ON THE FINITE DIFFERENCES OF A POLYNOMIAL

By I. H. Barkey

In this paper an apparently new and convenient method of finding the successive finite differences of a polynomial is considered. If operationally

$$\phi(u + r_1 r_2) = E^{r_1 r_2} \phi(u) = (1 + \Delta r_1)^{r_2} \phi(u)$$

then for any polynomial $f(x)$ of degree "n"

$$f(x) = p_0 x^n + p_1 x^{n-1} + \cdots + p_n$$
$$= p_0(x + a)^n + q_{11}(x + a)^{n-1} + \cdots + q_{1n}$$
$$E^a f(x) = p_0(x + a)^n + p_1(x + a)^{n-1} + \cdots + p_n$$
$$\Delta_a f(x) = (p_1 - q_{11})(x + a)^{n-1} + (p_2 - q_{12})(x + a)^{n-2} + \cdots + (p_n - q_{1n}).$$

Similarly, if $f_1(x) = \Delta a f(x)$, then

$$f_1(x) = (p_1 - q_{11})(x + 2a)^{n-1} + q_{22}(x + 2a)^{n-2} + \cdots + q_{2n}$$
$$E^a f_1(x) = (p_1 - q_{11})(x + 2a)^{n-1} + (p_2 - q_{12})(x + 2a)^{n-2} + \cdots + (p_n - q_{1n})$$
$$\Delta_a f_1(x) = (p_2 - q_{12} - q_{22})(x + 2a)^{n-2} + \cdots + (p_n - q_{1n} - q_{2n})$$

and so on for the higher orders, since $\Delta_a f_{s-1}(x) = \Delta_a^s f(x)$. In the practical application of this method, "a" may be conveniently taken as unity, and an abridged form of synthetic division employed. Thus, if

$$f(x) = 5x^4 + 3x^3 + 7x^2 - 2x + 3, \text{ then}$$

$$
\begin{array}{rrrr|rl}
5 + & 3 + & 7 - & 2 & + & 3 = f \\
 & - 2 + & 9 - & 11 & + & 14 \\
 & - 7 + & 16 - & 27 & & \\
 & - 12 + & 28 & & & \\
 & - 17 & & & & \\
\hline
 & 20 - & 21 + & 25 & - & 11 = f_1 \\
 & & - 41 + & 66 & - & 77 \\
 & & - 61 + & 127 & & \\
 & & - 81 & & & \\
\hline
 & & 60 - & 102 & + & 66 = f_2 \\
 & & & - 162 & + & 228 \\
 & & & - 222 & & \\
\hline
 & & & 120 & - & 162 = f_3 \\
 & & & & - & 282 \\
\hline
 & & & & & 120 = f_4.
\end{array}
$$

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As is evident from the darkened numerals, all figures to the right of the dotted line are redundant and may be omitted. From the above,

$$\Delta f(x) = 20(x+1)^3 - 21(x+1)^2 + 25(x+1) - 11$$

$$\Delta^2 f(x) = 60(x+2)^2 - 102(x+2) + 66$$

$$\Delta^3 f(x) = 120(x+3) - 162$$

$$\Delta^4 f(x) = 120.$$