A NEW EXPOSITION AND CHART FOR THE PEARSON SYSTEM OF FREQUENCY CURVES

By Cecil C. Craig

In the course of some years of teaching classes in mathematical statistics, the author has expanded the treatment of the Pearson system of frequency functions begun in the Handbook of Mathematical Statistics[1] into an exposition that he believes possesses marked advantages in unity, clarity, and elegance. This is accomplished by expressing the variable in standard units throughout and by making the two parameters $\alpha_3 (\alpha_3^2 = \beta_1, \alpha_4 = \beta_2$ in Pearson's notation) and

$$\delta = \frac{2\alpha_4 - 3\alpha_3^2 - 6}{\alpha_4 + 3}$$ [2]

fundamental in the discussion. The various formulae that arise are obtained directly and in a uniform manner and are relatively simple in form and easy to use. The criteria for the different members of the system of functions are expressed very simply in terms of α_3 and δ and the chart corresponding to the extension of the Rhind diagram given by Pearson[3] takes on a strikingly simple form.

Following the beginning made in the Handbook, the system of Pearson frequency functions are to be found among the solutions of the differential equation

$$\frac{1}{y}\frac{dy}{dt} = \frac{a - t}{b_0 + b_1 t + b_2 t^2}. \tag{1}$$

For those solutions $y = f(t)$ for which,

$$(b_0 + b_1 t + b_2 t^2)\, t^n f(t)\Big]_{t=r}^{s} = 0,$$

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[1] H. L. Rietz, Editor-in-Chief; Houghton-Mifflin Co., Boston (1924). See the chapter on Frequency Curves by H. C. Carver.

[2] The notation used is that of the Handbook, loc. cit., to which reference will be frequently made. The discussion of Robert Henderson, "Frequency Curves and Moments," Transactions of the Actuarial Society of America, Vol. VIII (1904), pp. 30–41, also proceeds along very similar lines, although Professor Carver was quite unaware of it when he wrote his chapter in the Handbook. The notation of the Handbook seems preferable however.

[3] Karl Pearson: Mathematical Contributions to the Theory of Evolution, XIX. Second Supplement to a Memoir on Skew Variation; Proc. Roy. Soc., A. Vol. 216 (1916), plate opposite p. 456.

if r and s are the extremes of the range of variation for t, and for which the first $n + 1$ moments over this range exist, the recursion formula for moments,

(2) $$\alpha_n a + n\alpha_{n-1}b_0 + (n + 1)\alpha_n b_1 + (n + 2)\alpha_{n+1}b_2 = \alpha_{n+1},$$

can be derived. Then setting $n = 0, 1, 2, 3$ we get the following expressions for the parameters, a, b_0, b_1, b_2 in terms of α_3 and δ:

(3) $$a = -\frac{\alpha_3}{2(1 + 2\delta)}, \qquad b_1 = \frac{\alpha_3}{2(1 + 2\delta)}$$
$$b_0 = \frac{2 + \delta}{2(1 + 2\delta)} \qquad b_2 = \frac{\delta}{2(1 + 2\delta)}$$ [4]

valid except when $\delta = -\frac{1}{2}$. Below note will be taken of those solutions for which the conditions imposed in deriving (2) are not satisfied. The case in which $\delta = -\frac{1}{2}$ will be included in the discussion of the transitional types of functions.

It is useful to note that

$$-2 < \delta < 2.$$

To show this, using a well-known device, we see that

$$\int_r^s f(t)\ (t^2 + \lambda t)^2\, \alpha t = \alpha_4 + 2\lambda\alpha_3 + \lambda^2$$

is never negative since $f(t) \geqq 0$, $r \leqq t \leqq s$, for any real λ. This requires that

$$\alpha_3^2 \leqq \alpha_4 .$$

But

$$-2 + \frac{4\alpha_4 - 3\alpha_3^2}{\alpha_4 + 3} = \delta = 2 - \frac{\alpha_3^2 + 4}{\alpha_4 + 3}$$

and the result follows. One consequence of this is that b_0 cannot vanish for any Pearson frequency function possessing moments of the fourth order.

Turning now to the integration of (1) and the development of the various forms of $f(t)$ that arise, it is useful to make the preliminary statements:

1. Over the range of variation of t, we must have $f(t) \geqq 0$.
2. The area under curve $y = f(t)$ over the range of variation must be finite. This being true then we always determine the constant of integration so that this area is unity.
3. The range in each case is taken as the maximum one for which (1) and (2) may be secured which contains the point, $t = 0$.
4. It is sufficient throughout to take $\alpha_3 \geqq 0$ since the curve for $\alpha_3 = -k$ is only a reflection of that for $\alpha_3 = k$ through the line $t = 0$.

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[4] See the Handbook, pp. 103, 104.

It seems best to follow the Handbook in disposing of three of the transitional types before proceeding to the main types of the system and then to the remaining transitional types.

The discussion is planned to embody a direct and uniform method of treatment, giving simple formulae for the calculation of the parameters in terms of α_3 and δ in each case, and noting the salient features of each type of curve. The criteria for each type are expressed in terms of α_3 and δ, which for the whole system permit a simple graphical representation by means of the chart found at the end of this article. The construction of this chart is made clear in the deviation of the criteria.

Transitional Type: The Normal Frequency Function: $\alpha_3 = \delta = 0$

In this case (1) reduces to,

$$\frac{1}{y}\frac{dy}{dt} = -t\,,$$

from which

$$\text{(N)} \qquad y = c\, e^{-\frac{t^2}{2}}$$

The range is, of course, $(-\infty, \infty)$ with $C = (2\pi)^{-\frac{1}{2}}$. On the chart, which we shall refer to as the (α_3^2, δ)-diagram, we see that this function corresponds to but a single point.

It may have the appearance of reasoning in a circle to use the values of the parameters given by (3), which were derived from (2), in solving (1) and then for the solution obtained examine the validity of (2). However, we may argue as follows: We will use the relations (3) as definitions of a, b_0, b_1, and b_2 in terms of α_3 and δ which are not yet defined. Using the values of a and the b's given by any choice of α_3 and δ, we solve (1). If the solution is such that for it (2) may be derived, then the relations (3) are valid when α_3 and δ have their usual meanings. For convenience let us denote the conditions for the validity of (2) by (A). It is obvious that conditions (A) are satisfied for

$$\text{(N)} \qquad f(t) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2}{2}}\,.$$

$$\textbf{Transitional types} \begin{cases} \textbf{III}, \ \alpha_3 \neq 0\,, & \delta = 0 \\ \textbf{X, if also } \alpha_3^2 = 4\,. \end{cases}$$

We get here (See the Handbook, loc. cit.):

$$\text{(III)} \qquad f(t) = \frac{A^{A^2}\, e^{-A^2}}{\Gamma(A^2)}\,(A + t)^{A^2-1}\, e^{-At}\,,$$

if $A = 2/\alpha_3$, the range being $(-A, \infty)$.

It is readily verified that, since $A^2 - 1 > -1$, conditions (A) are satisfied.

For $A^2 > 1$ (i.e., for $\alpha_3^2 < 4$) the curve is bell-shaped; for $A^2 < 1$ it is J-shaped with an infinite ordinate at $t = -A$. For the bell-shaped curve the mode falls at $t = -1/A$ and the mean—the mode $= 1/A = \alpha_3/2$.

For $A^2 = 1$, we have

$$f(t) = \frac{e^{-t}}{e}, \tag{X}$$

which represents a J-shaped curve with the range $(-1, \infty)$.

For $A^2 \neq 1$, the function has been designated type III, the special case as type X. On the (α_3^2, δ)-chart the points corresponding to type III functions fall on the line $\delta = 0$, the type X functions being represented by a single point on this line.

Turning now to the discussion of the three main types, we note that for $\delta \neq 0$, $b_2 \neq 0$ and that consequently the denominator on the right in (1) is always a quadratic which we can write in the form

$$b_2(t - r_1)\,(t - r_2)$$

in which neither r_1 nor r_2 can be zero (since $b_0 \neq 0$), and

$$\begin{aligned} r_1 &= \frac{-b_1 + \sqrt{b_1^2 - 4b_0b_2}}{2\,b_2} = \frac{-\alpha_3 - \sqrt{\alpha_3^2 - 4\delta(\delta + 2)}}{2\,\delta} = \frac{-\alpha_3 + \sqrt{D}}{2\,\delta}\,. \\ r_2 &= \frac{-\alpha_3 - \sqrt{D}}{2\,\delta}\,. \end{aligned} \tag{4}$$

Leaving aside the special case, $r_1 = r_2$, to be dealt with later, we can always solve (1) in the form

$$f(t) = C(t - r_1)^{m_1}(t - r_2)^{m_2} \tag{5}$$

with

$$\begin{aligned} m_1 &= \frac{a - r_1}{b_2\,(r_1 - r_2)} = \frac{1 + \delta}{\delta}\,\frac{\alpha_3}{\sqrt{D}} - \frac{1 + 2\,\delta}{\delta} \\ m_2 &= \frac{a - r_2}{b_2\,(r_2 - r_1)} = -\,\frac{1 + \delta}{\delta}\,\frac{\alpha_3}{\sqrt{D}} - \frac{1 + 2\,\delta}{\delta}\,. \end{aligned} \tag{6}$$

For $\delta < 0$, the r's are real and opposite in sign; for $\delta > 0$ and $\alpha_3^2 < 4\delta(\delta + 2)$, the r's are complex; and for $\delta > 0$ and $\alpha_3^2 > 4\delta(\delta + 2)$, the r's are real and of the same sign. These three conditions with the additional condition that $\alpha_3 \neq 0$ give rise respectively to the *main* types of frequency functions designated I, IV, and VI. The points corresponding to them fall in simply determined areas on the (α_3^2, δ)-chart. The boundaries of these areas, the curve,

$$(2 + 3\delta)\alpha_3^2 = 4(1 + 2\delta)^2\,(2 + \delta),$$

which intersects the type I and type VI areas, and the line,

$$\delta = -1/2$$

contain the points which correspond to the transitional types.

Main Type I. $\alpha_3 \neq 0, -1 < \delta < 0 \, [\delta \neq -\frac{1}{2}, (2 + 3\delta)\alpha_3^2 \neq 4(1 + 2\delta)^2 (2 + \delta)]$

For $\alpha_3 > 0$, we see that

$$r_1 < 0 < r_2 \text{ and that } |r_1| < |r_2| .$$

The range is taken to be (r_1, r_2) and (5) is written

$$\text{(I)} \qquad y = C(t - r_1)^{m_1}(r_2 - t)^{m_2} .$$

It is evident that the area under the curve over this interval is finite only when $m_1 + 1 > 0$ and $m_2 + 1 > 0$ and that if these inequalities hold moments of all orders exist. In this case also conditions (A) are satisfied. Now

$$m_1 + 1 = -\frac{1 + \delta}{\delta}\left(1 - \frac{\alpha_3}{\sqrt{D}}\right)$$

$$m_2 + 1 = -\frac{1 + \delta}{\delta}\left(1 + \frac{\alpha_3}{\sqrt{D}}\right),$$

and in the present case

$$1 \pm \frac{\alpha_3}{\sqrt{D}} > 0 .$$

Thus $m_1 + 1$ and $m_2 + 1$ are each > 0 only if $\delta > -1$. On the chart, then, the points for $\delta < -1$ correspond to no frequency functions,—they fall in the "Impossible Area."

Further the type I curve will be U-shaped, J-shaped, or bell-shaped if both m's are < 0, if the m's are opposite in sign, or if both are > 0. We have

$$m_1 = -\frac{1 + \delta}{\delta}\left(1 - \frac{\alpha_3}{\sqrt{D}}\right) - 1 .$$

Since for $-1 < \delta < -\frac{1}{2}$,

$$0 < -\frac{1 + \delta}{\delta} < 1 ,$$

we see that $m_1 < 0$ $(\alpha_3 > 0)$ for δ in this interval. For $-\frac{1}{2} < \delta < 0$, $m_1 > 0$ only if

$$-\frac{1 + \delta}{\delta}\left(1 - \frac{\alpha_3}{\sqrt{D}}\right) > 1 ,$$

which leads to the condition:

$$(2 + 3\delta)\alpha_3^2 < 4(1 + 2\delta)^2 (2 + \delta) .$$

Also,

$$m_2 = -\frac{1+\delta}{\delta}\left(1 + \frac{\alpha_3}{\sqrt{D}}\right) - 1$$

whence it is similarly seen that $m_2 > 0$ when $-\frac{1}{2} < \delta < 0$, and that generally $m_2 > 0$ only when

$$(2 + 3\delta)\alpha_3^2 < 4(1 + 2\delta)^2 (2 + \delta) .$$

Thus the curve,

$$(2 + 3\delta)\alpha_3^2 = 4(1 + 2\delta)^2 (2 + \delta) ,$$

being tangent to the line $\alpha_3^2 = 0$ at $\delta = -\frac{1}{2}$, divides the type I area on the chart into three parts: Above it lie the points corresponding to U-shaped curves, to the right of it the points corresponding to J-shaped curves, and below it the points corresponding to bell-shaped curves. (Note that for $\delta < -\frac{2}{3}$ the curves are always U-shaped.)

Since $r_2 - r_1 > 0$ and $b_2 \gtrless 0$ accordingly as $\delta \lesseqgtr -\frac{1}{2}$, it is readily verified that $r_1 < a < r_2$ only for U- or bell-shaped curves. The sign of a is always opposite to that of α_3 for curves with a mode. Finally the constant is determined by setting

$$C \int_{r_1}^{r_2} (t - r_1)^{m_1}(r_2 - t)^{m_2}\, dt = 1 ,$$

giving

$$C = \frac{1}{\beta(m_1 + 1, m_2 + 1)\,(r_2 - r_1)^{m_1+m_2+1}} .$$

Main Type IV: $\alpha_3 \neq 0$, $\delta > 0$, and $\alpha_3^2 < 4\delta(\delta + 2)$

In this case we write:

$$r_1 = \frac{-\alpha_3}{2\delta} + \frac{i\sqrt{-D}}{2\delta} = -r + is, \qquad r_2 = -r - is .$$

$$m_1 = -\frac{1+\delta}{\delta}\frac{\alpha_3}{\sqrt{-D}}\,i - \frac{1+2\delta}{\delta} = \frac{\nu i}{2} - m, \qquad m_2 = -\frac{\nu i}{2} - m .$$

With this notation (5) becomes

$$y = C[(t + r)^2 + s^2]^{-m}\left(\frac{t + r - is}{t + r + is}\right)^{\frac{\nu i}{2}},$$

and since,

$$\left(\frac{a - bi}{a + bi}\right)^{\frac{ci}{2}} = e^{c \tan^{-1} b/a} = e^{c(\pi/2 - \tan^{-1} a/b)},$$

the frequency function can be written,

$$\text{(IV)} \qquad y = Ce^{\nu\pi/2}[(t + r)^2 + s^2]^{-m} e^{-\nu \tan^{-1} \frac{t+r}{s}}$$

It is readily seen that $m > 0$, that ν is opposite in sign to α_3, that

$$e^{-\nu \tan^{-1} \frac{t+r}{s}}$$

can always be taken to lie between $e^{-\nu\pi/2}$ and $e^{\nu\pi/2}$, and that the range can now be taken $(-\infty, \infty)$.

In the previously discussed cases in which $\delta \leqq 0$, if the area under the curve was finite moments of all orders existed. In the present case, the area and the first four moments are always finite but this may fail to be true of moments of higher orders. For, since $0 < \delta < 2$,

$$m = \frac{1 + 2\delta}{\delta} > \frac{5}{2},$$

and the integral,

$$\int_{-\infty}^{\infty} t^n f(t)\, dt$$

for $f(t)$ given by (IV) will be finite for $n \leqq 4$ and infinite for $n = 5$ if $\delta \geqq 1$. In order for the n-th moment to exist we must have

$$2m > n + 1$$

or

$$\delta < \frac{2}{n - 3}.$$

Pearson designated as *heterotypic* those members of his system of frequency functions for which the eighth moment failed to exist. (In such a case the standard deviation of the fourth moment in samples would be infinite.) Setting $n = 8$, we get $\delta = 2/5$ as the deadline on the (α_3^2, δ)-chart.

It was apparent that conditions (A) were satisfied for $-1 < \delta < 0$. (It will appear below that the case in which $\delta = -\frac{1}{2}$ is no exception.) For $\delta > 0$ it will be seen that it is generally true, as in the present case, that the formulae (2) and (3) can be derived if α_{n+2} exists, i.e., if

$$\delta < \frac{2}{n - 1}.$$

To determine C, on setting the integral of (V) over the interval $(-\infty, \infty)$ equal to unity, we get

$$C = \frac{s^{2m-1}}{G(2m-2, \nu)}$$

in which

$$G(2m-2, \nu) = \int_0^\pi \sin^{2m-2}\varphi e^{\nu\varphi}\, d\varphi \text{ [5]} \qquad \left(\varphi = \frac{\pi}{2} - \tan^{-1}\frac{t+r}{s}\right).$$

Main Type VI: $\alpha_3 \neq 0, \delta > 0, \alpha_3^2 > 4\delta(\delta+2)\ [(2+3\delta)\alpha_3^2 \neq 4(1+2\delta)^2(2+\delta)]$

The conditions specify the remaining area on the chart. This may be left in the form

$$(5) \qquad y = C(t-r_1)^{m_1}(t-r_2)^{m_2}.$$

Now r_1 and r_2 are both opposite in sign to α_3, which, as usual, we will consider positive, and $|r_2| > |r_1|$. Always $m_2 < 0$ and $m_1 \gtrless 0$ accordingly as

$$(2+3\delta)\alpha_3^2 \lessgtr 4(1+2\delta)^2(2+\delta).$$

We note that

$$a - r_2 = b_2(r_2 - r_1)m_2 > 0,$$

since now $b_2 > 0$, and that

$$a - r_1 = b_2(r_1 - r_2)m_1$$

has the same sign as m_1. Finally $a < 0$.

Thus for $\alpha_3 > 0$ and $m_1 > 0$, the point $t = a$ on the axis of t lies to the right of both $t = r_1$ and $t = r_2$. Also

$$m_1 + m_2 = -\frac{2(1+2\delta)}{\delta} = -4 - \frac{2}{\delta}.$$

The range is taken (r_1, ∞), the curve being bell-shaped when $m_1 > 0$. If $m_1 < 0$, the curve is J-shaped, $t = a$ now lying to the left of $t = r_1$.

Since

$$m_1 + m_2 < -5, \text{ and } m_1 + 1 > 0,$$

the area and the first four moments always exist. In order for the n-th moment to be finite, we must have

$$-(m_1 + m_2) > n + 1$$

which is the same condition as in the case of the type IV function, giving the same deadline, $\delta = 2/5$.

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[5] Cf: Tables for Statisticians and Biometricians, Cambridge Univ. Press, Part I, 2nd edition (1924), p. lxxxi.

If the origin be shifted to the point, $t = r_2$, we have writing,

$$t - r_2 = z, \qquad r_1 - r_2 = \alpha,$$

for the type VI function the expression,

$$\text{(VI)} \qquad y = Cz^{m_2}(z - \alpha)^{m_1},$$

with the range (α, ∞). Finally

$$C = \frac{1}{\alpha^{m_1+m_2+1}\beta(m_1 + 1, -m_1 - m_2 - 1)}.$$

Transitional Type II: $\alpha_3 = 0, -1 < \delta < 0.$ $(\delta \neq -\frac{1}{2})$

In this case,

$$r_1 = -r_2 = \frac{\sqrt{D}}{\delta} < 0$$

$$m_1 = m_2 = -\frac{1 + 2\delta}{\delta} \gtrless 0 \quad \text{accordingly as } \delta \gtrless -\tfrac{1}{2}.$$

The frequency function is a special case of type I; setting,

$$-r_1 = r_2 = S$$

$$m_1 = m_2 = M,$$

we can write it in the form,

$$\text{(II)} \qquad y = C(S^2 - t^2)^M.$$

As in all cases in which $\alpha_3 = 0$, the curve is symmetrical about the mean.[6] As in the type I case, the area and moments do not exist for $\delta \leqq -1$; for $-1 < \delta < -\frac{1}{2}$, the curve is U-shaped; for $-\frac{1}{2} < \delta < 0$, it is bell-shaped. The range is, of course, $(-S, S)$.

Finally,

$$C = \frac{1}{(2S)^{2M+1}\beta(M + 1, M + 1)}.$$

Transitional Type VII; $\alpha_3 = 0, \delta > 0$

This function may be regarded as a special case of type IV, with

$$r = 0, \quad s = \frac{\sqrt{4\delta(\delta + 2)}}{2\delta} > 0, \quad \nu = 0, \quad \text{and} \quad m = \frac{1 + 2\delta}{\delta} > 0,$$

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[6] It follows at once from the recursion formula,

$$\alpha_{n+1} = \frac{n}{2 - (n - 2)\delta}[(2 + \delta)\alpha_{n-1} + \alpha_3\alpha_n],$$

obtained from setting the expressions (3) in (2), that on changing the sign of α_3, the signs of all the odd moments are changed.

and we write the function:

(VII) $$y = C(t^2 + s^2)^{-m}.$$

The type VII function may equally well be derived from the type II function by noting that

$$S = is \text{ and } M = -m.$$

The range is $(-\infty, \infty)$ however and for $\delta \geqq 2/5$ the function is heterotypic. Finally

$$C = \frac{s^{2m-1}}{\sqrt{2\pi}} \frac{\Gamma(m)}{\Gamma\left(\frac{2m-1}{2}\right)}.$$

Transitional Type V; $\alpha_3 \neq 0,\ \delta > 0,\ \alpha_3^2 = 4\delta(\delta + 2)$

Here

$$r_1 = r_2 = -r$$

and we return to (1) to derive the form of the function, writing it: (The type V can also be derived as a limiting form of type VI)

$$\frac{1}{y}\frac{dy}{dt} = \frac{a - t}{b_2(t + r)^2}.$$

On integration we get

$$y = C(t + r)^{-\frac{1}{b_2}} e^{-\frac{a+r}{b_2(t+r)}}$$

$$= C(t + r)^{-\frac{2(1+2\delta)}{\delta}} e^{-\frac{\alpha_3(1+\delta)}{\delta^2(t+r)}}.$$

(V) $$= C(t + r)^{-2m} e^{-\frac{2r(m-1)}{t+r}}$$

We note that r has the same sign as α_3 and that $m = 2 + 1/\delta$. The range is taken to be $(-r, \pm \infty)$ accordingly as $\alpha_3 \gtrless 0$. The curve is always bell-shaped. In order for the n-th moment to exist we must have as always when $\delta > 0$,

$$4 + 2/\delta > n + 1$$

leading to the same conclusions as in the type IV or VI case. Finally

$$C = \frac{[2r(m - 1)]^{2m-1}}{\Gamma(2m - 1)}.$$

Transitional Type VIII; $\alpha_3 = 0,\ \delta < -\frac{1}{2},\ (2 + 3\delta)\alpha_3^2 = 4(1 + 2\delta)^2(2 + \delta)$

The function is a special case of type I in which $m_1 < 0$ and $m_2 = 0$. But when $m_2 = 0$, $m_1 = -2m$, and the frequency function becomes

(VIII) $$y = C(t - r_1)^{-2m}.$$

The range is (r_1, r_2), the curve being J-shaped with an infinite ordinate at $t = r_1$ and a finite one at $t = r_2$. In this case,

$$C = \frac{1 - 2m}{(r_2 - r_1)^{1-2m}}. \qquad (1 - 2m > 1)$$

Transitional Type IX: $\alpha_3 \neq 0$, $-\frac{1}{2} < \delta < 0$, $(2 + 3\delta)\alpha_3^2 = 4(1 + 2\delta)^2(2 + \delta)$

We have another special type I function in which $m_1 = 0$ and $m_2 = -2m > 0$. The function is

(IX) $$y = C(r_2 - t)^{-2m}$$

the range still being (r_1, r_2), the curve being J-shaped with a finite ordinate at $t = r_2$. C has the same value as in the type VIII case.

Transitional Type XI; $\alpha_3 \neq 0$, $0 < \delta < 2/5$, $(2 + 3\delta)\alpha_3^2 = 4(1 + 2\delta)^2(2 + \delta)$

The function is a special type VI in which $m_1 = 0$, and $m_2 = -2m < 0$, and we may write it

(XI) $$y = C(t - r_2)^{-2m}$$

with the range still (r_1, ∞). The curve is J-shaped with a finite ordinate at $t = r_1$. Again,

$$C = \frac{2m - 1}{(r_1 - r_2)^{2m-1}} \qquad \left(2m - 1 = 3 + \frac{2}{\delta}\right)$$

Transitional Type XII: $\delta = -\frac{1}{2}$

If $\delta = -\frac{1}{2}$, the four linear equations derived from (2) from which the values of a, b_0, b_1, and b_2 in (3) are derived are inconsistent. We can however set the values (3) in the differential equation (1) and from its limiting form as $\delta \rightarrow -\frac{1}{2}$, derive the function appropriate to this case.

We obtain

$$\frac{1}{y}\frac{dy}{dt} = \frac{-\alpha_3 - 2(1 + 2\delta)t}{(2 + \delta) + \alpha_3 t + \delta t^2}$$

and if $\delta = -\frac{1}{2}$, this becomes

$$\frac{1}{y}\frac{dy}{dt} = \frac{2\alpha_3}{t^2 - 2\alpha_3 t - 3} = \frac{2\alpha_3}{(t - r_1)(t - r_2)}$$

with

$$r_1 = \alpha_3 - \sqrt{\alpha_3^2 + 3}, \qquad r_2 = \alpha_3 + \sqrt{\alpha_3^2 + 3}.$$

On integration,

$$y = C'(t - r_1)^{m_1}(t - r_2)^{m_2},$$

in which

$$m_1 = -\frac{\alpha_3}{\sqrt{\alpha_3^2 + 3}}, \qquad m_2 = \frac{\alpha_3}{\sqrt{\alpha_3^2 + 3}}.$$

We observe that ($\alpha_3 > 0$)

$$r_2 > 0 > r_1, \qquad |r_2| > |r_1|$$

$$m_2 = -m_1 > 0.$$

Taking the range to be (r_1, r_2), we write,

(XII) $$y = C\left(\frac{r_2 - t}{t - r_1}\right)^{m_2},$$

the curve being J-shaped. Here

$$C = \frac{1}{(r_2 - r_1)\,\beta(1 - m_2, 1 + m_2)}.$$

The values of the parameters and the form of the function can also be derived as a special type I function in which $\delta = -\frac{1}{2}$.

Finally we note that for $\alpha_3 = 0$, (XII) reduces to

$$y = C$$

thus including the rectangular distribution function among the Pearson system.

In the course of the above discussion a system of criteria for the various types of functions has been set up in terms of α_3 and δ, in terms of which in every case the parameters may be readily calculated. The (α_3^2, δ)-chart which makes these criteria visual is comparatively simple to construct and is strikingly simple in appearance. Besides the lines,

$$\delta = -1, \qquad \delta = -\frac{1}{2}, \qquad \delta = 0, \qquad \delta = \frac{2}{5}, \quad \text{and} \quad \alpha_3 = 0,$$

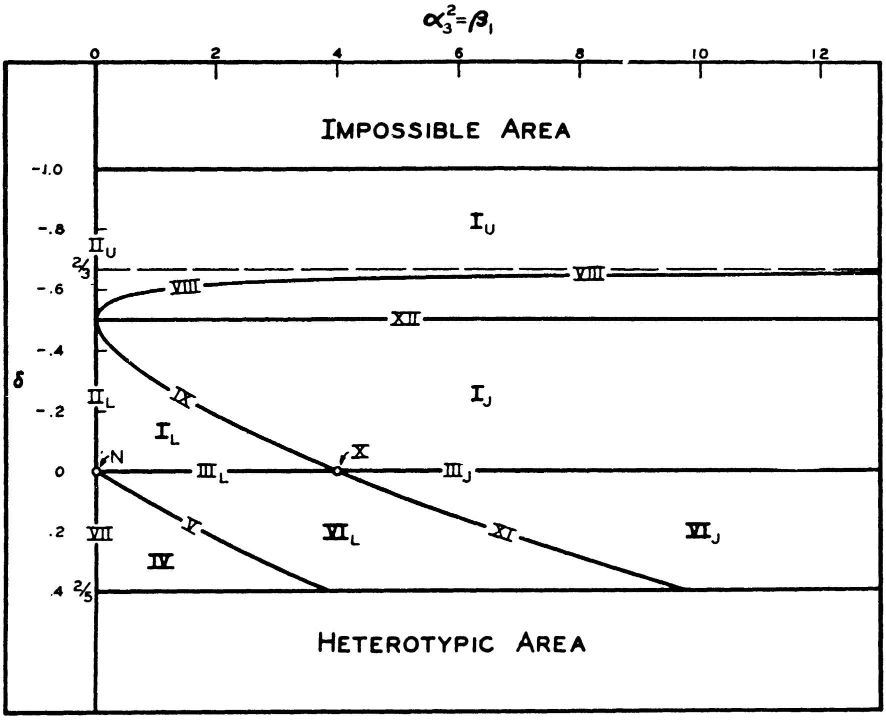
it contains only the curves

$$\alpha_3^2 = 4\delta(\delta + 2)$$

on which the points corresponding to the type V function lie, and the curve,

$$(2 + 3\delta)\alpha_3^2 = 4(1 + 2\delta)^2\,(2 + \delta)$$

on which the points corresponding to the functions of types VIII, IX, X, and XI are found. I must take occasion to express my thanks to Mr. Simon Yang who constructed this chart for me.



THE (α^2, δ) CHART FOR THE PEARSON SYSTEM OF FREQUENCY CURVES
(The subscript L refers to bell-shaped curves)

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