ON A METHOD FOR EVALUATING THE MOMENTS OF A BERNOULLI DISTRIBUTION[1]

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1. The moments (per unit frequency) of a frequency distribution have long been regarded as useful characteristics of the distribution. If we denote the moment about the arithmetic mean by μ, we have for the Bernoulli distribution

$$\mu_s = \sum_{x=0}^{n} (\bar{x})^s f(x),$$

where $\bar{x} = x - np$ and $f(x) = \binom{n}{x} p^x q^{n-x}$.

To evaluate the s-th moment about the arithmetic mean has always been a laborious task. Karl Pearson[2] gave the s-th moment about the arithmetic mean as,

$$\mu_s = \left[\frac{d^s}{dx^s}[qe^{px} + pe^{-qx}]^n\right]_{x=0}, \tag{1}$$

which he said at that time was perhaps the easiest expression for obtaining these moment coefficients by successive differentiation. Romanovsky,[3] however, was able to develop the recursion formula,

$$\mu_{s+1} = pq\left[ns\mu_{s-1} + \frac{d\mu_s}{dp}\right], \tag{2}$$

for the moments about the mean. Another relation for these moments is

$$\mu_{s+1} = \sum_{i=0}^{s-1} \binom{s}{i} [npq\mu_i - p\mu_{i+1}]. \tag{3}$$

Recently Kirkham[4] gave the expressions for the first eight moments which, however, are not in a form well adapted for numerical calculation on a machine.

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[1] Presented to the American Mathematical Society, January 2, 1936.

[2] Karl Pearson, *Biometrika*, vol. 12 (1918–1919), footnote, p. 270. This expression is obtained from the moment-generating function. Obviously this method is exceedingly impractical for numerical calculations.

[3] V. Romanovsky, "Note on the moments of the binomial $(p + q)^n$ about its mean," *Biometrika*, vol. 15 (1923). Recently this expression was given a simple proof by A. T. Craig (*Bull. Amer. Math. Soc.*, vol. 40, pp. 262–264) and extended to the Poisson case.

[4] W. J. Kirkham, "Moments about the arithmetic mean of a binomial frequency distribution," *Annals of Mathematical Statistics*, vol. VI, pp. 96–101.

2. It is the purpose of this paper to express the s-th moment about the arithmetic mean in the form

$$\mu_s = \sum_{t=1}^{t=s} F_{s,t}(n)p^t, \tag{4}$$

where $F_{s,t}(n)$ are determinable functions of n dependent on s and t. We note here that p and q are the probabilities of the success and failure of an event in a single trial.

Since we know that $\mu_2 = npq$ and $\mu_1 = 0$, it is evident that the part of (2) enclosed in [ ] will be of degree 2 less than $s + 1$ in p and hence (4) will satisfy as a representation of the moment.

3. To obtain a recursion formula for the functions $F_{s,t}(n)$ we differentiate (4) with respect to p. This gives

$$\frac{d\mu_s}{dp} = \sum_{t=1}^{s} tF_{s,t}(n)p^{t-1}.$$

By (2) we may then write

$$\begin{aligned}\sum_{t=1}^{s+1} F_{s+1,t}(n)p^t &= p(1-p)ns\sum_{t=1}^{s-1} F_{s-1,t}(n)p^t + p(1-p)\sum_{t=1}^{s} tF_{s,t}(n)p^{t-1} \\ &= ns\sum_{t=2}^{s} F_{s-1,t-1}(n)p^t - ns\sum_{t=3}^{s+1} F_{s-1,t-2}(n)p^t \\ &\quad + \sum_{t=1}^{s} tF_{s,t}(n)p^t - \sum_{t=2}^{s+1}(t-1)F_{s,t-1}(n)p^t.\end{aligned}$$

Since this is an identity in p, we have immediately the following recursion formula for determining $F_{s,t}(n)$:

$$\begin{aligned}F_{s,t}(n) &= n(s-1)F_{s-2,t-1}(n) - n(s-1)F_{s-2,t-2}(n) + tF_{s-1,t}(n) \\ &\quad - (t-1)F_{s-1,t-1}(n)\end{aligned} \tag{5}$$

in which

$$F_{0,0}(n) = 1; \quad \text{and} \quad F_{s,t}(n) = 0 \quad \text{for} \quad \begin{cases} t > s; \\ t < 1, s > 0; \\ t = 1, s = 1. \end{cases} \tag{6}$$

These definitions arise from the known values of the moments and the conditions imposed by the identity in p.

By means of (5) and (6) we are able to obtain very readily the values for $F_{s,t}(n)$ which are given in Table 1.

TABLE I

Values of $F_{s,t}(n)$

s	$F_{s,1}(n)$	$F_{s,2}(n)$	$F_{s,3}(n)$	$F_{s,4}(n)$
1	0	0	0	0
2	n	$-n$	0	0
3	n	$-3n$	$2n$	0
4	n	$-7n + 3n^2$	$12n - 6n^2$	$-6n + 3n^2$
5	n	$-15n + 10n^2$	$50n - 40n^2$	$-60n + 50n^2$
6	n	$-31n + 25n^2$	$180n - 180n^2 + 15n^3$	$-390n + 415n^2 - 45n^3$
7	n	$-63n + 56n^2$	$602n - 686n^2 + 105n^3$	$-2100n + 2590n^2 - 525n^3$
8	n	$-127n + 119n^2$	$1932n - 2394n^2 + 490n^3$	$-10206n + 13895n^2 - 3850n^3 + 105n^4$

s	$F_{s,5}(n)$	$F_{s,6}(n)$
1	0	0
2	0	0
3	0	0
4	0	0
5	$24n - 20n^2$	0
6	$360n - 390n^2 + 45n^3$	$-120n + 130n^2 - 15n^3$
7	$3360n - 4270n^2 + 945n^3$	$-2520n + 3234n^2 - 735n^3$
8	$25200n - 35700n^2 + 10990n^3 - 420n^4$	$-31920n + 46004n^2 - 14770n^3 + 630n^4$

s	$F_{s,7}(n)$	$F_{s,8}(n)$
1	0	0
2	0	0
3	0	0
4	0	0
5	0	0
6	0	0
7	$720n - 924n^2 + 210n^3$	0
8	$20160n - 29232n^2 + 9520n^3 - 420n^4$	$-5040n + 7308n^2 - 2380n^3 + 105n^4$

With this table it is a relatively easy task to evaluate the first eight moments with the aid of a calculating machine.

4. As an illustration of the preceding we propose to evaluate the first eight moments about the arithmetic mean for the binomial, $(.06785 + .93215)^{378}$ We first evaluate the coefficients $F_{s,t}(n)$.

TABLE II[5]

Values of $F_{s,t}(378)$

s	$F_{s,1}(378)$	$F_{s,2}(378)$	$F_{s,3}(378)$	$F_{s,4}(378)$	$F_{s,5}(378)$
1	0	0	0	0	0
2	378	−378	0	0	0
3	378	−1,134	756	0	0
4	378	426,006	−852,768	426,384	0
5	378	1,423,170	−5,696,460	7,121,520	−2,848,608
6	378	3,560,382	784,501,200	−2,371,307,400	2,374,868,160
7	378	7,977,690	5,573,275,090	−27,986,054,000	50,430,749,000
8	378	16,955,190	26,123,640,500	1,937,705,370,000	−7,986,171,610,000

s	$F_{s,6}(378)$	$F_{s,7}(378)$	$F_{s,8}(378)$
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	−791,622,720	0	0
7	−39,236,327,400	11,210,379,300	0
8	12,070,808,800,000	−8,064,644,270,000	2,016,161,070,000

Then running off the powers of p, we have:

$$p = .067\ 85 \qquad p^5 = .000\ 001\ 437\ 968\ 13$$
$$p^2 = .004\ 603\ 622\ 5 \qquad p^6 = .000\ 000\ 097\ 566\ 137\ 6$$
$$p^3 = .000\ 312\ 355\ 787 \qquad p^7 = .000\ 000\ 006\ 619\ 862\ 44$$
$$p^4 = .000\ 021\ 193\ 340\ 1 \qquad p^8 = .000\ 000\ 000\ 449\ 157\ 667$$

Applying (4) we have

[5] In this table, as well as in the one that follows, all values are correct to nine significant figures.

TABLE III

Values of $p^t F_{s,t}(378)$

s	2	3	4	5
$pF_{s,1}(378)$	25.6473	25.6473	25.6473	25.6473
$p^2F_{s,2}(378)$	−1.7401693	−5.2205079	1961.17087	6551.73743
$p^3F_{s,3}(378)$	0.	.2361410	−266.36702	−1779.32225
$p^4F_{s,4}(378)$	0.	0.	9.03650	150.92880
$p^5F_{s,5}(378)$	0.	0.	0.	−4.09621
$p^6F_{s,6}(378)$	0.	0.	0.	0.
$p^7F_{s,7}(378)$	0.	0.	0.	0.
$p^8F_{s,8}(378)$	0.	0.	0.	0.
μ_s	23.9071307	20.6629331	1729.48765	4944.89507

s	6	7	8
$pF_{s,1}(378)$	25.647	25.65	25.6
$p^2F_{s,2}(378)$	16390.655	36726.27	78055.3
$p^3F_{s,3}(378)$	245043.490	1740844.73	8159870.3
$p^4F_{s,4}(378)$	−50255.924	−593117.96	41066448.9
$p^5F_{s,5}(378)$	3414.985	72517.81	−11483860.3
$p^6F_{s,6}(378)$	−77.236	−3828.14	1177702.2
$p^7F_{s,7}(378)$	0.	74.21	−53386.8
$p^8F_{s,8}(378)$	0.	0.	905.6
μ_s	214541.617	1253242.57	38945760.8

This gives us the desired moments about the arithmetic mean of the binomial $(.06785 + .93215)^{378}$. These values may be rapidly checked by applying (3) to μ_8.

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