COMBINATORIAL FORMULAS FOR THE rth STANDARD MOMENT OF THE SAMPLE SUM, OF THE SAMPLE MEAN, AND OF THE NORMAL CURVE

By P. S. Dwyer

The standard moments of the normal curve are usually expressed by the two statements [1, p. 97]

$$\left\{\begin{array}{ll}\alpha_{2s} & = \dfrac{(2s)!}{2^s s!} \\ \alpha_{2s+1} & = 0\end{array}\right\}. \tag{1}$$

It is of some interest to note that these two statements may be generalized into a single statement by observing that $\frac{(2s)!}{2^s s!}$ is the number of ways in which $2s$ things can be grouped in pairs and that 0 is the number of ways in which $2s + 1$ things can be grouped in pairs. It is obvious that an odd number of things can not be grouped in pairs since there must be at least one unpaired unit. It is clear, too, that the number of orders in which $2s$ things can be grouped in pairs is $\binom{2s}{2}\binom{2s-2}{2}\binom{2s-4}{2}\cdots\binom{4}{2}\binom{2}{2}$ and this is $\frac{(2s)!}{2^s}$. However if the resulting paired groups (rather than the orders of grouping) are counted it is seen that each paired grouping is repeated $s!$ times so that $\frac{(2s)!}{2^s s!}$ represents the number of ways $2s$ things can be grouped in pairs. If we arbitrarily define the number of ways 0 things can be grouped in pairs to be 1 (or if we limit our theorem to values of $r > 0$) we may say "The rth standard moment of the normal curve is equal to the number of ways in which r things can be grouped in pairs."

As presented above the combination representation is used primarily as a means of unification of results. However, it is possible to derive the standard moments of the normal curve in such a way as to indicate the term $\frac{(2s)!}{2^s s!}$ early in the proof and to trace it throughout the proof. I follow the method outlined by H. C. Carver [2] in obtaining the normal distribution as the limit of the distribution of sample sums (or of sample means) though I use a somewhat different notation [3, p. 5]. If we let $\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ represent the number of ways in which r units can be collected with π_1 groups containing p_1 units, π_2 groups containing p_2 units, etc., then the multinomial theorem can be expressed as [3, p. 17]

$$(1)^r = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} (p_1^{\pi_1} \cdots p_s^{\pi_s}) \tag{2}$$

where the summation is taken over all possible partitions $p_1^{\pi_1} \cdots p_s^{\pi_s}$ of r and the expression $(p_1^{\pi_1} \cdots p_s^{\pi_s})$ represents the power product form [3, p. 14] which is $\pi_1!\pi_2! \cdots \pi_s!$ times the monomial symmetric function. If ρ represents the number of parts of the partition then

$$\rho = \pi_1 + \pi_2 + \cdots + \pi_s$$

while

$$r = p_1\pi_1 + p_2\pi_2 + \cdots + p_s\pi_s .$$

Now it can be shown from (2) in the case of infinite sampling that

$$\bar{\mu}_{r:(1)} = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} n^{(\rho)}(\bar{\mu}_{p_1})^{\pi_1} \cdots (\bar{\mu}_{p_s})^{\pi_s} \tag{3}$$

and since $\bar{\mu}_1 = 0$, it is only necessary to sum over all partitions which have no unit part. We have then, dividing by $[\bar{\mu}_{2:(1)}]^{\frac{1}{2}r} = [n\bar{\mu}_2]^{\frac{1}{2}r}$

$$\alpha_{r:(1)} = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} \frac{n^{(\rho)}}{n^{\frac{1}{2}r}} (\alpha_{p_1})^{\pi_1} \cdots (\alpha_{p_s})^{\pi_s}. \tag{4}$$

We have now a formula for the rth standard moment of the sample sum which is expressed essentially in combination notation since the quantity $\binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ represents the number of ways in which r units can be grouped to form π_1 groups containing p_1 units, π_2 groups containing p_2 units, etc. All non-unitary groupings of r are formed, each combinatorial coefficient is computed and multiplied by $n^{(\rho)}/n^{\frac{1}{2}r}$ times the product of the corresponding α's, and the sums are formed. It might be noted that the formula for the rth standard moment of the sample mean is identical with (4) while the corresponding finite sampling (without replacements) formula is

$$\alpha_{r:(1)} = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} \frac{N^\rho P_{p_1^{\pi_1} \cdots p_s^{\pi_s}}}{N^{\frac{1}{2}r} P_2^{\frac{1}{2}r}} (\alpha_{p_1})^{\pi_1} \cdots (\alpha_{p_s})^{\pi_s}. \tag{5}$$

The P's are defined in previous papers [2, p. 105–6][3, p. 113].

We obtain the formula for the rth standard moment of the normal curve by taking the limit of (4) as $n \to \infty$. (H. C. Carver has pointed out [2, p. 121] that this method of derivation imposes fewer restrictions than does the derivation from Hagen's hypothesis.) Each partition term will approach zero as n approaches infinity if $\rho < \frac{1}{2}r$. Now the only non-unitary partition in which ρ is not less than $\frac{1}{2}r$ is the partition $2^{\frac{1}{2}r}$ and we can have this partition only when r is even. Now the limit as n approaches infinity of $n^{(\rho)}/n^{\frac{1}{2}r}$ is unity and we have, in the limiting case

$$\alpha_r = \begin{cases} \binom{1^r}{2^{\frac{1}{2}r}} & \text{if } r \text{ is even.} \\ 0 & \text{if } r \text{ is odd.} \end{cases} \tag{6}$$

Since $\binom{1^r}{2^{\frac{1}{2}r}}$ is the number of ways r units can be grouped in pairs when r is even and since 0 is the number of ways r units can be grouped in pairs where r is odd, it follows that the rth standard moment of the normal curve is the number of ways in which r units can be grouped in pairs.

This development is of interest in that it makes possible the tracing of the value $\binom{1^r}{2^{\frac{1}{2}r}}$ back through the various stages of the development to the coefficient of $(2^{\frac{1}{2}r})$ in the power product expansion of the multinomial theorem.

REFERENCES

[1] H. C. CARVER, "Frequency Curves," Chapter VII of *Handbook of Mathematical Statistics*.

[2] H. C. CARVER, "Fundamentals of the theory of sampling," *Annals of Math. Stat.*, Vol. 1 (1930), pp. 101–121.

[3] P. S. DWYER, "Combined Expansions of Products of Symmetric Power Sums and of Sums of Symmetric Power Products with Application to Sampling," *Annals of Math. Stat.*, Vol. 9 (1938), pp. 1–47, 97–132.

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ON A METHOD OF SAMPLING[1]

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It is recorded that Diogenes fared forth with a lantern in his search for an honest man. History does not tell us how many dishonest men he encountered before he found the first honest one but, judging from the fact that he took his lantern, apparently he expected to have a long search. The general problem of sampling inspection, of which the above is a special case, can be stated as follows:

Given a lot, of size m, containing s items of a specified kind. If items are to be drawn without replacement until i of the s items have been drawn, how many drawings, on the average, will be necessary?

Uspensky[2] has solved a problem concerning balls in an urn, from which the answer to the above question can be obtained for the special case $i = 1$. For the general case, the distribution for the number n of the drawing in which the ith specified item appears, is given by terms of the series:

$$(1)\qquad \nu_0' = \sum_{n=1}^{m-s+i} \frac{C_{n-1,i-1}C_{m-n,s-i}}{C_{m,s}} = \sum_{n=0}^{\infty} \frac{C_{n-1,i-1}C_{m-n,s-i}}{C_{m,s}},$$

[1] Presented to The Institute of Mathematical Statistics, Dec. 27, 1938, at Detroit, Mich., as part of a paper, entitled "Remarks on two methods of sampling inspection."

[2] J. V. Uspensky, *Introduction to Mathematical Probability*, McGraw-Hill, New York, 1937, p. 178.