THE GENERALIZED PROBLEM OF CORRECT MATCHINGS

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A method common to many experimental and testing procedures in psychology and education is to require an individual to match, as best he can, members of one series of items with members of a second series of quite different items certain of which are in some sense true apposites of items in the first series. Thus the experimental psychology of personality has often investigated the ability of graphologists or laymen to pair samples of handwriting produced by a group of persons with, say, character-sketches of these same persons; and the excess of correct matchings thus produced over the number to be expected by chance has been used as evidence that the expressive movement of handwriting affords characteristics diagnostic of personal traits. Fortunately, the excesses experimentally obtained have often been so large as obviously to exclude the operation of chance alone. But many empirical results show small excesses only; and the interpretation of such findings has not hitherto been subjected to rigid statistical analysis.

The particular statistical problem resident in this experimental procedure is twofold, involving the estimation of the significance of (a) a given number of correct matchings produced by one individual, and (b) a given mean number of correct matchings produced by a group of individuals working with the same material independently.

Furthermore, two cases arise in practice: (1) the two series of items are of equal length, and each item in either series has a true apposite in the other series; or (2) the two series may be of unequal length, in which case the longer series contains not only a true apposite for each item of the shorter series, but, in addition, a certain number of extra, irrelevant items which cannot be correctly matched with any items in the shorter series. I have already given the solution to problems (a) and (b) for case (1).[1] But case (1) forms only a corollary of the more general case (2), to the solution of which this present paper is devoted.

(a) The Significance of a Given Number of Correct Matchings Resulting from a Single Trial

Let there be given a series of u x-items,

$$x_1, x_2, \cdots x_t, \cdots x_u$$

and a series of t y-items,

$$y_1, y_2, \cdots y_t.$$

[1] The Statistics of the Method of Correct Matchings, *Amer. Jour. Psychol.*, 46, 1934, 287–298.

Let $t \leqq u$, and let the first t x-items be in some sense true apposites of the correspondingly numbered y-items, so that if y_j be paired with x_j ($j = 1, 2, \cdots t$), this pairing will constitute a correct matching.

The first problem which arises is that of determining the probability that a single random arrangement of the t y-items against t of the x-items will result in exactly s ($= 0, 1, 2, \cdots t$) correct matchings.

We begin by putting the first s y-items in correspondence with their apposite x-items. Then the number of arrangements of the t y-items in which only these s are correctly matched is the number of arrangements of the remaining $t - s$ y-items against the remaining $u - s$ x-items such that no correct matchings occur. With respect to these items, let

$n =$ the number of all possible arrangements,

$n(Y_j) =$ the number of arrangements such that at least the j^{th} item is correctly matched with its apposite,

$n(Y_jY_k) =$ the number of arrangements such that at least both the j^{th} and k^{th} items are matched with their apposites, etc.;

and let

$n(\bar{Y}_j) =$ the number of arrangements such that at least the j^{th} item is not matched with its apposite,

$n(\bar{Y}_j\bar{Y}_k) =$ the number of arrangements such that at least the j^{th} and k^{th} items are not matched with their apposites, etc.

We have then to evaluate the expression $n(\bar{Y}_{s+1}\bar{Y}_{s+2} \cdots \bar{Y}_t)$, the number of arrangements of the items remaining, after setting s of them correctly matched, such that no further correct matchings occur.

Now it can be shown that[2]

$$
\begin{aligned}
n(\bar{Y}_{s+1}\bar{Y}_{s+2} \cdots \bar{Y}_t) = n \\
&- [n(Y_{s+1}) + n(Y_{s+2}) + \cdots + n(Y_t)] \\
&+ [n(Y_{s+1}Y_{s+2}) + n(Y_{s+1}Y_{s+3}) + \cdots + n(Y_{t-1}Y_t)] \\
&- [n(Y_{s+1}Y_{s+2}Y_{s+3}) + \cdots + n(Y_{t-2}Y_{t-1}Y_t)] \\
&+ \cdots \\
&+(-1)^t\, n(Y_{s+1}Y_{s+2} \cdots Y_t)\,.
\end{aligned}
$$

The value of the expressions on the right side of this equation can be determined as follows:

[2] H. Whitney, A Logical Expansion in Mathematics, *Bull. Amer. Math. Soc.*, 1932, 572–579.

The value of n is the number of ways in which $t - s$ items can be arranged against

$u - s$ items, which is $\dfrac{(u-s)!}{[(u-s)-(t-s)]!} = \dfrac{(u-s)!}{(u-t)!}$.

The value of the first bracket—the number of arrangements of these items such that some one of them is correctly matched—is derived by holding one of the items matched, which can be chosen in $t - s$ ways. This leaves $t - s - 1$ y-items, which can be arranged against the remaining $u - s - 1$ x-items in $(u - s - 1)!/(u - t)!$ ways. The product of these two expressions gives us for the value of the first bracket

$$[n(Y_{s+1}) + \cdots + n(Y_t)] = \frac{(t-s)!(u-s-1)!}{(u-t)!}.$$

To evaluate the second bracket, we hold two of the $t - s$ items matched, which can be chosen in $(t - s)!/[2!(t - s - 2)!]$ ways. There remains $t - s - 2$ y-items which can be arranged against the remaining $u - s - 2$ x-items in $(u-s-2)!/(u-t)!$ ways. The product of these two expressions gives us

$$[n(Y_{s+1}Y_{s+2}) + \cdots + n(Y_{t-1}Y_t)] = \frac{(t-s)!(u-s-2)!}{2!(t-s-2)!(u-t)!}.$$

Continuing thus, we develop the following series for the number of arrangements of t items against u items such that the first s are correctly matched:

$$n(\bar{Y}_{s+1}\bar{Y}_{s+2}\cdots\bar{Y}_t) = \frac{(u-s)!}{(u-t)!} - \frac{(t-s)!(u-s-1)!}{(u-t)!} + \frac{(t-s)!(u-s-2)!}{2!(t-s-2)!(u-t)!} - \cdots + (-1)^{t-s}\frac{(t-s)!(u-t)!}{(t-s)!(u-t)!}.$$

In order to express the number of arrangements, $N_{(s)}$, such that *any* s correct matchings occur, we must multiply the above series by $t!/[s!(t - s)!]$, which is the number of ways in which s items can be chosen from t items:

$$N_{(s)} = \frac{t!}{s!(t-s)!}\left[\frac{(u-s)!}{(u-t)!} - \frac{(t-s)!(u-s-1)!}{(u-t)!} + \cdots + (-1)^{t-s}\frac{(t-s)!(u-t)!}{(t-s)!(u-t)!}\right].$$

And in order to obtain the probability that a single random arrangement will result in exactly s correct matchings, we must further divide by $u!/(u - t!)$, which is the total number of ways in which t items can be arranged against u items. Calling this probability $P_{(s)}$, we have then

$$P_{(s)} = \frac{t!\,(u-t)!}{u!s!\,(t-s)!}\left[\frac{(u-s)!}{(u-t)!} - \frac{(t-s)!\,(u-s-1)!}{(u-t)!} + \cdots + (-1)^{t-s}\frac{(t-s)!\,(u-t)!}{(t-s)!\,(u-t)!}\right].$$

Finally, factoring $(t - s)!/(u - t)!$ out of all terms in the bracket, the series simplifies to[3]

$$P_{(s)} = \frac{t!}{s!u!}\left[\frac{(u-s)!}{0!(t-s)!} - \frac{(u-s-1)!}{1!(t-s-1)!} + \frac{(u-s-2)!}{2!(t-s-2)!} - \cdots + (-1)^{t-s}\frac{(u-t)!}{(t-s)!\,0!}\right]. \qquad (1)$$

In any practical situation, the significant question is not the probability that exactly s correct matchings shall occur, but the probability of *s or more* correct matchings. Obviously

$$P_{(s \text{ or more})} = P_{(s)} + P_{(s+1)} + \cdots + P_{(t)} .$$

whence, by equation (1),

$$\begin{aligned} P_{(s \text{ or more})} = {} & \frac{t!}{s!u!}\left[\frac{(u-s)!}{0!(t-s)!} - \frac{(u-s-1)!}{1!(t-s-1)!} + \frac{(u-s-2)!}{2!(t-s-2)!} - \cdots + (-1)^{t-s}\frac{(u-t)!}{(t-s)!0!}\right] \\ & + \frac{t!}{(s+1)!u!}\left[\frac{(u-s-1)!}{0!(t-s-1)!} - \frac{(u-s-2)!}{1!(t-s-2)!} + \cdots + (-1)^{t-s-1}\frac{(u-t)!}{(t-s-1)!0!}\right] \\ & + \frac{t!}{(s+2)!u!}\left[\frac{(u-s-2)!}{0!(t-s-2)!} - \cdots + (-1)^{t-s-2}\frac{(u-t)!}{(t-s-2)!0!}\right] \\ & + \cdots \\ & + \frac{t!}{t!u!}\left[\frac{(u-t)!}{0!0!}\right]. \end{aligned} \qquad (2)$$

Or, collecting terms in a form better suited to practical computation from tables of factorials and reciprocals,

$$\begin{aligned} P_{(s \text{ or more})} = \frac{t!}{u!}\Bigg\{ & \frac{(u-s)!}{(t-s)!}\left[\frac{1}{0!s!}\right] \\ & + \frac{(u-s-1)!}{(t-s-1)!}\left[\frac{1}{0!(s+1)!} - \frac{1}{1!s!}\right] \\ & + \frac{(u-s-2)!}{(t-s-2)!}\left[\frac{1}{0!(s+2)!} - \frac{1}{1!(s+1)!} + \frac{1}{2!s!}\right] \end{aligned}$$

[3] In the special case in which the series of x-items and the series of y-items are of the same length, whence $t = u$, equation (1) reduces to

$$P_{(s)} = \frac{1}{s!}\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^{t-s}\frac{1}{(t-s)!}\right].$$

$$+ \cdots$$

$$+ \frac{(u-t)!}{0!}\left[\frac{1}{0!t!} - \frac{1}{1!(t-1)!} + \frac{1}{2!(t-2)!} - \cdots + (-1)^{t-s}\frac{1}{(t-s)!s!}\right]\Bigg\}. \qquad (2')$$

(b) The Significance of a Given Mean Number of Correct Matchings Resulting from n Independent Trials

A frequent practical situation is that in which interest centers on the significance of the mean number of correct matchings achieved by a group of n individuals working independently with the same two series.

In order to determine the probability that the mean number of correct matchings, $\bar{s}$, resulting from n independent trials shall equal or exceed a given value, we are required to describe the distribution of the means of samples of size n drawn at random from a parent population in which the variable is $s(= 0, 1, 2, \cdots t)$ with relative frequencies $P_{(0)}, P_{(1)}, P_{(2)}, \cdots P_{(t)}$, given by equation (1). The tabulation of this parent distribution follows:

Table I: Distribution of s

s	*Relative frequency* $(= P_{(s)})$
0	$\frac{t!}{0!u!}\left[\frac{u!}{0!t!} - \frac{(u-1)!}{1!(t-1)!} + \frac{(u-2)!}{2!(t-2)!} - \frac{(u-3)!}{3!(t-3)!} + \cdots + (-1)^t\frac{(u-t)!}{t!0!}\right]$
1	$\frac{t!}{1!u!}\left[\frac{(u-1)!}{0!(t-1)!} - \frac{(u-2)!}{1!(t-2)!} + \frac{(u-3)!}{2!(t-3)!} - \cdots + (-1)^{t-1}\frac{(u-t)!}{(t-1)!0!}\right]$
2	$\frac{t!}{2!u!}\left[\frac{(u-2)!}{0!(t-2)!} - \frac{(u-3)!}{1!(t-3)!} + \cdots + (-1)^{t-2}\frac{(u-t)!}{(t-2)!0!}\right]$
...	...
t	$\frac{t!}{t!u!}\left[\frac{(u-t)!}{0!0!}\right]$

We now determine the first four moments, ν_1, ν_2, ν_3, and ν_4, of this distribution about the origin $s = 0$. Since, in general,

$$\nu_k = \sum_{s=0}^{t} [s^k \times (\text{Relative frequency of } s)] = \sum_{s=0}^{t} s^k P_{(s)},$$

the tabulation for the computation of any moment is as follows:

Table II: The Computation of the k^{th} Moment of the Distribution of s

s	$s^k P_{(s)}$
0	0
1	$\frac{1^k t!}{1!u!}\left[\frac{(u-1)!}{0!(t-1)!} - \frac{(u-2)!}{1!(t-2)!} + \frac{(u-3)!}{2!(t-3)!} - \cdots + (-1)^{t-1}\frac{(u-t)!}{(t-1)!0!}\right]$
2	$\frac{2^k t!}{2!u!}\left[\frac{(u-2)!}{0!(t-2)!} - \frac{(u-3)!}{1!(t-3)!} + \cdots + (-1)^{t-2}\frac{(u-t)!}{(t-2)!0!}\right]$
3	$\frac{3^k t!}{3!u!}\left[\frac{(u-3)!}{0!(t-3)!} - \cdots + (-1)^{t-3}\frac{(u-2)!}{(t-3)!0!}\right]$
...	...
t	$\frac{t^k t!}{t!u!}\left[\frac{(u-t)!}{0!0!}\right].$

Noting that $\frac{1^k}{1!} = \frac{1^{k-1}}{0!}, \frac{2^k}{2!} = \frac{2^{k-1}}{1!}, \cdots \frac{t^k}{t!} = \frac{t^{k-1}}{(t-1)!}$, and multiplying the terms in brackets by these factors, we develop Table III:

Table III

s	$s^k P_{(s)}$
	1st diagonal ↓ 2nd diagonal ↓ 3rd diagonal ↓ ... t^{th} diagonal ↓
0	0
1	$\frac{t!}{u!}\left[\frac{1^{k-1}(u-1)!}{0!0!(t-1)!} - \frac{1^{k-1}(u-2)!}{0!1!(t-2)!} + \frac{1^{k-1}(u-3)!}{0!2!(t-3)!} - \cdots + (-1)^{t-1}\frac{1^{k-1}(u-t)!}{0!(t-1)!0!}\right]$ (t terms)
2	$\frac{t!}{u!}\left[\frac{2^{k-1}(u-2)!}{1!0!(t-2)!} - \frac{2^{k-1}(u-3)!}{1!1!(t-3)!} + \cdots + (-1)^{t-2}\frac{2^{k-1}(u-t)!}{1!(t-2)!0!}\right]$ ($t-1$ terms)
3	$\frac{t!}{u!}\left[\frac{3^{k-1}(u-3)!}{2!0!(t-3)!} - \cdots + (-1)^{t-3}\frac{3^{k-1}(u-t)!}{2!(t-3)!0!}\right]$ ($t-2$ terms)
...	...
t	$\frac{t!}{u!}\left[\frac{t^{k-1}(u-t)!}{(t-1)!0!0!}\right]$ (1 term)

Since each series in brackets is one term shorter than the preceding series, the table forms a system of t diagonals. The sum which gives us ν_k may therefore be considered as the sum of these diagonals.

Now, from inspection, it is evident that the general diagonal is of the form

$$s^{\text{th}}\text{ diagonal} = \frac{t!(u-s)!}{u!(t-s)!}\left[\frac{s^{k-1}}{(s-1)!0!} - \frac{(s-1)^{k-1}}{(s-2)!1!} + \cdots + (-1)^{s-1}\frac{1^{k-1}}{0!(s-1)!}\right]$$

$$= \frac{t!(u-s)!}{u!(t-s)!}\left[\frac{1}{(s-1)!}\sum_{r=0}^{s-1}(-1)^r(s-r)^{k-1}\binom{s-1}{r}\right].$$

But it can be shown[4] that

$$\sum_{r=0}^{s-1}(-1)^r(s-r)^{k-1}\binom{s-1}{r} = 0 \qquad \text{when} \qquad k < s.$$

Whence

$$s^{\text{th}}\text{ diagonal} = 0 \qquad \text{when} \qquad k < s.$$

Therefore ν_k is given simply by the sum of the first k diagonals of Table III. Or, in general,

$$\nu_k = \frac{t!(u-1)!}{s!(t-1)!}\left[\frac{1^{k-1}}{0!0!}\right]$$

$$+ \frac{t!(u-2)!}{u!(t-2)!}\left[\frac{2^{k-1}}{1!0!} - \frac{1^{k-1}}{0!1!}\right]$$

$$+ \frac{t!(u-3)!}{u!(t-3)!}\left[\frac{3^{k-1}}{2!0!} - \frac{2^{k-1}}{1!1!} + \frac{1^{k-1}}{0!2!}\right]$$

$$+ \quad \cdots$$

$$+ \frac{t!(u-k)!}{u!(t-k)!}\left[\frac{k^{k-1}}{(k-1)!0!} - \frac{(k-1)^{k-1}}{(k-2)!1!} + \cdots + (-1)^{k-1}\frac{1^{k-1}}{0!(k-1!)}\right]. \tag{3}$$

To this equation we must, of course, add the condition $k \leqq t$.

[4] E. Netto, *Lehrbuch der Combinatorik*, Leipzig, 1901, 249, Formula 17.

Solving now for the first four moments, we have

$$\left.\begin{aligned}
\nu_1 &= \frac{t}{u}, \\
\nu_2 &= \frac{t}{u}\left[1 + \frac{t-1}{u-1}\right], \\
\nu_3 &= \frac{t}{u}\left[1 + 3\frac{t-1}{u-1} + \frac{(t-1)(t-2)}{(u-1)(u-2)}\right], \\
\nu_4 &= \frac{t}{u}\left[1 + 7\frac{t-1}{u-1} + 6\,\frac{(t-1)(t-2)}{(u-1)(u-2)} + \frac{(t-1)(t-2)(t-3)}{(u-1)(u-2)(u-3)}\right].
\end{aligned}\right\} \quad (4)$$

If now we define, for convenience,

$$a = \frac{t}{u},$$

$$b = \frac{t-1}{u-1},$$

$$c = \frac{t-2}{u-2},$$

$$d = \frac{t-3}{u-3},$$

we have, for the constants of the distribution of s,

$$\left.\begin{aligned}
\text{Mean} &= \nu_1 = a. \\
\mu_2 &= \nu_2 - \nu_1^2 \\
&= a(1+b) - a^2, \quad \text{whence } \sigma = \sqrt{a(1+b) - a^2}. \\
\mu_3 &= \nu_3 - 3\nu_1\nu_2 + 2\nu_1^3 \\
&= a(1 + 3b + bc) - 3a^2(1+b) + 2a^3 \\
\mu_4 &= \nu_4 - 4\nu_1\nu_3 + 6\nu_1^2\nu_2 - 3\nu_1^4 \\
&= a(1 + 7b + 6bc + bcd) - 4a^2(1 + 3b + bc) + 6a^3(1+b) - 3a^4.
\end{aligned}\right\} \quad (5)$$

From these constants we can determine the skewness and kurtosis of the distribution of s,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}, \quad \text{and} \quad \beta_2 = \frac{\mu_4}{\mu_2^2}. \quad (6)$$

Now it is known that the means of samples of size n drawn from a parent population with constants β_1 and β_2 are distributed in such a way that

$$\beta_{1(\text{means})} = \frac{\beta_1}{n}, \qquad \text{and} \qquad \beta_{2(\text{means})} = 3 + \frac{\beta_2 - 3}{n}. \tag{7}$$

Therefore, having determined the beta-constants for the distribution of s, we can determine the beta-constants of the distribution of $\bar{s}$, the mean number of correct matchings resulting from n independent trials.

Now when $t = u \geqq 4$, we have

$$a = b = c = d = 1,$$

and equations (5) give us for the distribution of s

$$\left.\begin{aligned} \text{Mean} &= 1, \\ \mu_2 &= 1, \\ \mu_3 &= 1, \\ \mu_4 &= 4. \end{aligned}\right\} \quad \text{whence} \quad \begin{cases} \beta_1 = 1 \\ \beta_2 = 4 \end{cases}$$

and therefore, for the constants of the distribution of $\bar{s}$, we have, by equations (7),

$$\beta_1 = \frac{1}{n}, \qquad \text{and} \qquad \beta_2 = 3 + \frac{1}{n},$$

which indicates a positively skewed and leptokurtic distribution. The effect of increasing u and holding t constant is to increase the skewness, as shown in the following table for $t = 5$:

t	u	β_1
5	5	$\frac{1}{n}$
5	6	$\frac{1.05}{n}$
5	7	$\frac{1.16}{n}$
5	8	$\frac{1.31}{n}$
5	9	$\frac{1.46}{n}$

The degrees of skewness and kurtosis met with in practical cases of matching with any considerable number of judges (n) are such that a Pearson Type III distribution curve gives a reasonably good fit to the distribution of mean numbers of correct matchings. If, therefore, we have to determine the significance of any obtained mean number of correct matchings, we may resort to Salvosa's tables[5] of the area under the Type III curve.

As a concrete example of the application of this method let us imagine that 10 judges have arranged 5 character sketches against 8 specimens of handwriting, 5 of which are true apposites of the sketches. Let the total number of correct matchings achieved by this group be 12, whence the mean number per judge is 1.2. We have, then,

$$\bar{s} = 1.2, \qquad n = 10,$$

$$t = 5, \qquad u = 8, \qquad \text{whence} \qquad a = \frac{t}{u} = .625,$$

$$b = \frac{t-1}{u-1} = .571,$$

$$c = \frac{t-2}{u-2} = .500.$$

We now find the mean, standard deviation, and β_1 of the distribution of $\bar{s}$, as follows:

The mean of the distribution of $\bar{s}$ is, by sampling theory, the same as the mean of the distribution of s:

$$\text{Mean} = a = .625.$$

The second moment of the distribution of $\bar{s}$ is, by sampling theory, $\frac{1}{n}$ times the second moment of the distribution of s; whence, by equation (5),

$$\text{Standard deviation} = \sqrt{\frac{1}{10}[a(1+b) - a^2]} = .243.$$

And, by equations (5) and (7),

$$\beta_1 = \frac{1}{10}\frac{[a(1+3b+bc) - 3a^2(1+b) + 2a^3]^2}{[a(1+b) - a^2]^3} = .131.$$

Now the obtained mean number of correct matchings was 1.2, and the next lower number which could have occurred (corresponding to a total of 11 instead of 12 for the group of judges) is 1.1. The lower boundary of the class-interval whose midpoint is $\bar{s} = 1.2$ is therefore 1.15; and it is the area above this boundary under the curve of $\bar{s}$ in which we are interested.

[5] L. R. Salvosa, Tables of Pearson's Type III function, *Ann. Math. Statist.*, 1, 1930, 191–198.

The deviation of this boundary from the mean of $\bar{s}$ is

$$1.15 - .625 = .525\,,$$

and this deviation expressed in terms of the standard deviation gives

$$\frac{.525}{.243} = 2.16\,.$$

Entering Salvosa's table for the deviation 2.16 and skewness $= \sqrt{\beta_1} = .36$, we find by interpolation that so good a performance should be expected by chance only about 23 times in 1000.