with the associated indicial equation

$$f(x) = x^4 - .398x^3 + .220x^2 - .013x - .027 = 0. \tag{13}$$

Its roots have been computed and are known to be less than unity in absolute value. This may be verified by computing

$$\begin{aligned} \pi_0 &= 0.782 > 0 \\ \pi_1 &= 3.338 > 0 \\ \pi_2 &= 5.398 > 0 \\ \pi_3 &= 4.878 > 0 \\ \pi_4 &= 1.604 > 0 \\ T_2 &= 14.204 > 0 \\ T_3 &= 43.177 > 0 \end{aligned} \tag{14}$$

To compute the same results by cross-multiplication the work is arranged as follows:

$$\begin{array}{ccc} \pi_0 & \pi_2 & \pi_4 \\ .782 & 5.398 & 1.604 \\ \pi_1 & \pi_3 & \\ 3.338 & 4.878 & \\ \pi_1\pi_2 - \pi_0\pi_3 & \pi_3\pi_4 - 0 & \\ 14.204 & 7.824 & \\ \pi_3(\pi_1\pi_2 - \pi_0\pi_3) - \pi_1\pi_3\pi_4 & & \\ 43.177 & & \end{array} \tag{15}$$

It may be remarked that the presence of a negative coefficient anywhere in the table is an immediate indication of instability, and that there is no necessity to continue the computation until a negative sign appears in a leading coefficient. This fact often saves much labor.

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# VALUES OF MILLS' RATIO OF AREA TO BOUNDING ORDINATE AND OF THE NORMAL PROBABILITY INTEGRAL FOR LARGE VALUES OF THE ARGUMENT

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A pair of simple inequalities is proved which constitute upper and lower bounds for the ratio $R_x$[1], valid for $x > 0$. The writer has failed to encounter these inequalities in the literature, hence it seems worthwhile to present them for whatever value they may have.

[1] J. P. Mills, "Table of ratio: area to bounding ordinate, for any portion of the normal curve." *Biometrika Vol. 18* (1926) pp. 395–400. Also Pearson's tables, Part II, Table III.

The function $R_x$ is defined by

$$R_x = e^{x^2/2} \int_x^\infty e^{-t^2/2}\, dt. \tag{1}$$

The following relations between $R = R_x$ and its derivatives are easily established by direct differentiations and substitutions:

$$\frac{dR}{dx} = xR - 1, \tag{2}$$

$$\frac{d^2R}{dx^2} = x\frac{dR}{dx} + R = \frac{x^2+1}{x}\frac{dR}{dx} + \frac{1}{x}, \tag{3}$$

$$\frac{d^3R}{dx^3} = \left(1 + \frac{2}{x^2+1}\right)x\frac{d^2R}{dx^2} - \frac{2}{x^2+1}. \tag{4}$$

Also by ordinary rules

$$R_x > 0, \tag{5}$$

$$\lim_{x\to\infty} xR_x = 1. \tag{6}$$

1°. Suppose that at any point $x_1 > 0$, $x_1R > 1$. Then by (2) $dR/dx > 0$, and $R_x$ would continue to increase with increasing $x$: still more, $xR_x$ would continue to increase, hence we should have $xR_x > 1$ for $x \geqq x_1$, which contradicts (6). Therefore we find $xR_x \leqq 1$ for $x > 0$, and

$$R_x \leqq \frac{1}{x}, \tag{7}$$

which establishes the required upper inequality.

2°. Suppose that at any point $x_2 > 0$, $d^2R/dx^2 < 0$. Then by (4) $d^3R/dx^3 = (d/dx)(d^2R/dx^2) < 0$ at this point. Since these derivatives are continuous this implies that for all $x > x_2$, $d^2R/dx^2 < [d^2R/dx^2]_{x=x_2} < 0$. Then we get the inequalities, for $x > x_2$

$$\frac{dR}{dx} < \left[\frac{dR}{dx}\right]_2 + (x - x_2)\left[\frac{d^2R}{dx^2}\right]_2 < \left[\frac{dR}{dx}\right]_2$$

$$R < R_{x_2} + (x - x_2)\left[\frac{dR}{dx}\right]_2 + \tfrac{1}{2}(x - x_2)^2\left[\frac{d^2R}{dx^2}\right]_2$$

where $[\;]_2$ indicates evaluation at $x = x_2$. Since $[d^2R/dx^2]_2 < 0$, this implies that for sufficiently large $x$, $R_x < 0$, which contradicts (5). It follows then that (3) is positive, and substitution of (2) gives

$$R_x \geqq \frac{x}{x^2+1}. \tag{8}$$

We combine (7) and (8) in the double inequality:

$$\frac{x}{x^2+1} \leqq R_x \leqq \frac{1}{x}, \qquad \text{if } x \geqq 0. \tag{9}$$

This gives for the probability integral the corresponding inequality

$$\frac{x}{x^2+1} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \leqq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt \leqq \frac{1}{x} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{10}$$

It can easily be shown (for $x > 0$) that equalities in (9) and (10) are impossible.