{y,z} = \frac{N^{\frac{N}{2}}}{2^{\frac{N-2}{2}}\,\Gamma\left(\frac{N-1}{2}\right)\,\pi^{1/2}}$$

It is pertinent to prove now that Δf_1 equals Δf_2.

Letting $$w = \frac{N}{2} v^2 = \frac{N}{2}\left(\frac{y}{z}\right)^2,$$

then $$dw = N v\, dv = -\frac{Ny^2}{z^3} dz.$$

Substituting in (10), we have,

$$\int_0^{\sigma y} e^{-\frac{N}{2}v^2} v^{N-2}\, dv = \frac{2^{\frac{N-3}{2}}}{N^{\frac{N-1}{2}}} \int_0^{\frac{Ny^2}{2\sigma^2}} e^{-w} w^{\frac{N-3}{2}} dw$$

Substituting in (11), we have,

$$\int_\sigma^\infty e^{-\frac{N}{2}\left(\frac{y}{z}\right)^2} z^{-N}\, dz = \frac{2^{\frac{N-3}{2}}}{N^{\frac{N-1}{2}}\, y^{N-1}} \int_0^{\frac{Ny^2}{2\sigma^2}} e^{-w} w^{\frac{N-3}{2}} dw.$$

Thus

(13) $$\Delta f_1 = \frac{N^{\frac{1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)\pi^{1/2}} \int_B^\infty e^{-\frac{N}{2}y^2} dy \int_0^{\frac{Ny^2}{2\sigma^2}} e^{-w} w^{\frac{N-3}{2}} dw = \Delta f_2 .$$

Noting that B equals $1.96/\sqrt{N}$, it would seem logical to conclude from the general form of equation (13) that Δf approaches a limit of .025 as N increases. We have not yet succeeded in proving this explicitly.

Numerical evaluation of the double integral for Δf presents difficulties. These may be overcome by applying a succession of reduction formulas to the series of single integrals in powers of y^2 obtained from the integration with respect to w. For example, when $N = 5$, $B = 0.8765$, $b = 1.3882$, and

$$\Delta f = \frac{\sqrt{5}}{\sqrt{2\pi}} \int_B^\infty e^{-\frac{5}{2}y^2} dy \int_0^{\frac{5y^2}{2b^2}} e^{-w}\, w\, dw$$

$$= \frac{\sqrt{5}}{\sqrt{2\pi}} \left[\int_B^\infty e^{-\frac{5}{2}y^2} dy - \frac{5}{2b^2} \int_B^\infty e^{-\frac{5}{2}y^2(1+\frac{1}{b^2})} y^2 dy - \int_B^\infty e^{-\frac{5}{2}y^2(1+\frac{1}{b^2})} dy \right]$$

$$= .025 - \frac{\sqrt{5}\; B\, e^{-\frac{5B^2}{2}(1+\frac{1}{b^2})}}{2\sqrt{2\pi}\;(b^2+1)} - \frac{\sqrt{5}}{\sqrt{2\pi}} \left[1 + \frac{1}{2(b^2+1)}\right] \int_B^\infty e^{-\frac{5}{2}y^2(1+\frac{1}{b^2})} dy$$

$$= .025 - .0072 - \frac{b(2b^2+3)}{2(b^2+1)^{3/2}\sqrt{2\pi}} \int_{\frac{B}{b}\sqrt{5b^2+5}}^\infty e^{-\frac{w^2}{2}} dw$$

$$= .0178 - .0074 = .0104 \;.$$

Values for the frequency volumes Δf (corresponding to the area bCD in Figure 2) are given as column (4) of Table I for the chosen values of N. The differences between these values and .025 provide the magnitudes of the frequency volumes corresponding to BCb and dCD. The latter volumes, which are necessarily equal, are given in column (5) of the same table. In columns (6) and (7) the values in columns (4) and (5) respectively are expressed as percentages of the limiting value, .025.

We have not succeeded as yet in expressing any of these proportional frequencies as simple equations in terms of N only. In Figure 3, however, a graph of the relationship is plotted, based on the data of Table I. The vertical scale on the left gives the proportional frequency beyond the two planes passing through C. By following the dotted lines to the scale on the right vertical margin, the percentage error ($100\ dCD/.025$) with which we are concerned may be read off directly.

TABLE I

Data for evaluation of volumes truncated by the planes passing through C (Fig. 1), for different sizes of sample, where C corresponds to the .025 points of y and z.

(1)	(2)	(3)	(4)	(5)	(6)	(7)
N	B	b	Volumes*		Volumes* as % of .025	
			bCD	$BCb=dCL$	bCD	$BCb=dCD$
3	1.1316	3.042	.0064	.0186	25.6	74.4
5	.8765	1.388	.0104	.0146	41.6	58.4
7	.7408	.999	.0126	.0124	50.4	49.6
9	.6533	.815	.0142	.0108	56.8	43.2
11	.5910	.705	.0151	.0099	60.4	39.6
13	.5436	.629	.0160	.0090	64.0	36.0
15	.5061	.573	.0166	.0084	66.4	33.6
17	.4754	.530	.0171	.0079	68.4	31.6
19	.4497	.495	.0176	.0074	70.4	29.6
21	.4277	.466	.0179	.0071	71.6	28.4
25	.3920	.421	.0185	.0065	74.0	26.0
29	.3640	.387	.0190	.0060	76.0	24.0
99	.1970	.201	.0216	.0034	86.4	13.6

PRACTICAL TESTS

In order to test the accuracy of the above deductions when applied to a supply which is grouped into fairly fine categories, two sampling studies were made. Samples of 5 individuals each were drawn in both cases. The first study dealt with a much used supply of two anthropometric measures which conform fairly well to the "normal" curve in their distributions. The second study

* Volumes (of frequency) follow the notation of Figure 2.

used as a supply a *theoretical* "normal" bivariate frequency surface, seriated into classes. These studies will be referred to as Series I and II.

Series I. From the table provided by MacDonell (1902) on the associated variation of stature (to the nearest inch) and length of the left middle finger (to the nearest millimeter) in 3000 British criminals, the measurements were transferred to 3000 numbered Denison metal-rim tags from which the cords had been removed. After thorough checking and mixing of these circular disks, samples of 5 tags each were drawn at random until the supply was exhausted. Unfortunately, three of these samples were erroneously returned to a receiving box before being copied, and the records of 597 samples only are available. For these, the statistics y and z were calculated for each variable, and frequency surfaces for joint occurrence of y and z were prepared in which the statistics for stature and finger length were first considered separately, then combined. After calculating the correlation coefficient, the frequencies of the opposite quadrants were added so as to provide the seriation without regard to the signs of y and z. The actual number of cases falling beyond the planes of truncation corresponding to the 2.5% points were then counted and the proportional frequencies tabled.

Series II. From the tables of the probability integral of the "normal" correlation surface prepared by Lee and others (see Pearson, 1931*b*) a correlation table of total frequency of 1000 approximately was prepared for the case where the correlation is .5, using .3 σ as the unit of classification in both directions. Modification of the fractional frequencies to the nearest whole number yielded a table in which N equalled 998, r equalled .5003 and the two standard deviations equalled .9914 (Sheppard's correction applied). Samples of 5 were drawn by working systematically through the tables of random numbers provided by Tippett (1927), 2043 samples being so secured. These samples were treated as in the case of Series I.

The actual correlation surfaces secured for the joint occurrence of y with v and y with z may be illustrated by scatter diagrams prepared from the data of Series II. These are given as Figures 4 and 5, the variates in the latter case being considered without regard to sign. Both conform very well indeed to the theoretical contour diagrams presented earlier (Figures 1 and 2).

For the correlation between y and z (signs *not* ignored), Pearson (1931*a*) has determined theoretically that, for N equal to 5, r should equal $+ .8862$. We find the following results for our two series:

Variable	Series I	Series II
1	.8744	.8797
2	.7883	.8869
1 + 2	.8270	.8832

For Series II the agreement with theory is splendid. The wider deviations from the theoretical value in Series I are probably due, in part, to the less pertectly "normal" nature of the supply distributions.

The inadequacy of the correlation coefficient as a descriptive measure of such a "non-normal" surface as that for y and z will be apparent at once from an inspection of figure 5. Discordance of the two variables increases rapidly as their values increase to such an extent that, for N equal to 5, values of z beyond the customary level of significance provide exceedingly poor bases of prognostication concerning the true significance of the deviation in the mean, despite the fairly high value of the correlation coefficient.

In Table II the frequencies beyond the chosen levels of significance for y and z, separately and jointly, are given for both series. The empirical frequencies are given in Roman type in the whole numbers, and as proportions in parentheses. The theoretical values are given in italics in the last column for comparison. The agreement is very good in every case, the deviation of observed values from the theoretical being well within the range of error assignable to random sampling effects.

TABLE II

Comparison of actual and theoretical frequencies beyond the given levels of significance in the practical tests

Series	I	II	Theoretical
Total frequency	1194 (1)	4096 (1)	*1*
Frequency beyond 5% level for (a) y alone (b) z alone (c) y and z together	 56(.0469) 59(.0494) 22(.0184)	 206(.0504) 191(.0467) 79(.0193)	 *.05* *.05* *.0208*
Maximum inefficiency of z test	62.7%	58.6%	*58.4%*

SUMMARY

"Student's" distribution has been very widely used in the analysis of small samples in order to determine the probability that the deviation of a mean is ascribable to errors of random sampling. Most workers appear to have lost sight of the fact that the distribution is that of a ratio, in which both the numerator and denominator must be expected to vary independently. It is quite erroneous to ascribe the probability of such a ratio to the value taken by the numerator alone.

The rationality of segregation according to any given "level of significance" using "Student's" distribution may be analyzed by considering the joint distributions due to errors of sampling in the means, standard deviations, and the ratio of these two for samples of any given size, N. Theoretical evaluation of the percentage of irrationally segregated samples is given herein for the odd values of N from 3 to 29 and for $N = 99$, using the 5% level of significance. This percentage falls in a curvilinear manner as N increases, a few values being 75% for $N = 3$, 58% for $N = 5$, 33% for $N = 15$, and 14% for $N = 99$. The so-called "large" samples, then, are open to a considerable error of this kind. These

results have been verified by two extensive sampling tests for the case where $N = 5$.

Results such as those given herein stress again the dangers attendant upon the drawing of deductions of practical importance from a single sample of small size. When only a single sample is available it is certainly desirable that the statistical analysis should depend not merely upon most likely estimates of needed parameters, but also upon those of less probability which might readily be true and which guard against the erroneous segregation of possibly insignificant deviations.

LITERATURE CITED

FISHER, R. A.
1915. Frequency distribution of the values of the correlation coefficient in samples from an indefinitely large population. Biometrika 10: 507-521.

FISHER, R. A.
1925. Application of "Student's" distribution. Metron 5: 2-32.

MACDONELL, W. R.
1902. On criminal anthropometry. Biometrika 1: 177-227.

"STUDENT"
1908. The probable error of a mean. Biometrika 6: 1-25.

"STUDENT"
1915. Tables for estimating the probability that the mean of a unique sample of observations lies between $-\infty$ and any given distance of the mean of the population from which the sample is drawn. Biometrika 11: 414-417.

PEARSON, KARL
1931a. Some properties of "Student's" z : Correlation, regression and scedasticity of z with the mean and standard deviation of the sample. Biometrika 23: 1-9.

PEARSON, KARL
1931b. Tables for statisticians and biometricians. Part II. Cambridge University Press, England. pp. ccl + 262.

TIPPETT, L. H. C.
1927. Random sampling numbers. Cambridge University Press, England. pp. viii + 26.

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FIGURE 1

Theoretical frequency surface for y and v, separately and jointly, for $N = 5$.

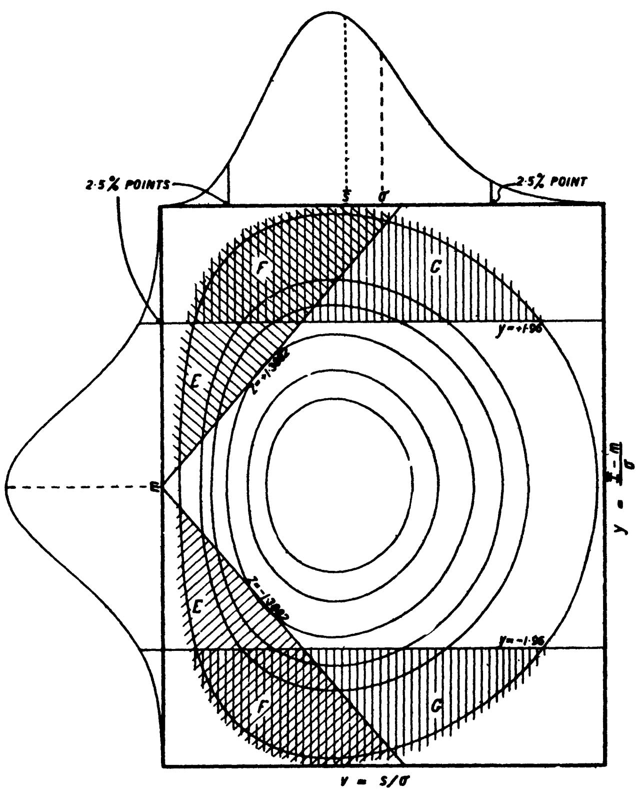

FIGURE 2

Theoretical frequency distributions of y and z, separately and jointly, for $N = 5$. (Contours for the joint distribution are approximate only and the intervals between them do not correspond to the same increment of frequency.)

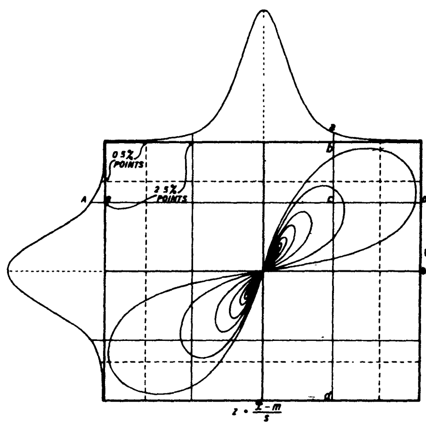

FIGURE 3

Curve to illustrate the increase in correct segregation of means by the z test as N increases.

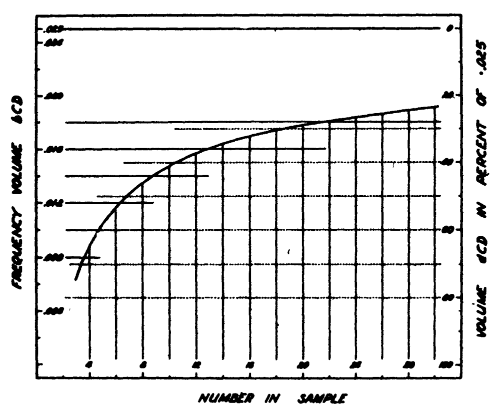

FIGURE 4

Frequency surface for the joint occurrence of y and v as secured in Series II.

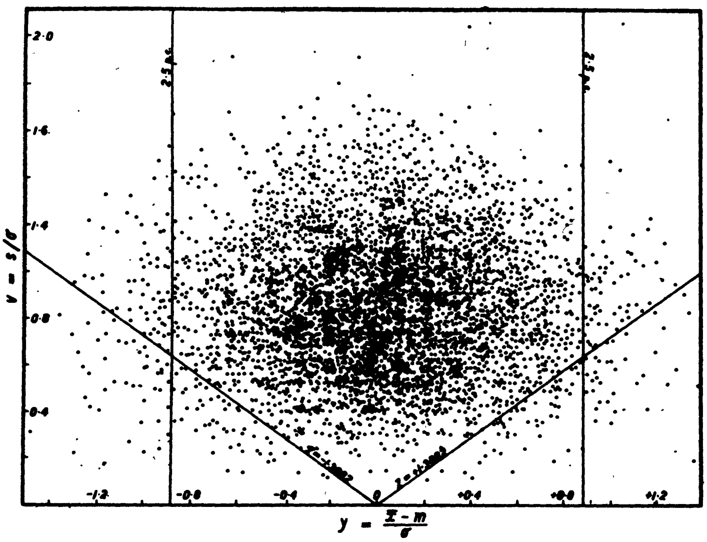

FIGURE 5

Frequency surface for the joint occurrence of y and z as secured in Series II.