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THE NUMERICAL COMPUTATION OF THE PRODUCT OF CONJUGATE IMAGINARY GAMMA FUNCTIONS

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The difference equation

$$\frac{f_{x+1}}{f_x} = \frac{x^2 + c_1 x + c_2}{x^2 + c_3 x + c_4} \tag{1}$$

was used by Professor Harry C. Carver [1] as the basis for graduating frequency distributions in a manner analogous to the use of the differential equation

$$\frac{1}{y}\frac{dy}{dx} = \frac{a - x}{b_0 + b_1 x + b_2 x^2}$$

in the Pearson system of frequency curves. In order to determine a particular f_x by Professor Carver's method it was necessary to perform the complete graduation from the lower limit of the range up to and including the required f_x. When x is large and only isolated values of f_x are required it seems desirable to have a method for computing f_x directly, and the present note seeks to accomplish this purpose.

It is well known [2] that the difference equation

$$\frac{f_{x+1}}{f_x} = a\frac{(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)}{(x - \beta_1)(x - \beta_2) \cdots (x - \beta_m)} \tag{2}$$

has the solution

$$f_x = w_x a^x \frac{\Gamma(x - \alpha_1) \cdots \Gamma(x - \alpha_n)}{\Gamma(x - \beta_1) \cdots \Gamma(x - \beta^m)}, \tag{3}$$

where w_x is a periodic function of x ($w_x = w_{x+n} = \cdots = k$) and $\Gamma(x + 1)$ for x, a positive real number may be defined in the usual manner by the second Euler integral

$$\Gamma(x + 1) = \int_0^\infty t^x e^{-t}\, dt \tag{4}$$

which obeys the recursion formula

$$\Gamma(x + 1) = x\Gamma(x). \tag{5}$$

When x is a positive integer

$$\Gamma(x + 1) = x!. \tag{6}$$

Equation (1) is seen to be a special case of (2) for $n = m = 2$ and accordingly, the solution may be written as

$$f_x = K\frac{\Gamma(x - \alpha_1)\Gamma(x - \alpha_2)}{\Gamma(x - \beta_1)\Gamma(x - \beta_2)}, \tag{7}$$

where α_1 and α_2 are roots of $x^2 + c_1x + c_2 = 0$ and β_1 and β_2 are roots of $x^2 + c_3x + c_4 = 0$. The following simple examples illustrate three special cases of this solution.

I. All α's and β's are integers.

$$\frac{f_{x+1}}{f_x} = \frac{2(x^2 + 9x + 20)}{x^2 + 5x + 6}$$

has the solution

$$f_x = K2^x\frac{\Gamma(x + 4)\Gamma(x + 5)}{\Gamma(x + 2)\Gamma(x + 3)}$$

which, with the aid of recursion formula (5) can readily be verified by direct substitution.

II. Either the α's and/or the β's are real irrational numbers.

$$\frac{f_{x+1}}{f_x} = \frac{x^2 + 5x + 6}{x^2 + 3x + 1}$$

has the solution

$$f_x = K\frac{\Gamma(x + 2)\Gamma(x + 3)}{\Gamma[x + \frac{1}{2}(3 - \sqrt{5})]\Gamma[x + \frac{1}{2}(3 + \sqrt{5})]}$$

which, with the aid of the recursion formula (5) can also be verified by direct substitution.

III. Either the α's and/or the β's are complex.

$$\frac{f_{x+1}}{f_x} = \frac{x^2 + 8x + 17}{x^2 + 10x + 29}$$

has the solution

$$f_x = K\frac{\Gamma(x + 4 + i)\Gamma(x + 4 - i)}{\Gamma(x + 5 + 2i)\Gamma(x + 5 - 2i)}.$$

Since the recursion formula (5) is also valid for complex arguments [3], this solution can be verified by direct substitution just as in the first two cases.

The evaluation of f_x for a given x in cases I and II involves only computation of quantities of the form $\Gamma(x)$ which can be accomplished through the use of existing tables of Gamma Functions for small values of x and through application of Stirling's formula for large values of x. Evaluation of f_x in case III, however, involves the computation of quantities of the form $\Gamma(u + iv)\Gamma(u - iv)$, a problem which seems to have escaped previous attention. The remainder of the present discussion will center about this quantity.

The Gamma Function for a real positive argument has been defined by equation (4), but for the present purposes, it is more expedient to use the definition

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)} \tag{8}$$

which is valid for all values of the complex argument z except at the poles ($z = -1; z = -2$, etc.). The above definition is equivalent to (4) at all points where (4) is valid [3].

From equation (8), it immediately follows that $\Gamma(u + iv)\Gamma(u - iv)$ is a real number. In fact, we have

$$\Gamma(u + iv)\Gamma(u - iv) = \lim_{n\to\infty} \frac{(n!)^2 n^{2u}}{[u^2 + v^2][(u+1)^2 + v^2]\cdots[(u+n)^2 + v^2]}.$$

We now develop a formula applicable in evaluating this quantity when u is a sufficiently small positive integer. As a consequence of equation (8) it can be shown that [3]

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}. \tag{9}$$

Let $z = iv$ in the above equation and we immediately obtain the result

$$\Gamma(iv)\Gamma(-iv) = \frac{2\pi v^{-1}}{e^{\pi v} - e^{-\pi v}}. \tag{10}$$

When u is a positive integer, we may write

$$\Gamma(u + iv) = (\overline{u - 1} + iv)(\overline{u - 2} + iv)\cdots(iv)\Gamma(iv), \tag{11}$$

$$\Gamma(u - iv) = (\overline{u - 1} - iv)(\overline{u - 2} - iv)\cdots(-iv)\Gamma(-iv). \tag{12}$$

The product of (11) by (12) gives

$$\Gamma(u + iv)\Gamma(u - iv) = v^2(v^2 + 1)\cdots(v^2 + \overline{u - 1}^2)\Gamma(iv)\Gamma(-iv)$$

which upon substitution of the value found in Equation (10) for $\Gamma(iv)\Gamma(-iv)$ becomes

$$\Gamma(u + iv)\Gamma(u - iv) = \frac{2\pi v}{e^{\pi v} - e^{-\pi v}} \prod_{r=1}^{u-1} (v^2 + r^2). \tag{13}$$

To obtain a result that is applicable when u is not a positive integer, we make use of Stirling's formula for complex arguments. Lipschitz [4] proves

$$\text{(14)}\quad \begin{aligned} \operatorname{Log} \Gamma(z) &= \log \sqrt{2\pi} + (z - \tfrac{1}{2}) \log z \\ &\quad - z + (-1)^m \sum_{m=0}^{\infty} \frac{B_{2m+1}}{(2m+1)(2m+2)} \frac{1}{z^{2m+1}} \end{aligned}$$

and that the remainder after the mth term is

$$v_{m+1} = \frac{(-1)^{m+1} B_{2m+3}}{(2m+3)(2m+4)} \frac{1}{z^{2m+3}} (\epsilon + \epsilon' i),$$

where $\epsilon < 1$; $\epsilon' < 1$. B_{2m+1} designates the Bernoulli numbers. ($B_1 = \frac{1}{6}$; $B_3 = \frac{1}{30}$; $B_5 = \frac{1}{42}$; etc.) We are thus able to write

$$\text{(15)}\quad \begin{aligned} \operatorname{Log} \Gamma(u + iv) &= \log \Gamma(Re^{i\varphi}) \\ &= \log \sqrt{2\pi} + (Re^{i\varphi} - \tfrac{1}{2})(\log R + i\varphi) \\ &\quad - Re^{i\varphi} + \sum_{m=0}^{\infty} \frac{(-1)^m B_{2m+1}}{(2m+1)(2m+2)} \frac{e^{-(2m+1)i\varphi}}{R^{2m+1}}, \end{aligned}$$

where $\varphi = \tan^{-1} \dfrac{v}{u}$ and $R = \sqrt{u^2 + v^2}$;

$$\text{(16)}\quad \begin{aligned} \operatorname{Log} \Gamma(u - iv) &= \log \Gamma(Re^{-i\varphi}) \\ &= \log \sqrt{2\pi} + (Re^{-i\varphi} - \tfrac{1}{2})(\log R - i\varphi) \\ &\quad - Re^{-i\varphi} + \sum_{m=0}^{\infty} \frac{(-1)^m B_{2m+1}}{(2m+1)(2m+2)} \cdot \frac{e^{(2m+1)i\varphi}}{R^{2m+1}}. \end{aligned}$$

Adding (15) and (16), we obtain

$$\begin{aligned} \operatorname{Log} \Gamma(u + iv)\Gamma(u - iv) &= \log 2\pi + (e^{i\varphi} + e^{-i\varphi})R \log R - \log R \\ &\quad + Ri\varphi(e^{i\varphi} - e^{-i\varphi}) - R(e^{i\varphi} + e^{-i\varphi}) \\ &\quad + \sum_{m=0}^{\infty} \frac{(-1)^m B_{2m+1}}{(2m+1)(2m+2)} (e^{(2m+1)i\varphi} + e^{-(2m+1)i\varphi}) \frac{1}{R^{2m+1}} \end{aligned}$$

which upon being simplified becomes

$$\text{(17)}\quad \begin{aligned} &\operatorname{Log} \Gamma(u + iv)\Gamma(u - iv) \\ &\quad = \log 2\pi + (2u - 1) \log R - 2(\varphi v + u) + 2\psi(R, \varphi), \end{aligned}$$

where

$$\text{(18)}\quad \psi(R, \varphi) = \sum_{m=0}^{\infty} \frac{(-1)^m B_{2m+1}}{(2m+1)(2m+2)} \frac{1}{R^{2m+1}} \cos (2m + 1)\varphi.$$

This result is somewhat similar to that obtained by Karl Pearson [5] in connection with the evaluation of the $G_{(r\nu)}$ integrals of his Type IV frequency curve. If $R > 1$, the expansion of $\psi(R, \varphi)$ is asymptotic and the greatest numerical value that the mth term can have is

$$\frac{B_{2m+1}}{(2m+1)(2m+2)} \cdot \frac{1}{R^{2m+1}}.$$

Thus according to Lipschitz results, the error committed in dropping all terms after the mth will not exceed: $\pm \frac{B_{2m+1}}{(2m+1)(2m+2)} \frac{1}{R^{2m+1}}$. The following table gives an indication of the size of the error:

Terms omitted after	Error committed in $\psi(R, \varphi)$ less than
1st	$\pm .0833\ 3333/R$
2nd	$\pm .0027\ 7777/R^3$
3rd	$\pm .0007\ 9365/R^5$
4th	$\pm .0005\ 9524/R^7$
5th	$\pm .0008\ 4175/R^9$.

It is now obvious that formula (18) will give satisfactory results whenever R is sufficiently large. The degree of accuracy required together with the value of R will determine the number of terms of $\psi(R, \varphi)$ to be computed.

We now turn to the solution of the example under Case III and proceed to calculate f_4, f_{15}, and f_{150} when $f_0 = 29$. We may write

$$K = 29 \frac{\Gamma(5+2i)\Gamma(5-2i)}{\Gamma(4+i)\Gamma(4-i)}.$$

Application of formula (13) gives

$$\Gamma(5+2i)\Gamma(5-2i) = 244.043\ 648,$$

$$\Gamma(4+i)\Gamma(4-i) = 27.202\ 292,$$

from which, $K = 260.171\ 676$,

$$f_4 = 260.171\ 676 \frac{\Gamma(8+i)\Gamma(8-i)}{\Gamma(9+2i)\Gamma(9-2i)}.$$

Again making use of formula (13) we have

$$f_4 = 260.171\ 676 \cdot \frac{22{,}243{,}314}{1{,}020{,}258{,}635} = 5.6722,$$

$$f_{15} = 260.171\ 676 \frac{\Gamma(19+i)\Gamma(19-i)}{\Gamma(20+2i)\Gamma(20-2i)}$$

Since R is fairly large in this instance, formula (17) is used and all terms of $\psi(R, \varphi)$ after the first are dropped. This result gives

$$\log \Gamma(19+i)\Gamma(19-i) = 31.5892\ 259,$$

$$\log \Gamma(20+2i)\Gamma(20-2i) = 34.0812\ 782.$$

Accordingly, $\log f_{15} = 9.9232\,071 - 10$

and $f_{15} = .8379.$

By the same method f_{150} is calculated and we find $f_{150} = .008723$.

As a check on the accuracy of the results obtained in the above computations, values of f_x for x ranging from 1 to 15 were computed, using the given equation as a recursion formula. That is

$$f_1 = \frac{17}{29} f_0 = 17, \qquad f_2 = \frac{26}{40} f_1 = 11.05, \qquad \text{etc.}$$

These results are given in the following table, and it is to be noted that the values in the table for f_4 and f_{15} agree with those previously computed by use of formulas contained in this paper. For obvious reasons, no attempt was made to compute the value of f_{150} by this method.

TABLE I

x	f_x	x	f_x	x	$f(x)$
0	29.0000	5	4.3375	10	1.6228
1	17.0000	6	3.4200	11	1.3961
2	11.0500	7	2.7633	12	1.2135
3	7.7142	8	2.2779	13	1.0644
4	5.6722	9	1.9092	14	0.9411
				15	0.8379

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