THE DOOLITTLE TECHNIQUE

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1. Introduction. Most authors who have presented the Doolittle method, from Doolittle [1] down to the present, have not given any formal proof that the solution is valid in the general case. They usually are content with a form describing the various steps of a Doolittle solution.

The author has recently shown [2] that the Doolittle method can be abbreviated to a technique which is also an abbreviation, essentially, of the method of single division and its abbreviation which Aitken called the "Method of Pivotal Condensation" [3]. It appears at once that the validity of the Doolittle method follows from the validity of the method of single division—a validity which is readily established.

However one may desire a "proof" which is based directly on the Doolittle technique without referring to other methods of solution. It is the chief purpose of this paper to present such a proof. It is accomplished by the introduction of a notation which precisely describes the conventional Doolittle process and by proving that this process results in a system of equations whose prediagonal terms are zero. It is a secondary purpose of the paper to emphasize the advantages of the Abbreviated Doolittle method and to explain and illustrate minor variations in the conventional Doolittle technique.

2. The Abbreviated Doolittle solution. We first direct our attention to the essential parts of a Doolittle solution and these are the last two rows of each matrix of the standard Doolittle presentation. The additional rows in the standard presentation are rows of products which are used solely for the purpose of finding the two bottom rows of each matrix and they need not be recorded, if a computing machine is available, since the essential information is present in the two bottom rows. Doolittle [1] did not have calculating machines (he used multiplication tables) but he put the important information in Table A and carefully segregated the supplementary information in Table B. With reference to this he wrote [1]

"It is to be observed that the numbers in Table B have but a single use while those in Table A are used over and over, and where the number of equations is large, it is of great advantage that they should be thus tabulated by themselves in a form compact and easy of reference."

For purposes of proof, as well as for purposes of calculation if a computing machine is available, it is only necessary to utilize the forward part of the Abbreviated Doolittle solution which is the equivalent of the Doolittle Table A.

A four variable illustration of the Abbreviated Doolittle technique is presented in Table I. The successive equations are indicated by number, as is customary, and the operation which defines the equation is specified. The actual operation is indicated more explicitly by the notation of column 3 and this is discussed in the next section.

The presentation of Table I introduces one variation from the standard Doolittle method. The division is made by the diagonal coefficient of each row rather than by its negative. One may still use the old technique, if he prefers, but it is felt that one can subtract products as easily as he can add products with modern machines equipped with automatic negative multiplication. In addition the entries of the equivalent rows then have the same signs and, too, it is not necessary to take the time to change the signs of the second rows. This variation uses the same division method as the method of single division [2] and as the method of pivotal condensation [3] so that the abbreviated form of these methods is, essentially, the same as the abbreviated form of the Doolittle method.

The application of this technique leads at each step to a coefficient for each variable. However if the process is to lead from our four equations in four unknowns, to three in three, to two in two, to one in one, it follows that all the entries to the left of the diagonal, which we may call prediagonal entries, must be zero. That this is true in the general case is the objective of the proofs of later sections.

3. A notation for and description of the Doolittle technique. A main contribution of the present article is the use of a notation which describes the Doolittle technique. As long as the Doolittle process is described loosely by means of "operations" it is difficult to be precise in defining quantities which appear in the calculation, but when a notation is used which is definite enough to permit expansion in terms of the original coefficients, some sort of proof may be available. The present notation bears some resemblance to that suggested by Gauss [4], though Gauss used letters to indicate the primary subscripts and numbers to indicate the number of secondary subscripts and his notation was directly applicable to the sums of least squares theory rather than to symmetric equations in general.

We wish to find the solution of the equations

$$(1) \qquad \sum_{i=1}^{n} a_{ij} x_i = a_{n+1,j}, \qquad j = 1, 2, \cdots, n$$

where the matrix of the coefficients is symmetric. We do this by obtaining auxiliary equations which feature a decreasing number of variables. No serious restriction is made if we assume that the variables x_1, x_2, x_3, etc., are eliminated successively. The Doolittle technique may then be described as follows: We take the first equation of (1) and divide by its leading coefficient, a_{11}, to get

TABLE I

Abbreviated Doolittle technique; forward solution

Eq.	Operation	Notation	x_1	x_2	x_3	x_4	
I		a_{i1}	a_{11}	a_{21}	a_{31}	a_{41}	a_{51}
II		a_{i2}	a_{12}	a_{22}	a_{32}	a_{42}	a_{52}
III		a_{i3}	a_{13}	a_{23}	a_{33}	a_{43}	a_{53}
IV		a_{i4}	a_{14}	a_{24}	a_{34}	a_{44}	a_{54}
V	I repeated	a_{i1}	a_{11}	a_{21}	a_{31}	a_{41}	a_{51}
VI	V divided by a_{11}	$b_{i1} = \dfrac{a_{i1}}{a_{11}}$	1	b_{21}	b_{31}	b_{41}	b_{51}
VII	II—b_{21} V	$a_{i2\cdot1} = a_{i2} - a_{i1}b_{21}$	$a_{12\cdot1}$	$a_{22\cdot1}$	$a_{32\cdot1}$	$a_{42\cdot1}$	$a_{52\cdot1}$
VIII	VII divided by $a_{22\cdot1}$	$b_{i2\cdot1} = \dfrac{a_{i2\cdot1}}{a_{22\cdot1}}$	$b_{12\cdot1}$	1	$b_{32\cdot1}$	$b_{42\cdot1}$	$b_{52\cdot1}$
IX	III—b_{31} (V)—$b_{32\cdot1}$ VII	$a_{i3\cdot12} = a_{i3} - a_{i1}b_{31} - a_{i2\cdot1}b_{32\cdot1}$	$a_{13\cdot12}$	$a_{23\cdot12}$	$a_{33\cdot12}$	$a_{43\cdot12}$	$a_{53\cdot12}$
X	IX divided by $a_{33\cdot12}$	$b_{i3\cdot12} = \dfrac{a_{i3\cdot12}}{a_{33\cdot12}}$	$b_{13\cdot12}$	$b_{23\cdot12}$	1	$b_{43\cdot12}$	$b_{53\cdot12}$
XI	IV—b_{41} V—$b_{42\cdot1}$ VII—$b_{43\cdot12}$ IX	$a_{i4\cdot123} = a_{i4} - a_{i1}b_{41} - a_{i2\cdot1}b_{42\cdot1} - a_{i3\cdot12}b_{43\cdot12}$	$a_{14\cdot123}$	$a_{24\cdot123}$	$a_{34\cdot123}$	$a_{44\cdot123}$	$a_{54\cdot123}$
XII	XI divided by $a_{44\cdot123}$	$b_{i4\cdot123} = \dfrac{a_{i4\cdot123}}{a_{44\cdot123}}$	$b_{14\cdot123}$	$b_{24\cdot123}$	$b_{34\cdot123}$	1	$b_{54\cdot123}$

$$\sum_{i=1}^{n} b_{i1} x_i = b_{n+1,1}, \qquad \text{where} \quad b_{i1} = \frac{a_{i1}}{a_{11}}, \tag{2}$$

and we then form

$$\sum_{i=1}^{n} a_{i2\cdot 1} x_i = a_{n+1,2\cdot 1} \qquad \text{with} \quad a_{i2\cdot 1} = a_{i2} - a_{i1} b_{21}. \tag{3}$$

We then divide by $a_{22\cdot 1}$ and get

$$\sum_{i=1}^{n} b_{i2\cdot 1} x_i = b_{n+1,2\cdot 1} \qquad \text{with} \quad b_{i2\cdot 1} = \frac{a_{i2\cdot 1}}{a_{22\cdot 1}}. \tag{4}$$

We next form

$$\sum_{i=1}^{n} a_{i3\cdot 12} x_i = a_{n+1,3\cdot 12} \qquad \text{with} \quad a_{i3\cdot 12} = a_{i3} - a_{i1} b_{31} - a_{i2\cdot 1} b_{32\cdot 1}, \tag{5}$$

and

$$\sum_{i=1}^{n} b_{i3\cdot 12} x_i = b_{n+1,3\cdot 12} \qquad \text{with} \quad b_{i3\cdot 12} = \frac{a_{i3\cdot 12}}{a_{33\cdot 12}}. \tag{6}$$

This process is continued so that, in general, we have

$$\sum_{i=1}^{n} a_{ij\cdot 12\cdots j-1} x_i = a_{n+1,j\cdot 12\cdots j-1}, \qquad j = 1, 2, \cdots, n \tag{7}$$

and

$$\sum_{i=1}^{n} b_{ij\cdot 12\cdots j-1} x_i = b_{n+1,j\cdot 12\cdots j-1}, \qquad j = 1, 2, \cdots, n \tag{8}$$

with

$$\begin{aligned} a_{ij\cdot 12\cdots j-1} = a_{ij} &- a_{i1} b_{j1} - a_{i2\cdot 1} b_{j2\cdot 1} - a_{i3\cdot 12} b_{j3\cdot 12} - \cdots \\ &- a_{i,j-2\cdot 12\cdots j-3} b_{j,j-2\cdot 12\cdots j-3} - a_{i,j-1\cdot 12\cdots j-2} b_{j,j-1\cdot 12\cdots j-2} \end{aligned} \tag{9}$$

and

$$b_{ij\cdot 12\cdots j-1} = \frac{a_{ij\cdot 12\cdots j-1}}{a_{jj\cdot 12\cdots j-1}}. \tag{10}$$

It is to be noted that the n equations (1) are transformed by this process to the n auxiliary equations of (7) or (8). The solutions of (1) are also solutions of these auxiliary equations since the auxiliary equations are linear combinations of (1). It is our purpose to show that the prediagonal coefficients of these auxiliary equations are always 0 so that these auxiliary equations feature a decreasing number of variables.

We may use the term primary subscripts to indicate the first two subscripts and the term secondary subscripts to indicate the later subscripts which specify the order of elimination of the variables. The "order" of the coefficient is then equal to the number of secondary subscripts.

The formula (9) gives the matrix of the final Doolittle set of equations. At each stage of the reduction one can write down a formula for all the elements in the matrix at that stage. Thus one can write the coefficients of order h, $a_{ij\cdot 12\cdots h}$, in terms of coefficients of order less than h,

$$\begin{aligned} a_{ij\cdot 12\cdots h} = a_{ij} - a_{i1}b_{j1} - a_{i2\cdot 1}b_{j2\cdot 1} - \cdots \\ - a_{i,h-1\cdot 12\cdots h-2}b_{j,h-1\cdot 12\cdots h-2} - a_{ih\cdot 12\cdots h-1}b_{jh\cdot 12\cdots h-1}\,. \end{aligned} \tag{11}$$

It follows at once that

$$\begin{aligned} a_{ij\cdot 12\cdots h} &= a_{ij\cdot 12\cdots h-1} - a_{ih\cdot 12\cdots h-1}\,b_{jh\cdot 12\cdots h-1} \\ &= a_{ij\cdot 12\cdots h-1} - \frac{a_{ih\cdot 12\cdots h-1}\,a_{jh\cdot 12\cdots h-1}}{a_{hh\cdot 12\cdots h-1}}\,. \end{aligned} \tag{12}$$

4. Some theorems on the interchangeability of subscripts. Our main objective is to prove that the prediagonal terms are zero. In order to do this we first prove some theorems dealing with the primary and secondary subscripts.

THEOREM 1: *The value of $a_{ij\cdot\cdots h}$ is not changed if the primary subscripts are interchanged.* This theorem which might be stated "The matrix of the coefficients of a given order is symmetric" follows from the symmetry of the matrix of coefficients of zero order. We can show that the symmetry of the matrix having coefficient of order h follows at once from the symmetry of the matrix having coefficients of order $h - 1$ by comparing the value $a_{ij\cdot\cdots h}$ with that of $a_{ji\cdot\cdots h}$ obtained by dual substitution in (12). Since the matrix of zero order coefficients is symmetric by hypothesis, it follows that the matrices of the coefficients of order 1, 2, 3, 4, etc., are in turn symmetric.

THEOREM 2: *Any pair of consecutive secondary subscripts may be interchanged without changing the value of the coefficient.* This theorem indicates that, within prescribed limits, the order of elimination does not have any effect on the result.

Consider the coefficient $a_{ij\cdot\cdots kl\cdots}$ having r secondary subscripts before the k and s secondary subscripts after the l and consider the corresponding coefficient $a_{ij\cdot\cdots lk\cdots}$ which results from an interchange of k and l. These coefficients can be expressed by continued use of (12) in terms of coefficients of order $r + 2$. The resulting expansion of $a_{ij\cdot\cdots kl\cdots}$ is equivalent to that of $a_{ij\cdot\cdots lk\cdots}$ with the interchange of the l and the k. It follows that the theorem is true if $a_{ij\cdot\cdots lk} = a_{ij\cdot\cdots kl}$. Now a double application of (12) to $a_{ij\cdot\cdots lk}$ leads to the expansion in terms of coefficients of order r (using the notation $a_{ij\cdot}$ to indicate the coefficient of the r-th order).

$$a_{ij\cdot\cdots kl} = a_{ij\cdot} - \frac{a_{ik\cdot}a_{jk\cdot}}{a_{kk\cdot}} - \frac{\left(a_{il\cdot} - \dfrac{a_{ik\cdot}a_{lk\cdot}}{a_{kk\cdot}}\right)\left(a_{jl\cdot} - \dfrac{a_{jk\cdot}a_{lk\cdot}}{a_{kk\cdot}}\right)}{a_{ll\cdot} - \dfrac{a_{lk\cdot}^2}{a_{kk\cdot}}}\,. \tag{13}$$

Then $a_{ij\cdot\cdots lk}$ is expanded similarly, the difference is formed and found to be zero. It follows that the theorem is true.

The application of Theorem 2 with the continued interchange of successive secondary subscripts in all possible ways leads at once to

THEOREM 3: *The secondary subscripts may be interchanged in all possible ways without changing the value of the coefficient.* This theorem might be stated "The value of the resulting coefficient is independent of the order of elimination." This is the sort of result one would expect to find and indeed, some may feel that it is intuitively evident, but this formal proof is presented for those who desire a more rigorous approach.

Theorem 3 enables us to prove Theorem 4 which may be stated: *The value of* $a_{ij.12 \cdots n}$ *is always zero if at least one of the secondary subscripts is equal to one of the primary subscripts.*

Suppose i is this subscript. Then by Theorem 3, i may be placed in the final position. Now by (12) we have

$$a_{ij.\cdots i} = a_{ij.\cdots} - \frac{a_{ij.\cdots}\, a_{ii.\cdots}}{a_{ii.\cdots}} = 0.$$

A similar statement holds if j appears among the secondary subscripts.

5. The vanishing of the prediagonal entries. As an application of Theorem 4 we can show that the prediagonal entries are identically zero and this is exactly what is needed to establish the validity of the forward Doolittle process. It is to be noted that the prediagonal entries are of form $a_{ij.12 \cdots j-1}$ with $i < j$. Then i must equal one of the secondary subscripts and the term is zero.

It follows that no entries need be made to the left of the diagonal in the Abbreviated Doolittle solution and, indeed, no entries need be made in the original matrix below the main diagonal. A numerical problem is presented in the next section.

6. Illustration. The Abbreviated Doolittle technique is illustrated in Table II. This illustration is essentially an illustration of a previous article [2] and serves as the basis, in a later section, for expansion into the standard Doolittle solution. The check is shown in the right hand column and the back solution is indicated. The check entries for the first matrix are obtained by adding the entries in the row to the main diagonal and then adding the entries in the column. All other check entries are obtained by adding the entries in the row.

The solution is easily made once it is understood and results from continued application of formula (9). For example

$$a_{54.123} = a_{54} - a_{51}b_{41} - a_{52.1}b_{42.1} - a_{53.12}b_{43.12}$$

and this is

$$a_{54.123} = .8000 - (.2000)(.6000) - (.3200)(.1905) - (.4619)(-.1612) = .6935$$

(see the underscored entries of Table II). Terms of this sort are easily computed if a calculating machine, and especially so if one equipped with automatic positive and negative multiplication, is available. The back solution too is easily accomplished with a machine. It is only necessary to substitute in turn in each of the "b" equations. Thus the value of x_1 is $\frac{a_{54\cdot123}}{a_{44\cdot123}} = b_{54\cdot123}$, the value of x_2 is $b_{53\cdot12} - b_{43\cdot12}b_{54\cdot123} = b_{53\cdot124}$, that of x_3 is $b_{52\cdot1} - b_{42\cdot1}b_{54\cdot123} - b_{32\cdot1}b_{53\cdot124} = b_{52\cdot134}$, etc. The back solution of the check is treated similarly.

7. A variation in technique. Before proceeding with the presentation of a standard Doolittle solution it seems wise to indicate another possible variation in the technique in addition to the division by the diagonal coefficient rather than its negative. It is possible to obtain the Doolittle solution by using the fixed entry from the first of the equivalent rows in place of using the fixed "b" entry and the variable "a". This results from the fact that

$$a_{ik\cdots}\,b_{jk\cdots} = a_{jk\cdots}\,b_{ik\cdots}\left(= \frac{a_{ik\cdots}\,a_{jk\cdots}}{a_{kk\cdots}}\right). \tag{14}$$

Thus in Table II the value $a_{54\cdot123}$ can be obtained with the use of

$$a_{54\cdot123} = a_{54} - a_{41}b_{51} - a_{42\cdot1}b_{52\cdot1} - a_{43\cdot12}b_{53\cdot12}$$

as readily as with the use of

$$a_{54\cdot123} = a_{54} - a_{51}b_{41} - a_{52\cdot1}b_{42\cdot1} - a_{53\cdot12}b_{43\cdot12}\,.$$

See the boxed entries of Table II.

There seems to be no real choice between these techniques. The fixed "b" is traditional in the standard Doolittle solution while the abbreviation of the method of single division leads to a fixed "a". The point to be emphasized here is that either the fixed "a" or the fixed "b" can be used. Also (14) is used in the next section in supplying details for the check portion of a standard Doolittle method.

8. The standard Doolittle method. If no computing machine is available or if a more detailed solution is desired, it is preferable to record the individual products of (9) and thus arrive at the standard Doolittle method. (The division by the diagonal coefficient rather than its negative is not a fundamental difference.) The standard Doolittle method, from this point of view, is an expanded form of the Abbreviated Doolittle method with more details added. Its validity then follows from the validity of the Abbreviated Doolittle method. While it is not true that all prediagonal terms vanish in the standard Doolittle method, and this fact complicates the check by row sums, yet the prediagonal $a_{ij\cdots}$ (and $b_{ij\cdots}$) are all zero.

The standard Doolittle method is presented in Table III. Some remarks should be made about the non-recorded terms, the two check solutions, and the back solution.

The blanks (—) indicate non zero entries which are usually not presented in a

Doolittle solution. They should be considered however if the first check method is to be used.

The first check method, which is the logical extension of the check method of the Abbreviated Doolittle solution, has been outlined by Ezekial [5]. The row sum is the sum of all the entries in the row whether recorded or not. In order to check, it is necessary to add these unrecorded entries, and they are available

TABLE II

Abbreviated Doolittle Solution; illustration

x_1	x_2	x_3	x_4		Check
1.0000	.4000	.5000	.6000	.2000	2.7000
—	1.0000	.3000	.4000	.4000	2.5000
—	—	1.0000	.2000	.6000	2.6000
—	—	—	1.0000	.8000	3.0000
1.0000	.40000	.5000	$\boxed{.6000}$	.2000	2.7000
1.0000	.40000	.5000	.6000	$\boxed{.2000}$	2.7000
	.8400	.1000	$\boxed{.1600}$	.3200	1.4200
	1.0000	.1190	.1905	$\boxed{.3810}$	1.6905
		.7381	$\boxed{-.1190}$	.4619	1.0810
		1.0000	−.1612	$\boxed{.6258}$	1.4646
			.5903	.6935	1.2837
			1.0000	1.1748	2.1747
		1.0000		.8152	1.8152
	1.0000			.0602	1.0602
1.0000				−.9366	.0635

in the columns above if we make use of formula (12). Thus, if we wish to check the value $\sum_{i=1}^{5} a_{i1}b_{41} = 1.6200$, we have

$$a_{11}b_{41} + a_{21}b_{41} + a_{31}b_{41} + a_{41}b_{41} + a_{51}b_{41} =$$

$$a_{41} + a_{41}b_{21} + a_{41}b_{31} + a_{41}b_{41} + a_{51}b_{41} =$$

$$.6000 + .2400 + .3000 + .3600 + .1200 = 1.6200.$$

Another check method, which is recommended by Peters and Van Voorhis [6] sums the entries in the row only over those columns which are to be recorded.

This is presented as check method 2 of Table III. As is to be expected, the check values of the a's and b's of the last two rows of each matrix are in agreement.

It might be noted that one may use the first check method without checking the intermediate steps (the sums for each row) if he checks the sums for the last two rows of each matrix.

TABLE III

Doolittle solution, with checks

Notation	x_1	x_2	x_3	x_4		Check Method 1	Check Method 2		
a_{i1}	1.0000	.4000	.5000	.6000	.2000	2.7000	2.7000		
a_{i2}	—	1.0000	.3000	.4000	.4000	2.5000	2.1000		
a_{i3}	—	—	1.0000	.2000	.6000	2.6000	1.8000		
a_{i4}	—	—	—	1.0000	.8000	3.0000	1.8000		
a_{i1}	1.0000	.4000	.5000	.6000	.2000	2.7000	2.7000		
b_{i1}	1.0000	.4000	.5000	.6000	.2000	2.7000	2.7000		
a_{i2}	—	1.0000	.3000	.4000	.4000	2.5000	2.1000		
$a_{i1}b_{21}$	—	.1600	.2000	.2400	.0800	1.0800	.6800		
$a_{i2\cdot 1}$		.8400	.1000	.1600	.3200	1.4200	1.4200		
$b_{i2\cdot 1}$		1.0000	.1190	.1905	.3810	1.6905	1.6905		
a_{i3}	—	—	1.0000	.2000	.6000	2.6000	1.8000		
$a_{i1}b_{31}$	—	—	.2500	.3000	.1000	1.3500	.6500		
$a_{i2\cdot 1}b_{32\cdot 1}$		—	.0119	.0190	.0381	.1690	.0690		
$a_{i3\cdot 12}$			.7381	−.1190	.4619	1.0810	1.0810		
$b_{i3\cdot 12}$			1.0000	−.1612	.6258	1.4646	1.4646		
a_{i4}	—	—	—	1.0000	.8000	3.0000	1.8000		
$a_{i1}b_{41}$	—	—	—	.3600	.1200	1.6200	.4800		
$a_{i2\cdot 1}b_{42\cdot 1}$		—	—	.0305	.0610	.2705	.0914		
$a_{i3\cdot 12}b_{43\cdot 12}$			—	.0192	−.0745	−.1743	−.0553		
$a_{i4\cdot 123}$				.5903	.6935	1.2838	1.2839		
$b_{i4\cdot 123}$				1.0000	1.1748	2.1748			
$b_{i3\cdot 124}$			1.0000	−.1894	.8152	1.81532	−.3506		
$b_{i2\cdot 134}$		1.0000	.0970	.2238	.0602	1.0602	.4143	.2160	
$b_{i1\cdot 234}$	1.0000	.0241	.4076	.7049	−.9366	.0634	1.3049	.9076	.4241

The back solution is carried out as in Table II. If no computing machine is available or if the detailed steps are desired they may be indicated as in Table III. The entries in the box under the x_4 column are respectively $b_{54.123}b_{43.12}$, $b_{54.123}b_{42.1}$, and $b_{54.123}b_{41}$. Those in the preceding column are $b_{53.124}b_{32.1}$ and $b_{53.124}b_{31}$. The other entry is $b_{52.134}b_{21}$. The values of the coefficients are obtained by subtracting these row entries from the constant term of the corresponding "b" equation. Thus, $b_{53.124} = (.6258) - (-.1894)$; $b_{52.134} =$

(.3810) − .0970 − .2238, etc. The back solution of check method 1 agrees with that of check method 2. A form for accomplishing the back solution of the check is indicated at the right. It is not necessary to complete the back solution of the check if it is not desired, and indeed, there are some who feel that the use of the row sum check is unnecessary with modern computing machines [7]. The basic check is substitution in the original equations.

9. Summary. The chief purpose of this paper is to show that the Doolittle technique actually leads to a set of equations featuring a decreasing number of unknowns. This is accomplished by the introduction of an appropriate notation to describe the process and the establishment of certain theorems which serve to validate the process. These theorems are of some interest aside from the application made here. It is a secondary purpose of this paper to emphasize the practicability and theoretical advantages (relative ease of calculating, theoretically more accurate, less chance for numerical error, less recording, less time consuming, more compact, and more easily checked) of the Abbreviated Doolittle method and to explain and illustrate possible variations in technique in the forward and check (by row sums) portions of the standard Doolittle solution. It should be noted that the notation suggested is very useful in providing an easy development of various theorems used in multiple and partial correlation studies, the presentation of which is not the purpose of the present paper.

REFERENCES

[1] M. H. Doolittle, "Method employed in the solution of normal equations and the adjustment of a triangulation," *U. S. Coast and Geodetic Survey Report* (1878), pp. 115–120.

[2] P. S. Dwyer, "The solution of simultaneous equations," *Psychometrika*, Vol. 6 (1941), pp. 101–129.

[3] A. C. Aitken, "Studies in practical mathematics I. The evaluation, with applications, of a certain triple product matrix," *Roy. Soc. Edin. Proc.*, Vol. 57 (1937), pp. 172–181.

[4] C. F. Gauss, "Supplementum theoriae combinationis observationum erroribus minimis obnoxiae," *Werke*, Vol. 4 (1873), pp. 69–71.

[5] Mordecai Ezekial, *Methods of Correlation Analysis*. John Wiley and Sons, Inc., New York (1930), pp. 362–364.

[6] C. C. Peters, and W. R. Van Voorhis, *Statistical Procedures and Their Mathematical Bases*, McGraw Hill (1940), pp. 228–229.

[7] A. K. Kurtz, "The use of the Doolittle method in obtaining related multiple correlation coefficients," *Psychometrika*, Vol. 1 (1936), pp. 45–51.