# ON CERTAIN COEFFICIENTS USED IN MATHEMATICAL STATISTICS

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## I. Introduction

(1.1) We have studied here certain coefficients arising in interpolation, numerical differentiation and integration formulas in order to establish explicit expansions for these coefficients in the form of a finite summation. Ordinarily they are obtained by means of recursion relations, which necessarily demand the building up of a complete table in order to find the desired set of coefficients. By using the methods described in this paper, we are able to calculate any desired set independent of the ones which precede it in the table. In the literature we find two other expansions of the *difference quotients of zero*, one by Jeffery[1] and one by Boole.[2] Our expansion for the *differential quotients of zero* is the same as one obtained by Jeffery,[3] however the proof is more elementary and simple.

The Bernoulli numbers also find a wide range of application in many finite integration formulas, and hence our attention was drawn to the discussion of certain coefficients which occur in the study of these functions.[4] As in the cases mentioned above these coefficients are likewise ordinarily obtained by recursion formulas, but by our expansions they may be obtained directly.

## II. Difference Quotients of Zero

(2.1) It is our purpose here to show that this difference quotient of zero, $\Delta^m 0^n$, may be expressed by the following summation:

$$\Delta^m 0^n = m! \sum \left(\frac{m}{m-1}\right)^{a_{m-1}} \left(\frac{m-1}{m-2}\right)^{a_{m-2}} \cdots \left(\frac{3}{2}\right)^{a_2} \left(\frac{2}{1}\right)^{a_1} \tag{1}$$

where $a_1, a_2, \cdots, a_{m-1} = 0, 1, 2, \cdots, n - m$ and $a_1 \geqq a_2 \geqq \cdots \geqq a_{m-1} \geqq 0$. Obviously the number of terms in the summation is the number of combinations of $n - m + 1$ things taken $m - 1$ together where repetitions are allowed.

(2.2) By means of the recursion relation[5]

$$\Delta^m 0^n = m\,\Delta^m 0^{n-1} + m\,\Delta^{m-1} 0^{n-1} \tag{2}$$

[1] Henry M. Jeffery, "On a method of expressing the combinations and homogeneous products of numbers and their powers by means of differences of nothing." *Quarterly Journal of Pure and Applied Mathematics*, vol. 4 (1861), pp. 364 ff.

[2] George Boole, *A Treatise on the Calculus of Finite Differences*, (Stechert, N. Y.), p. 20.

[3] Loc. cit.

[4] Steffensen, *Interpolation* (Williams & Wilkins, Baltimore), p. 125.

[5] L. M. Milne-Thompson, *Calculus of Finite Differences*, (Macmillan), p. 36, sec. 2.53, (2).

we are able to build up a table of values. By substitution it can be shown that (1) satisfies the values of this table except when $m = 0, 1$ and for $m > n$, for then the summation becomes meaningless. We therefore define the summation to have the value 0 for $m = 0, n > 0$ and for $m > n$, and the value 1 for $m = 1$. We exhibit one substitution below. When $m = 3$ and $n = 4$,

$$\Delta^3 0^4 = 3!\left\{\left(\frac{3}{2}\right)^0\left(\frac{2}{1}\right)^0 + \left(\frac{3}{2}\right)^0\left(\frac{2}{1}\right)^1 + \left(\frac{3}{2}\right)^1\left(\frac{2}{1}\right)^1\right\} = 36 .$$

(2.3) Taking (2), we proceed by repeated application of the recursion formula and finally we have

$$\Delta^m 0^n = m^{n-m}\,\Delta^m 0^m + \sum_{d=m}^{n-1} m^{n-d}\,\Delta^{m-1}0^d ,$$

which since $\Delta^m 0^m = m!$,[6] becomes

$$\Delta^m 0^m = m^{n-m} m! + \sum_{d=m}^{n-1} m^{n-d}\,\Delta^{m-1}0^d . \tag{3}$$

We will now prove (1). Proceeding by induction we assume (1) true for $m - 1$. Hence from (3) we have

$$\Delta^m 0^n = m^{n-m} m! + \sum_{d=m}^{n-1} m^{n-d}(m-1)! \sum \left(\frac{m-1}{m-2}\right)^{a_{m-2}} \cdots \left(\frac{3}{2}\right)^{a_2}\left(\frac{2}{1}\right)^{a_1} ,$$

where $a_1, a_2, \cdots, a_{m-2} = 0, 1, 2, \cdots, d - m + 2$ and $a_1 \geqq a_2 \geqq \cdots \geqq a_{m-2} \geqq 0$. This becomes

$$\Delta^m 0^n = m^{n-m} m! + m! \sum_{d=m}^{n-1} m^{n-d-1} \sum \left(\frac{m-1}{m-2}\right)^{a_{m-2}} \cdots \left(\frac{3}{2}\right)^{a_2}\left(\frac{2}{1}\right)^{a_1} . \tag{4}$$

Using the symbol $\Sigma\Sigma$ for the double summation of (4), we may write

$$\begin{aligned}\sum\sum = \sum_{d=m}^{n-1} m^{n-d-1}\Bigg\{&\left(\frac{m-1}{m-2}\right)^0 \cdots \left(\frac{3}{2}\right)^0\left(\frac{2}{1}\right)^0 + \left(\frac{m-1}{m-2}\right)^0 \cdots \left(\frac{3}{2}\right)^0\left(\frac{2}{1}\right)^1 \\ &+ \left(\frac{m-1}{m-2}\right)^0 \cdots \left(\frac{3}{2}\right)^1\left(\frac{2}{1}\right)^1 + \cdots \\ &+ \left(\frac{m-1}{m-2}\right)^{d-m}\left(\frac{m-2}{m-3}\right)^{d-m+1} \cdots \left(\frac{3}{2}\right)^{d-m+1}\left(\frac{2}{1}\right)^{d-m+1} \\ &+ \left(\frac{m-1}{m-2}\right)^{d-m+1} \cdots \left(\frac{2}{1}\right)^{d-m+1}\Bigg\} \\ = \sum_{d=m}^{n-1} \Bigg\{&\left(\frac{m}{m-1}\right)^{n-d-1} \cdots \left(\frac{3}{2}\right)^{n-d-1}\left(\frac{2}{1}\right)^{n-d-1}\end{aligned}$$

[6] Milne-Thompson, loc. cit.

$$+\left(\frac{m}{m-1}\right)^{n-d-1}\cdots\left(\frac{3}{2}\right)^{n-d-1}\left(\frac{2}{1}\right)^{n-d}+\cdots$$

$$+\left(\frac{m}{m-1}\right)^{n-d-1}\left(\frac{m-1}{m-2}\right)^{n-m-1}\left(\frac{m-2}{m-3}\right)^{n-m}\cdots\left(\frac{2}{1}\right)^{n-m}$$

$$+\left(\frac{m}{m-1}\right)^{n-d-1}\left(\frac{m-1}{m-2}\right)^{n-m}\cdots\left(\frac{2}{1}\right)^{n-m}\Bigg\}.$$

Now, $m^{n-m}=\left(\frac{m}{m-1}\right)^{n-m}\left(\frac{m-1}{m-2}\right)^{n-m}\cdots\left(\frac{3}{2}\right)^{n-m}\left(\frac{2}{1}\right)^{n-m}$, and also $d$ varies from $m$ to $n-1$. Hence by including $m^{n-m}$ under the summation we are able to replace the double summation by a single one and have

$$\Delta^m 0^n = m!\sum\left(\frac{m}{m-1}\right)^{a_{m-1}}\left(\frac{m-1}{m-2}\right)^{a_{m-2}}\cdots\left(\frac{3}{2}\right)^{a_2}\left(\frac{2}{1}\right)^{a_1}$$

where $a_1, a_2, \cdots, a_{m-1} = 0, 1, 2, \cdots, n-m$ and $a_1 \geqq a_2 \geqq \cdots \geqq a_{m-1} \geqq 0$. Hence (1) is proved.[7]

## III. Differential Quotients of Zero

(3.1) In Markoff's formula for numerical differentiation we meet coefficients of the type $D^m0^{(n)}$. We will show here that this differential quotient of zero may be expressed by the following finite sum:

$$D^m0^{(n)} = (-1)^{n-m}\,m!\sum(p_1p_2\cdots p_{n-m}) \tag{5}$$

where $p_1 > p_2 > \cdots > p_{n-m} > 0$ take on values from $1, 2, \cdots, n-1$. Obviously the number of terms in the expansion will be the same as the number of combinations of $n-1$ things taken $n-m$ together without repetitions.

(3.2) By means of the recursion formula[8]

$$D^m0^{(n)} = (1-n)\,D^m0^{(n-1)} + m\,D^{m-1}0^{(n-1)} \tag{6}$$

we are able to build up a table of values. By substitution it can easily be shown that (5) satisfies the values of the table when $n > m > 0$. For the other values the summation is meaningless, hence we define it to have the value 1 for $m = n > 0$; and the value 0 for $m > n$ and $m = 0$. When $m = 2$ and $n = 4$, we have

$$D^20^{(4)} = (-1)^{4-2}\,2!\,\{(3\cdot 2)+(3\cdot 1)+(2\cdot 1)\} = 22\,,$$

which is the same value as found by (6).

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[7] Our expansion may be shown to be equal to that of Jeffery's cited in the introduction, which is $\Delta^m0^{m+n} = m!\,\xi^m0^{m+n}$, where $\xi^m0^{m+n}$ expresses the sum of all the homogeneous products of $n$ dimensions which can be formed by the first $m$ natural numbers and their powers. The proof of Jeffery's expansion involves the use of complicated symbolic operators, while our proof uses elementary notions only.

[8] Steffensen, op. cit., p. 57, 58, (12) and (14).

(3.3) Returning to (6), we obtain by its repeated application:

$$D^m0^{(n)} = (-1)^{n-m}(n-1)^{(n-m)} D^m0^{(m)} + m \sum_{a=0}^{n-m-1} (-1)^a (n-1)^{(a)} D^{m-1}0^{(n-a-1)}$$

or, since $D^m0^{(m)} = m!$,

$$D^m0^{(n)} = (-1)^{n-m}(n-1)^{(n-m)} m! + m \sum_{a=0}^{n-m-1} (-1)^a (n-1)^{(a)} D^{m-1}0^{(n-a-1)} \quad (7)$$

In proving (5), we proceed by induction, assuming (5) true for $m - 1$; hence by (7) we have

$$D^m0^{(n)} = (-1)^{n-m}(n-1)^{(n-m)} m! + m! \sum_{a=0}^{n-m-1} (-1)^{n-m} (n-1)^{(a)} \sum(p_1 p_2 \cdots p_{n-m-a}) \quad (8)$$

where $p_1 > p_2 > \cdots > p_{n-m-a} > 0$ take the values $1, 2, \cdots, n - a - 2$. Expanding the double sum of (8) we have

$$\sum\sum = \sum_{p_i=1}^{n-2} (p_1 \cdots p_{n-m}) + \sum_{p_i=1}^{n-3} (n-1)(p_1 \cdots p_{n-m-1}) + \sum_{p_i=1}^{n-4} (n-1)(n-2)(p_1 \cdots p_{n-m-2}) + \cdots + \sum_{p_i=1}^{m-1} (n-1)(n-2)\cdots(m+1)(p_1) \quad (9)$$

in which $p_1 > p_2 > \cdots > p_s > 0$ always holds, where

$$s = n - m, n - m - 1, \cdots, 2, 1$$

in turn.

Upon inspection, it is evident that (9) contains all the terms of (5) with the exception of $(n-1)(n-2)\cdots(m+1)m$. Hence, since by definition $(n-1)^{(n-m)} = (n-1)\cdots(m+1)m$, we may include the first term on the right-hand side of (8) under the summation and then we have proved (5).[9]

## IV. The Coefficient $G_n^{(r)}$

(4.1) In discussing the Bernoulli numbers and the Bernoulli polynomials, Steffensen[10] makes use of the relation:

$$B_{2r}(x) = (-1)^r \sum_{n=0}^{r} G_n^{(r)} z^{r-n} \quad (10)$$

[9] Jeffery's expansion referred to in the introduction is $D^m0^{(n)} = \zeta^m0^{(n)}$, where $\frac{(-1)^{n-m}\zeta^m0^{(n)}}{m!}$ expresses the sum of the combinations of the first $n - 1$ natural numbers taken $n - m$ together. The remarks made above under article 2.3 concerning symbolic operators also apply here *mutatis mutandis*.

[10] Op. cit., p. 125, (24); cf. also Jacobi's theorem. *Journal für reine und angewandte Mathematik* (Crelle's Journal), vol. 12, pp. 268–269.

where $z = x - x^2$. We wish here to show that the coefficient $G_n^{(r)}$, ordinarily found by means of recursion formulas, may be obtained from the following summation:

$$G_n^{(r)} = (2r)^{(2n)} \sum_{N_n=3}^{r-n+1} [N_n] \sum_{N_{n-1}=3}^{N_n+1} [N_{n-1}] \cdots \sum_{N_1=3}^{N_2+1} [N_1] \tag{11}$$

where $[N] = (N)^{(2)}/(2N)^{(4)}$. Obviously the summation has no meaning for $n = 0$, nor for $r < n + 2$. Therefore it will be necessary to make definitions or devise other schemes for meeting this difficulty.

Steffensen[11] shows that

$$G_0^{(r)} = 1 \quad \text{for} \quad r \geqq 0; \qquad G_{r-1}^{(r)} = 0 \quad \text{for} \quad r > 1; \tag{12}$$

and likewise he gives the following recursion relation:

$$(2r - 2n)^{(2)} G_n^{(r)} = (2r)^{(2)} G_n^{(r-1)} + (r - n + 1)^{(2)} G_{n-1}^{(r)}. \tag{13}$$

In accordance with (12), we define the sum of (11) to be equal to 1 for $n = 0$, and to be equal to 0 for $n = r - 1$, when $r > 1$. By means of the recursion formula (13), Steffensen[12] gives a table of values of $G_n^{(r)}$, which (11) may be easily shown to satisfy. From this table we have the value $G_3^{(6)} = 10$. Using this as an example of the expansion, we have by (11):

$$\begin{aligned} G_3^{(6)} &= (12)^{(6)} \sum_{N_3=3}^{4} [N_3] \sum_{N_2=3}^{N_3+1} [N_2] \sum_{N_1=3}^{N_2+1} [N_1] \\ &= (12)^{(6)} \langle [3]\{[4]([5] + [4] + [3]) + [3]([4] + [3])\} \\ &\qquad + [4]\{[5]([6] + [5] + [4] + [3]) + [4]([5] + [4] + [3]) + [3]([4] + [3])\}\rangle \\ &= 10. \end{aligned}$$

(4.2) Before proving the general case, we will prove by induction that

$$G_1^{(r)} = (2r)^{(2)} \sum_{N_1=3}^{r} [N_1] \tag{14}$$

Assuming (14) true for $r - 1$, we have by (12) and (13)

$$G_1^{(r)} = (2r)^{(2)} \sum_{N_1=3}^{r-1} [N_1] + (2r)^{(2)} [r] = (2r)^{(2)} \sum_{N_1=3}^{r} [N_1].$$

Hence (14) is valid.

(4.3) We shall prove (11) with respect to $r$. By repeated application of (13), we have

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[11] Op. cit., p. 125.

[12] Op. cit., p. 126.

$$\begin{aligned}
G_n^{(r)} &= \{(2r)^{(2)}/(2r-2n)^{(2)}\}G_n^{(r-1)} + \{(2r)^{(2)}(r-n+1)^{(2)}/(2r-2n+2)^{(4)}\}G_{n-1}^{(r-1)} \\
&\quad + \{(2r)^{(2)}(r-n+1)^{(2)}(r-n+2)^{(2)}/(2r-2n+4)^{(6)}\}G_{n-2}^{(r-1)} + \cdots \\
&\quad + \{(2r)^{(2)}(r-n+1)^{(2)} \cdots (r-1)^{(2)}/(2r-2)^{(2n)}\}G_1^{(r-1)} \\
&\quad + \{(r-n+1)^{(2)} \cdots (r)^{(2)}/(2r-2)^{(2n)}\}G_0^{(r)} \\
&= (2r)^{(2n)} \sum_{N_n=3}^{r-n} [N_n] \cdots \sum_{N_1=3}^{N_2+1} [N_1] \\
&\quad + (2r)^{(2n)}[r-n+1] \sum_{N_{n-1}=3}^{r-n+1} [N_{n-1}] \cdots \sum_{N_1=3}^{N_2+1} [N_1] \\
&\quad + (2r)^{(2n)}[r-n+1][r-n+2] \sum_{N_{n-2}=3}^{r-n+2} [N_{n-2}] \cdots \sum_{N_1=3}^{N_2+1} [N_1] + \cdots \\
&\quad + (2r)^{(2n)}[r-n+1][r-n+2] \cdots [r-1] \sum_{N_1=3}^{r-1} [N_1] \\
&\quad + (2r)^{(2n)}[r-n+1] \cdots [r].
\end{aligned}$$

It is evident from inspection that this is nothing but an expanded form of (11), hence (11) is proved with respect to $r$.

(4.4) Proceeding in the same way as above to prove induction with respect to $n$, we have again by repeated application of (13)

$$\begin{aligned}
G_n^{(r)} &= \{(r-n+1)^{(2)}/(2r-2n)^{(2)}\}G_{n-1}^{(r)} + \{(2r)^{(2)}(r-n)^{(2)}/(2r-2n)^{(4)}\}G_{n-1}^{(r-1)} \\
&\quad + \{(2r)^{(4)}(r-n-1)^{(2)}/(2r-2n)^{(6)}\}G_{n-1}^{(r-2)} \\
&\quad + \cdots + \{(2r)^{(2r-2n-4)}(3)^{(2)}/(2r-2n)^{(2r-2n-2)}\}G_{n-1}^{(n+2)} \\
&= (2r)^{(2n)}[r-n+1] \sum_{N_{n-1}=3}^{r-n+2} [N_{n-1}] \cdots \sum_{N_1=3}^{N_2+1} [N_1] \\
&\quad + (2r)^{(2n)}[r-n] \sum_{N_{n-1}=3}^{r-n+1} [N_{n-1}] \cdots \sum_{N_1=3}^{N_2+1} [N_1] \\
&\quad + \cdots + (2r)^{(2n)}[4] \sum_{N_{n-1}=3}^{5} [N_{n-1}] \cdots \sum_{N_1=3}^{N_2+1} [N_1] \\
&\quad + (2r)^{(2n)}[3] \sum_{N_{n-1}=3}^{4} [N_{n-1}] \cdots \sum_{N_1=3}^{N_2+1} [N_1].
\end{aligned}$$

From this latter equation, (11) follows immediately and therefore the proof is complete.

(4.5) Bernoulli numbers may be expressed in terms of this coefficient $G_n^{(r)}$, as is shown by Steffensen,[13] in the following way

$$B_{2r} = (-1)^r G_r^{(r)} \tag{15}$$

[13] Op. cit., p. 125, (27).

which we shall express in terms of (11). However as (11) is meaningless for $n = r$, we obtain the relation

$$(2r + 2)^{(2)} G_r^{(r)} = -(2)^{(2)} G_{r-1}^{(r+1)} \quad \text{for} \quad r > 0, \tag{16}$$

which follows immediately from (12) and (13), and thereby obviate this difficulty. Hence, by (11), (15) and (16), we can write

$$B_{2r} = \{(-1)^{(r+1)}(2r)!/(4)^{(2)}\} \sum_{N_{r-1}=3}^{3} [N_{r-1}] \sum_{N_{r-2}=3}^{N_{r-1}+1} [N_{r-2}] \cdots \sum_{N_1=3}^{N_2+1} [N_1] \tag{17}$$

We note here that the definitions of the summation, given in 4.1, likewise hold.

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