tion $\theta_i = \theta_i(\theta'_1, \cdots, \theta'_l)$ $(i = 1, \cdots, l)$ leads to the equally valid representation of the family

$$p'(x_1, \cdots, x_n \mid \theta'_1, \cdots, \theta'_l) = p[x_1, \cdots, x_n \mid \theta_1(\theta'_1, \cdots, \theta'_l), \cdots, \theta_l(\theta'_1, \cdots, \theta'_l)].$$

Is a set of statistics sufficient with respect to the first representation also sufficient with respect to the second? The answer is partly in the affirmative and is given by the following proposition.

THEOREM II. *If the set of algebraically independent statistics* $T_1, \cdots, T_m$ *is sufficient with regard to the parameters* $\theta_1, \cdots, \theta_q$ *and the probability law* $p(x_1, \cdots, x_n \mid \theta_1, \cdots, \theta_q, \cdots, \theta_l)$, *it is also sufficient with regard to* $\theta'_1, \cdots, \theta'_q$ *and any other representation* $p'(x_1, \cdots, x_n \mid \theta'_1, \cdots, \theta'_q, \cdots, \theta'_l)$ *of the same probability law provided* θ'_i $(i = 1, \cdots, q)$ *are independent functions of* $\theta_1, \cdots, \theta_q$ *only and* θ'_j $(j = q + 1, \cdots, l)$ *are functions of* $\theta_{q+1}, \cdots, \theta_l$ *only.*

PROOF: The proof of the theorem is obvious. We are given the fact that $p(x_1, \cdots, x_n \mid \theta_1, \cdots, \theta_q, \cdots, \theta_l) = p(T_1, \cdots, T_m \mid \theta_1, \cdots, \theta_q) \cdot \phi(x_1, \cdots, x_n; \theta_{q+1}, \cdots, \theta_l)$. Since the θ'_i $(i = 1, \cdots, q)$ are functions of $\theta_1, \cdots, \theta_q$ only and the θ'_j $(j = q + 1, \cdots, l)$ are functions of $\theta_{q+1}, \cdots, \theta_l$ only, it follows that $\theta_i = \theta_i(\theta'_1, \cdots, \theta'_q)$ $(i = 1, \cdots, q)$ and $\theta_j = \theta_j(\theta'_{q+1}, \cdots, \theta'_l)$ $(j = q + 1, \cdots, l)$. Consequently,

$$(4) \quad \begin{aligned} &p'(x_1, \cdots, x_n \mid \theta'_1, \cdots, \theta'_q, \cdots, \theta'_l) \\ &\qquad = p'(T_1, \cdots, T_m \mid \theta'_1, \cdots, \theta'_q) \cdot \phi'(x_1, \cdots, x_n; \theta'_{q+1}, \cdots, \theta'_l) \end{aligned}$$

and the theorem is established.

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NOTE ON THE MOMENTS OF A BINOMIALLY DISTRIBUTED VARIATE

BY W. D. EVANS

J. A. Joseph, has given two interesting triangular arrangements of numbers, the second of which is reproduced herewith as Table 1.[1] The successive rows in this table are the coefficients in the expansion of x^n as a function of the factorials $x^{(i)}$, using the notation of the calculus of finite differences. For example,

$$x^4 = x^{(4)} + 6x^{(3)} + 7x^{(2)} + x,$$

where

$$x^{(i)} = x(x - 1)(x - 2) \cdots (x - i + 1).$$

Joseph points out that the coefficients may be used to generate the numbers of Laplace.

[1] J. A. Joseph, "On the Coefficients of the Expansion of $X^{(n)}$," *Annals of Math. Stat.*, Vol. X (1939), p. 293.

A general expression defining any of the coefficients in terms of its place of occurrence in Table 1 may be set up. If we denote by $F_c(r)$ the number in row r and column c of the table, we have

$$F_c(r) = \sum_{1}^{r-c+1} k_1 \sum_{1}^{k_1} k_2 \sum_{1}^{k_2} k_3 \cdots \sum_{1}^{k_{c-2}} k_{c-1} \qquad (r \geq c). \tag{1}$$

This expression is of additional interest since the numbers defined by it are likewise the coefficients in the expression of the z-th moment about the origin of a binomially distributed variate in terms of the probability of the variate and the size of the sample in which it is contained. For example, it may be easily

TABLE 1

	1	2	3	4	5	$\cdots$	c
1	1						
2	1	1					
3	1	3	1				
4	1	6	7	1			
5	1	10	25	15	1		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
r	$F_1(r)$	$F_2(r)$	$F_3(r)$	$F_4(r)$	$F_5(r)$	$\cdots$	$F_c(r)$

verified that if α is such a variate, p its probability of occurrence, and n the size of the sample in which it is contained,

$$E(\alpha)^2 = n^{(2)}p^2 + np$$
$$E(\alpha)^3 = n^{(3)}p^3 + 3n^{(2)}p^2 + np$$
$$E(\alpha)^4 = n^{(4)}p^4 + 6n^{(3)}p^3 + 7n^{(2)}p^2 + np$$

and so on.

Ordinarily, computation of the higher moments of a binomially distributed variate is a tedious process of repeated differentiation. However, equation (1) immediately permits us to generalize the foregoing expressions to give the z-th moment of α as follows:

$$E(\alpha)^z = \sum_{i=0}^{z-1} n^{(z-i)} p^{z-i} \sum_{1}^{z-i} k_1 \sum_{1}^{k_1} k_2 \cdots \sum_{1}^{k_{i-1}} k_i. \tag{2}$$

It will be noted that when $c - 1$ in equation (1) and i in equation (2) are equal to zero, the repeated summations vanish to be replaced by the value one.

By means of equation (2) much of the labor usually involved in expressing the z-th moment about the origin of a binomially distributed variate in terms of n and p may be avoided.

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