NOTE ON NUMERICAL EVALUATION OF DOUBLE SERIES[1]

1. The Euler-Maclaurin summation formula has been extended to two variables by Dr. Sheppard,[2] and Mr. Irwin,[3] to determine cubature formulas. A more complicated two-dimensional form was given by Baten[4] involving product polynomials, for which a remainder term was also calculated. The purpose of this note is to apply the simpler formula to the numerical evaluation of double series of positive terms. The method may be extended to multiple series of order $p > 2$. If the double series converges one may sum by rows (or columns), using the ordinary sum formula twice. The method is to take out a rectangular block of mn terms and then apply the formula to the remaining terms. By taking m and n sufficiently large one may cause the series resulting from the formula to converge sufficiently rapidly to obtain the sum to the desired number of decimal places. For practical work the error may be estimated because of the asymptotic character of the series involved in the Euler-Maclaurin formula.

Write this in the form

$$(1)\quad \sum_a^{s-1} f(x) = \int_a^s f(x)dx + \tfrac{1}{2}f(a) - \tfrac{1}{2}f(s) - \frac{f'(a) - f'(s)}{12} + \frac{f'''(a) - f'''(s)}{720} - \frac{f^{\text{V}}(a) - f^{\text{V}}(s)}{30240} + \frac{f^{\text{VII}}(a) - f^{\text{VII}}(s)}{1209600} - \cdots + (-1)^r B_r \frac{f^{(2r-1)}(a) - f^{(2r-1)}(s)}{(2r)!} + \cdots.$$

If $s \to \infty$ one has accordingly in the ordinary case of convergence

$$(2)\quad \sum_a^{\infty} f(x) = \int_a^{\infty} f(x)dx + \tfrac{1}{2}f(a) - \frac{f'(a)}{12} + \frac{f'''(a)}{720} - \frac{f^{\text{V}}(a)}{30240} + \cdots.$$

Now define $v(x) = \sum_{y=b}^{\infty} u(x, y) = \int_b^{\infty} u(x, y)\,dy + \tfrac{1}{2}u(x, b) - \frac{u_y(x, b)}{12} + \frac{u_{y^3}(x, b)}{720} - \cdots$ and $w(y) = \sum_{x=a}^{\infty} u(x, y) = \int_a^{\infty} u(x, y)dx + \tfrac{1}{2}u(a, y) - \frac{u_x(a, y)}{12} + \frac{u_{x^3}(a, y)}{720} - \cdots$, then $\sum_{x=1}^{\infty}\sum_{y=1}^{\infty} u(x, y) = \sum_{x=1}^{a-1}\sum_{y=1}^{b-1} u(x, y) + \sum_{x=1}^{\infty} v(x) + \sum_{y=1}^{b-1} w(y)$

$$(3)\quad = \int_1^{\infty} v(x)dx + \tfrac{1}{2}v(1) - \frac{v'(1)}{12} + \frac{v'''(1)}{720} - \frac{v^{\text{V}}(1)}{30240} + \cdots + \sum_{x=1}^{a-1}\sum_{y=1}^{b-1} u(x, y) + \int_1^b w(y)dy - \tfrac{1}{2}w(b) + \tfrac{1}{2}w(1) + \frac{w'(b) - w'(1)}{12} - \frac{w'''(b) - w'''(1)}{720} + \cdots.$$

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[1] Presented to the Society, Nov. 30, 1934.

[2] W. F. Sheppard, "Some Quadrature Formulae," Proc. London Math. Soc., Vol. xxxii, 1900.

[3] J. O. Irwin, "Tracts for Computers," No. X, Cambridge Univ. Press, 1923, On Quadrature and Cubature.

[4] W. D. Baten, "A Remainder for the Euler-Maclaurin summation formula in two independent variables," Amer. Journal of Math., Vol. 54, 1932, pp. 265–275.

Instead of this one may use $\sum_{x=1}^{a-1}\sum_{y=1}^{b-1} u(x, y) + \sum_{y=1}^{\infty} w(y) + \sum_{x=1}^{a-1} v(x)$. The scheme of the double series may be illustrated by a sketch of a quadrant of the xy-plane in which the point (x, y) represents the term $u(x, y)$.

Evidently by taking a combination of results from (3) one may evaluate quite readily such finite sums as $\sum_{x=p}^{q}\sum_{y=r}^{t} u(x, y)$ where q and t are large.

As an illustration of (3) consider $\sum\sum (x^2 + a^2y^2)^{-2}$. Here one needs to evaluate the integral of the summand. The transformations $x = ay \tan \theta$ and $y = 1/t$ lead to a form which may be integrated by parts. The more complicated form $\sum\sum (ax^2 + 2bxy + cy^2)^{-s}$ for the case in which $s > 3/2$, $a > 1$, might be handled by using $x = 1/t$ and approximate integration by Simpson's rule.

Take as a second example $\sum\sum (x + y)^{-p}$, $p > 2$. The case of $p = 4$ was carried out by taking $a = b = 10$ in (3) and carrying the computation to twelve decimals. The series involved converge rapidly and a result was obtained which differed by 2 in the 12th place from the true value $0.119\ 733\ 669\ 448^+$. By summing diagonally one may convert this to the simple series $\sum_{1}^{\infty} z(z + 1)^{-4}$ or $\sum_{2}^{\infty} (s - 1)s^{-4} = \sum_{1}^{\infty} (s^{-3} - s^{-4})$. The method of summation diagonally may be extended to $\sum\sum (x + ay)^{-p}$, $p > 2$, $a > 0$, by the applications of the Euler-Maclaurin sum formulas (1), (2) in succession after a triangular array of terms have been omitted.

The form $\sum\sum x^{-p}y^{-q}$ can be written as the product of the single series $(\sum x^{-p})(\sum y^{-q})$.

2. Another method of numerical evaluation is the analog of that used for single series by the author.[5] Instead of rectangles one has right prisms of square or rectangular cross-section. Instead of shifting the rectangles one unit to the right to determine upper and lower bounds the prisms are shifted diagonally so that they go effectively one unit in each variable. In the case of a square base each prism is moved along the 45° line one diagonal unit length. For the lower bound instead of trapezoids one uses truncated prisms. For example, the prism of height u_{mn} is cut by two planes, one determined by the upper vertices u_{mn}, $u_{m,n+1}$, $u_{m+1,n}$ and the other by the upper vertices $u_{m+1,n}$, $u_{m,n+1}$, $u_{m+1,n+1}$ of the truncated prism. The surface $z = u(m, n)$ passes through all the upper corners of the truncated prisms. Each prism is composed of two truncated triangular prisms. Now the volume of such a triangular prism is the arithmetic mean of its vertical edges multiplied by the area of its base.

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[5] "A New Method for Finding the Numerical Sum of an Infinite Series," Amer. Math. Monthly, vol. XL, No. 9, Nov., 1933, pp. 537–542.

Hence the difference in volume between the truncated rectangular prism mentioned above and the prism of uniform hight $z = u_{mn}$ can be shown to be

$$(4)\qquad (5u_{mn} - u_{m+1,\,n+1} - 2u_{m+1,\,n} - 2u_{m,\,n+1})/6.$$

Let us consider series whose corresponding surfaces do not rise above these truncated prisms. This sort of truncated prism differs less from the volume under the surface than the one formed by the diagonal joining the other pair of upper vertices and planes through it for upper faces. The lower bound for the remainder is the volume under the surface extending to infinity in the m and n directions plus the sum of these differences. Accordingly one determines as the lower bound for the remainder $R_{m-1,\,n-1}$ after summing a rectangular array $\sum_{i=1}^{m-1}\sum_{j=1}^{n-1} u_{i,\,j}$ the form

$$(5)\qquad \begin{aligned}&(2u_{m,1} + 2u_{1,n} - 5u_{m,n})/6 + \tfrac{1}{2}\sum_{i=1}^{\infty} u_{m+i,1} + \tfrac{1}{2}\sum_{j=1}^{\infty} u_{1,n+j} + \tfrac{1}{2}\sum_{i=1}^{m} u_{i,n}\\ &+ \tfrac{1}{2}\sum_{j=1}^{n} u_{m,j} + \int_1^{\infty}\int_m^{\infty} u_{m,n}\,dmdn + \int_n^{\infty}\int_1^{m} u_{m,n}\,dmdn < R_{m-1,n-1}.\end{aligned}$$

The upper bound may likewise be given as follows:

$$(6)\qquad R_{m-1,n-1} < S + T + \int_{n-1}^{\infty}\int_{m-1}^{\infty} u_{m,n}\,dmdn - k\left[\sum_{j=n-1}^{\infty} u_{m-1,j} + \sum_{i=m}^{\infty} u_{i,n-1}\right]$$

where

$$(7)\qquad S = \sum_{i=m}^{\infty}\sum_{j=1}^{n-1} u_{ij}, \qquad T = \sum_{j=n}^{\infty}\sum_{i=1}^{m-1} u_{ij},$$

$$(8)\qquad k = \frac{\int_{n-1}^{\infty}\int_{m-1}^{\infty} u_{m,n}\,dmdn - \int_n^{\infty}\int_m^{\infty} u_{m,n}\,dmdn - \sum_{j=n-1}^{\infty} u_{m-1,j} - \sum_{i=m}^{\infty} u_{i,m-1}}{u_{m-1,n-1} + u_{m,n-1}}.$$

An alternate definition of k is

$$(9)\qquad k' = \left[\int_{n-1}^{n}\int_{m-1}^{m} u_{m,n}\,dmdn - u_{m,n}\right] \div (u_{m-1,n-1} - u_{m,n}).$$

An illustration is afforded by $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (m + 1)^{-4}$ for which $k = .45614$, $k' = .44536$ when $m = n = 10$ in (8), (9). In this case (5) gave an error of -14×10^{-6} and (6) an error of 10^{-6}.

S and T may be evaluated by the method published in the Monthly.[6]

One must assume that k increases with m and n. It is evident that for this

[6] Loc. cit.

method and for the one in the Monthly differentiability is not required but only integrability, conditions less restrictive than those required by the Euler-Maclaurin summation formulas. It is also clear that the method may be extended to multiple series of positive terms of multiplicity greater than two.

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