ON TESTING THE HYPOTHESIS THAT TWO SAMPLES HAVE BEEN DRAWN FROM A COMMON NORMAL POPULATION

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1. **Introduction.** This paper is devoted to the problem of testing the hypothesis that two samples of 2, 3 and 4 variables, and of equal size, have been drawn from a common unspecified normal population. It is, in a certain sense, a continuation of J. W. Fertig's papers [1, 2] which were devoted to the problem of testing the hypothesis that one or more samples of n variables have been drawn from a completely or partially specified normal population.

For the sake of application to biological research, it is important to have means of determining whether two samples may have come from a common population even if this population is unknown. Moreover, it is often imperative to test two samples with respect to all their variables simultaneously. Much valuable information may be lost if the variables are tested individually. One has to consider not only the fact that two samples which differ almost significantly from each other in each variable separately might be significantly different if the probabilities would be combined, but one has to take account of the possible correlations between the variables which are completely disregarded if the tests are applied to each variable separately. It is not difficult to imagine two samples of two variables with identical means and variances and significantly different correlation coefficients.

J. Neyman and E. S. Pearson [3] have investigated the problem of testing statistical hypotheses in general. They have developed the method of likelihood ratios. It is beyond the scope of the present paper to give an account of this theory; we have to restrict ourselves to statements concerning the fundamental concepts we are going to apply to our specific problem.

A sample with one variable and of size N can be regarded as a point in an N-dimensional space. The acceptance or rejection of a hypothesis concerning this sample will depend on whether or not the point representing the sample is contained in certain critical regions determined by the hypothesis and by the statistical criterion that is to be applied. The choice of the critical regions is of fundamental importance; its implications have been thoroughly discussed by Neyman and Pearson. These authors found a useful criterion for testing the hypothesis that a sample was drawn from a specified member of an admissible set of populations by introducing the ratio of the likelihood that the sample was drawn from the specified population to the maximum value of the likelihood for all populations in the admissible set (Cf. §2). This ratio λ can vary between

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0 and 1. The association between values of λ and the credibility of the hypothesis in question is such that the greater the value of λ the greater the degree of tenability of the hypothesis. λ = constant defines a surface in the sample space. These surfaces are the contours of the critical regions associated with the acceptance or rejection of the hypothesis. A hypothesis is rejected as untenable if λ is so small that

$$\int_0^\lambda P(\lambda)\, d\lambda < \alpha,$$

where α is some value arbitrarily small, say .01 or .05 and $P(\lambda)$ is the distribution of λ if the hypothesis is true.

This method of testing hypotheses is evidently not restricted to one sample with one variable, nor is it restricted to *simple* hypotheses. A simple hypothesis is one which is associated with one completely specified population. A composite hypothesis is one which is associated with a subset of the admissible populations. For example, the hypothesis that a sample with n variables has been drawn from a normal population with means $a_1, a_2, \cdots a_n$ whatever may be the variances and correlation coefficients is a composite hypothesis. Such is the hypothesis that two or more samples have come from a common but unspecified population.

The problem of several samples with one variable was discussed by Neyman and Pearson [4, 5]; the problem of several samples with two variables by Pearson and Wilks [6]. In another paper Wilks [7] derived formulas for λ and the moments of $P(\lambda)$ for the most general case of k samples of n variables. For the sake of practical applications it is necessary to have tables for the limits of significance of λ. Such tables have been prepared for samples with one variable by Neyman and Pearson and for completely or at least partially specified populations and more than one variable by Fertig [1, 2]. The present paper contains tables for the case of 2, 3 and 4 variables and a common unspecified normal population. Since the case of two variables has been theoretically solved by Pearson and Wilks we shall have to compare our results with those of the above authors who derived the distribution of $P(\lambda)$ but did not compute tables.

Our procedure is the following: We start with the moments of $P(\lambda)$ as given by Wilks and approximate the distribution of $\lambda^{1/N}$ by a suitable function. Then we compute the limits of significance for this approximating function. This procedure was originally suggested by Neyman and Pearson and was applied with some modifications by Fertig.

§2 contains the definition of the likelihood ratio λ; §3 deals with the moments of its distribution for the case of a common unspecified population. In §4 we introduce the approximating distribution $y = Cx^{p-1}(1 - x)^{q-1}$ and discuss the determination of the parameters p and q. In §5 we give an independent derivation of the formula obtained by Pearson and Wilks for $P(\lambda)$ for the case of two samples with two variables and compare our approximation with the exact formula. §6 deals with the determination of the limits of significance and contains the tables. §7 is devoted to an example.

2. Definition of λ. Let C_π denote the probability of obtaining a given sample from a population π. C will depend on the parameters of the population and the data of the sample. Let Ω be the set of all admissible populations and ω a subset of Ω which corresponds to a certain hypothesis that is to be tested. Intuitively one would consider a hypothesis tenable or plausible if it gives a high probability density for the given sample if compared with other possible hypotheses. Following this reasoning Neyman and Pearson defined the likelihood of a hypothesis as the ratio of $\underset{\pi \epsilon \omega}{\text{Max}}\, C_\pi$ to $\underset{\pi \epsilon \Omega}{\text{Max}}\, C_\pi$. In the special case which we propose to investigate, the populations are assumed to be normal. We wish to test the hypothesis that two given samples have come from a *common* unspecified population. Hence λ, the likelihood of this hypothesis, is the maximum likelihood that the samples have come from a common normal population divided by the maximum likelihood that the samples have come from any two normal populations.

The value of λ can be expressed by the variates of the samples by means of the following formula [Cf. [7] p. 489]

$$\lambda = \left[\frac{S_1}{S_0}\right]^{\frac{1}{2}N_1} \left[\frac{S_2}{S_0}\right]^{\frac{1}{2}N_2}, \tag{1}$$

where S_1 and S_2 are the generalized variances[2] of the samples and S_0 is the generalized variance of the sample obtained by pooling the two given samples. N_1 is the size of the first sample, N_2 the size of the second. In case of equal samples to which we shall restrict ourselves $N_1 = N_2 = N$; thus

$$\lambda^{1/N} = \frac{S_1^{\frac{1}{2}} S_2^{\frac{1}{2}}}{S_0}. \tag{2}$$

3. The Moments of the λ Distribution. The distribution of λ depends on the number of variables, the number and the size of the samples and on the kind of hypothesis that is to be tested; e.g. that the samples have come from a common unspecified population. This distribution has been evaluated for the case of equal samples of one and two variables and our hypothesis concerning a common unspecified population. The general form of this distribution is still unknown and even the known formula for two variables is not very suitable for computation. Therefore we shall follow the procedure introduced by Neyman and Pearson [4] and we shall use the known moments of the unknown distribution

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[2] The generalized variance of a sample is a determinant, the elements of which are the variances and covariances. Thus, for two variables x and y the generalized variance $S = S_x^2 S_y^2(1 - r^2)$; where S_x^2 and S_y^2 denote the variances of x and y respectively, r the correlation coefficient.

function $P(\lambda)$ in order to construct an approximation to $P(\lambda)$. For two equal samples of n variables the moments of $P(\lambda)$ about the origin are [Cf. [7] p. 490]

$$(3) \qquad M_h = 2^{nNh} \prod_{i=1}^{n} \left\{ \frac{\Gamma\left(\frac{N(1+h)-i}{2}\right)}{\Gamma\left(\frac{N-i}{2}\right)} \right\}^2 \frac{\Gamma\left(\frac{2N-i}{2}\right)}{\Gamma\left(\frac{2N(1+h)-i}{2}\right)}$$

$$\text{for } h = 1, 2, 3, \cdots.$$

Equation (2) readily suggests that we should compute or approximate the distribution of $\lambda^{1/N}$ rather than that of λ. Let μ_2 denote the h-th moment of $P(\lambda^{1/N})$ then $\mu_h = M_{h/N}$ follows immediately from the definition of $M_h = \int_0^1 \lambda^h P(\lambda) d\lambda$. Hence in order to obtain the μ's we have to replace Nh by h in (3).

$$(4) \qquad \mu_h = 2^{nh} \prod_{i=1}^{n} \left\{ \left[\frac{\Gamma\left(\frac{N+h-i}{2}\right)}{\Gamma\left(\frac{N-i}{2}\right)} \right]^2 \frac{\Gamma\left(\frac{2N-i}{2}\right)}{\Gamma\left(\frac{2N+2h-i}{2}\right)} \right\}.$$

This expression can be much simplified for all given values of h and n. However, there is no need for such simplification, because one has to compute the first moments only. All higher moments can be expressed by means of the first moments for various N's. The dependence of μ_h on N is evident from (4), we shall indicate it by writing $\mu_h(N)$. The ratio of two subsequent moments is

$$(5) \quad \frac{\mu_{h+1}(N)}{\mu_h(N)} = 2^n \prod_{i=1}^{n} \left\{ \left[\frac{\Gamma\left(\frac{N+h+1-i}{2}\right)}{\Gamma\left(\frac{N+h-i}{2}\right)} \right]^2 \frac{\Gamma\left(\frac{2(N+h)-i}{2}\right)}{\Gamma\left(\frac{2(N+h+1)-i}{2}\right)} \right\}$$
$$= \mu_1(N+h).$$

Equation (5) contains an important relation of the moments. In fact from (5) follows:

$$(6) \qquad \begin{cases} \mu_2(N) = \mu(N)\mu(N+1), \\ \mu_3(N) = \mu(N)\mu(N+1)\mu(N+2), \\ \vdots \\ \mu_h(N) = \mu(N)\mu(N+1) \cdots \mu(N+h-1), \end{cases}$$

where the 1's from $\mu_1(N)$'s have been omitted. This last group of equations holds for any number of variables. Thus we have to compute $\mu(N)$ for each N, then multiplication gives the higher moments.

For[3] $n = 2$, $$\mu(N) = \frac{(N-2)^2}{(N-1)(N-\frac{1}{2})}$$

For $n = 3$, $$\mu(N) = 2^3\left[\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-3}{2}\right)}\right]^2 \frac{1}{(N-\frac{1}{2})(N-1)(N-\frac{3}{2})}.$$

For $n = 4$, $$\mu(N) = \frac{(N-2)^2(N-4)^2}{(N-1)(N-\frac{1}{2})(N-2)(N-\frac{3}{2})}.$$

4. **Approximation of the distribution of $\lambda^{1/N}$.** Following the procedure of Neyman and Pearson we shall use the moments computed in the previous section for the fitting of a Pearson frequency curve to the unknown distribution $P(\lambda^{1/N})$. Since $0 \leqq \lambda \leqq 1$ it is natural to fit a frequency curve of the following type

$$y = Cx^{p-1}(1-x)^{q-1}, \tag{7}$$

where $$C = \frac{1}{B(p, q)} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}.$$

The first two moments are sufficient to determine the two parameters p and q. The moments of the distribution (7) are readily computed:

$$\begin{aligned} \nu_1 &= \frac{p}{p+q}, \\ \nu_2 &= \nu_1 \frac{p+1}{p+q+1} = \frac{p}{p+q} \cdot \frac{p+1}{p+q+1}. \end{aligned} \tag{8}$$

In general:

$$\frac{\nu_{h+1}}{\nu_h} = \frac{p+h}{p+q+h}. \tag{9}$$

Equation (9) corresponds to equation (5) since one can write $\nu = \nu(p, q)$, then (9) becomes

$$\frac{\nu_{h+1}(p, q)}{\nu_h(p, q)} = \nu(p+h, q). \tag{10}$$

At first sight the similarity of equations (5) and (10) would suggest that one should choose p and q so that $\nu(p+h, q) = \mu(N+h)$, for $h = 0, 1, 2, 3, \cdots$. However, this cannot be done because the equations which express the equality of the first two moments:

$$\nu(p, q) = \mu(N) \tag{11}$$

$$\nu(p+1, q) = \mu(N+1) \tag{12}$$

[3] The case $n = 1$ is omitted here since it has been treated by Neyman and Pearson [4].

determine p and q *completely*. The quantities $\nu(p + h, q)$ can be only *approximately* equal to $\mu(N + h)$ for $h > 1$. The goodness of fit may be tested by comparing the third and fourth moments.

The advantage of equations (5) and (10) is that once p and q have been computed one does not have to compare

$$\nu_3 = \nu_2 \nu(p + 3, q)$$

with

$$\mu_3 = \mu_2 \mu(N + 3),$$

but since $\nu_2 = \mu_2$ one can compare $\nu(p + 3, q)$ with $\mu(N + 3)$. Similarly the comparison of the fourth moments can be replaced by comparing $\nu(p + 4, q)$ with $\mu(N + 4)$. It has to be remembered that once the sequence of $\mu(N)$'s has been computed for all N's, each of its terms can be used four times in the determination and the checking of p's and q's.

The general procedure for the determination of p and q is to compute the $\mu(N)$'s first and then solve the equations (11) and (12): i.e.

$$\frac{p}{p + q} = \mu(N), \tag{13}$$

$$\frac{p + 1}{p + q + 1} = \mu(N + 1). \tag{14}$$

The solution of these equations is:

$$p = \mu(N) \frac{1 - \mu(N + 1)}{\mu(N + 1) - \mu(N)}, \tag{15}$$

$$q = \left[\frac{1}{\mu(N)} - 1\right] p. \tag{16}$$

As N increases $\mu(N)$ approaches 1 from below; $\mu(N + 1) - \mu(N)$ will be very small. E.g., for $n = 2$ as N varies from 30 to 50 $\mu(N)$ increases from .9164 to .9499. It is easily seen that small errors in μ may produce much larger errors in p and q. For $n = 4$ it was necessary to compute μ to nine decimal places to get p and q to three decimal places accurately. For $n = 2$ equations (13) and (14) become

$$\frac{p}{p + q} = \frac{(N - 2)^2}{(N - 1)(N - \frac{1}{2})}, \tag{17}$$

$$\frac{p + 1}{p + q + 1} = \frac{(N - 1)^2}{N(N + \frac{1}{2})}. \tag{18}$$

These can be solved explicitly

$$p = (N - 2)\left[1 - \frac{3(N - 1)}{5N^2 - 9N + 1}\right], \tag{19}$$

$$q = 2.5 + \frac{4.5}{5N^2 - 9N + 1} \tag{20}$$

The last two equations enable us to compute p and q directly and thus avoid the more laborious computation by means of the $\mu(N)$'s. For $n = 3$ and 4, however, such a short cut was not found. The computation of μ's for $n = 4$ was facilitated by the following relation:

$$_4\mu(N) = {_2\mu(N)} \frac{(N-4)^2}{(N-2)(N-\frac{3}{2})}, \tag{21}$$

where the suffices denote the number of variables in the problem to which the μ's refer. Thus the computed values of $_2\mu(N)$ were again used. Eight-place logarithms were used in the computation of $_3\mu(N)$ from the formula at the end of §3.

5. **The distribution of $\lambda^{1/N}$ for two variables.** For two variables it is possible to evaluate the distribution of $\lambda^{1/N}$ or some other suitable power of λ directly from the moments. Pearson and Wilks (Cf. [6] pp. 364–368) derived the distribution of $\lambda^{2/N}$ for this case. Their method was adapted to the treatment of more general problems than ours. It is possible to derive the distribution of $\lambda^{1/N}$ in our special case more directly:

For $n = 2$ the moments of $\lambda^{1/N}$ are:

$$\mu_h = 2^{2h}\left[\frac{\Gamma\left(\frac{N+h-1}{2}\right)\Gamma\left(\frac{N+h-2}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-2}{2}\right)}\right]^2 \frac{\Gamma(N-\frac{1}{2})\Gamma(N-1)}{\Gamma(N+h-\frac{1}{2})\Gamma(N+h-1)}, \tag{22}$$

$$h = 1, 2, 3, \cdots.$$

Applying the following transformation formula[4]

$$\Gamma(z)\Gamma(z+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2z-1}}\Gamma(2z) \tag{23}$$

to

$z_1 = \frac{1}{2}(N+h-2)$, $\quad z_2 = \frac{1}{2}(N-2)$, $\quad z_3 = N-1$ and $z_4 = N+h-1$

(22) becomes

$$\mu_h = 2^{2h}\left[\frac{\Gamma(N+h-2)}{\Gamma(N-2)}\right]^2 \frac{\Gamma(2N-2)}{\Gamma(2N+2h-2)}. \tag{24}$$

Thus μ_h is of the form $F(N+h)/F(N)$ with

$$F(N) = 2^{2N}\frac{\Gamma(N-2)^2}{\Gamma(2N-2)}.$$

[4] Cf. Whittaker and Watson. Modern Analysis, 4th ed., p. 240.

Our problem is to find a function $P(t)$ such that

$$\mu_h = \int_0^1 t^h P(t)\, dt, \tag{25}$$

$$\text{for } h = 0, 1, 2, \cdots .$$

This problem is solved if we can find a function of N and t, say, $p(N, t)$ such that

$$F(N + h) = \int_0^1 t^h p(N, t)\, dt \tag{26}$$

$$\text{for } h = 0, 1, 2, \cdots .$$

N and h enter the left side of equation (26) symmetrically. The same must be true for the right side. Hence $p(N, t)$ must have the form $t^N f(t)$ where $f(t)$ is independent of N. If then (26) is satisfied for all N and $h = 0$, it is also satisfied for all N and all h.

Let us now examine $F(N)$. Applying again the transformation (23) we can bring it to the form:

$$\begin{aligned} F(N) &= 2^3 \frac{\Gamma(N-2)\sqrt{\pi}}{\Gamma(N-\frac{1}{2})(N-2)} = 2^4 \frac{\Gamma(N-2)\Gamma(\frac{3}{2})}{(N-2)\Gamma(N-\frac{1}{2})} \\ &= \frac{2^4}{N-2} B(N-2, \tfrac{3}{2}). \end{aligned} \tag{27}$$

Now $B(N - 2, \frac{3}{2})$ can be represented as an integral of the desired type

$$B(N-2, \tfrac{3}{2}) = \int_0^1 t^{N-3}(1-t)^{\frac{1}{2}}\, dt. \tag{28}$$

We set $p(N, t) = 2^4 t^{N-3} g(t)$ and seek to determine $g(t)$ so that (26) will be satisfied for all N and for $h = 0$: i.e.,

$$F(N) = \frac{2^4}{N-2} B(N-2, \tfrac{3}{2}) = 2^4 \int_0^1 t^{N-3} g(t)\, dt. \tag{29}$$

An integration by parts with $g(1) = 0$ gives

$$\frac{2^4}{N-2} B(N-2, \tfrac{3}{2}) = -\frac{2^4}{N-2} \int_0^1 t^{N-2} g'(t)\, dt. \tag{30}$$

This equation evidently holds for all N if and only if

$$-tg'(t) = \sqrt{1-t}. \tag{31}$$

This differential equation is readily solved by the substitution of $y = 1 - t$. In fact it becomes

$$\frac{dg(y)}{dy} = \frac{2y^2}{1-y^2} = 2[y^2 + y^4 + y^6 + \cdots]. \tag{32}$$

Hence

$$g = \log \frac{1+y}{1-y} - 2y = 2\left[\log \frac{1+\sqrt{1-t^2}}{\sqrt{t}} - \sqrt{1-t^2}\right]. \tag{33}$$

The complete solution for the distribution function is

$$dP(t) = \frac{\Gamma(2N-2)}{2^{2N-4}[\Gamma(N-2)]^2}\, t^{N-3}\left[\log \frac{1+\sqrt{1-t^2}}{\sqrt{t}} - \sqrt{1-t^2}\right] dt \tag{34}$$

with $t = \lambda^{1/N}$ in accordance with the formula of Pearson and Wilks.[5] Integration by parts gives

$$\begin{aligned} P(\lambda^{1/N} < t) &= \frac{\Gamma(2N-2)}{2^{2N-5}\Gamma(N-1)\Gamma(N-2)} \cdot \\ &\left\{t^{N-2}\left[\log \frac{1+\sqrt{1-t^2}}{\sqrt{t}} - \sqrt{1-t^2} + \tfrac{1}{2}\int_0^t y^{N-3}\sqrt{1-y}\,dy\right\}.\right. \end{aligned} \tag{35}$$

One can use this last equation to determine the limits of significance. However, this was not done when the tables of this paper were computed. The approximation of the distribution function by the function described in equation (7) was deemed sufficient and the use of the tables of the incomplete beta function greatly facilitated the computation.

In concluding this section we wish to demonstrate the goodness of approximation of the exact distribution function by a function of the type $Ct^{p-1}(1-t)^{q-1}$ with p and q given by equations (19) and (20).

For small values of t the shape of the curve is determined by the exponent of t, which is exactly $N - 3$ for the distribution function and nearly $N - 3 - \frac{3}{8}$ for the approximating function. For large t; i.e., small $1 - t$, the exponent of $(1 - t)$ is the determining factor. By (32) we have

$$g(\sqrt{1-t}) = 2\left[\frac{(1-t)^{\frac{3}{2}}}{3} + \frac{(1-t)^{\frac{5}{2}}}{5} + \cdots\right],$$

or approximately $\frac{2}{3}(1-t)^{\frac{3}{2}}$. For the approximating curve $q - 1 = \frac{3}{2} + O(1/N^2)$ which is even better agreement. It is easily seen that the goodness of approximation increases with N.

6. Determination of the Levels of Significance. The final task was to compute the values of x which satisfy the equations:

$$I_x(p, q) = \frac{1}{B(p, q)}\int_0^x t^{p-1}(1-t)^{q-1}\,dt = \alpha,$$

with $\alpha = .01$ and $\alpha = .05$. This was done by interpolation in the *Tables of the Incomplete Beta Function* [8]. In these tables the argument x increases by steps of .01. A value x_0 was determined by inspection, so that $I_{x_0}(p, q) < \alpha$ but $I_{x_1}(p, q) > \alpha$ where $x_1 = x_0 + .01$. The values of $I_{x_0}(p, q)$ and $I_{x_1}(p, q)$ were

[5] Cf. [6] p. 368 Equation 60. ($l^2 = t$).

determined by interpolation with respect to p and q, using the two-dimensional Everett formula, neglecting fourth and higher differences. x was then determined by linear interpolation. It is worth while to mention that the terms of second order in Everett's formula decreased quite rapidly as N increased. Once this was noticed some labor has been saved by not computing the terms of second order for values of N between 30 and 50 but by estimating the second order terms from those obtained for $N = 30, 40$ and 50.

Levels of Significance of $\lambda^{1/N}$

Sample Size N	2 Variables		3 Variables		4 Variables	
	1%	5%	1%	5%	1%	5%
10	.395	.507	.238	.328	.122	.184
11	.437	.546	.282	.374	.153	.217
12	.475	.579	.323	.414	.198	.270
13	.508	.610	.360	.451	.233	.308
14	.537	.634	.393	.483	.267	.343
15	.563	.656	.423	.512	.298	.375
16	.586	.676	.451	.537	.328	.405
17	.607	.694	.476	.561	.355	.432
18	.626	.710	.500	.582	.380	.456
19	.644	.724	.521	.601	.404	.479
20	.660	.737	.541	.619	.426	.501
22	.687	.760	.576	.650	.466	.538
24	.711	.779	.606	.676	.501	.571
26	.731	.795	.632	.699	.532	.599
28	.749	.809	.655	.719	.560	.624
30	.765	.821	.675	.736	.584	.646
32	.778	.832	.694	.752	.606	.666
34	.791	.842	.710	.765	.626	.683
36	.802	.850	.724	.778	.644	.700
38	.811	.858	.738	.789	.660	.714
40	.820	.865	.750	.799	.676	.727
42	.828	.871	.761	.808	.689	.739
44	.836	.877	.771	.816	.702	.750
46	.843	.882	.780	.824	.713	.760
48	.849	.887	.789	.831	.724	.769
50	.854	.891	.796	.837	.734	.773

7. **An Example.** The problem chosen to illustrate the use of the tables is taken from a study on insulin-treated schizophrenic patients of the Worcester State Hospital. It was attempted to differentiate between those patients who recovered after treatment and those who did not recover. Blood constituents and blood pressure were determined among other variables.

The variables in this example are designated as x = blood phosphorus, y = cholesterol in mg./100 cc., z = blood pressure in mm. Hg. The statistics for the 10 "recovered" patients are:

$$S_x^2 = 2.222 \qquad S_y^2 = 376.50 \qquad S_z^2 = 51.97$$

$$r_{12}S_xS_y = -1.121 \qquad r_{13}S_xS_z = -8.217 \qquad r_{23}S_yS_z = 12.51$$

For "not-recovered" 10 patients

$$S_x^2 = 3.120 \qquad S_y^2 = 816.19 \qquad S_z^2 = 96.32$$

$$r_{12}S_xS_y = 26.23 \qquad r_{13}S_xS_z = 2.92 \qquad r_{23}S_yS_z = 65.78$$

For the total group of 20

$$S_x^2 = 3.034 \qquad S_y^2 = 609.02 \qquad S_z^2 = 83.09$$

$$r_{12}S_xS_y = 10.41 \qquad r_{13}S_xS_z = -.845 \qquad r_{23}S_yS_z = 15.99$$

These values give for the sample variances 17,462; 168,628; and 143,904, respectively.
Hence

$$\lambda^{1/10} = \frac{\sqrt{17,462 \times 168,628}}{143,904} = .377.$$

The 5% limit of significance is .328, hence the two groups do not differ significantly from each other.

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