ASYMPTOTIC FORMULAS FOR SIGNIFICANCE LEVELS OF CERTAIN DISTRIBUTIONS

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1. Introduction. The purpose of this paper is to derive asymptotic formulas for the significance levels, or per cent points, of certain well-known statistical distributions.[1] Although we restrict ourselves here to two distributions, those of Chi-Square and of Student's t, it will be apparent that the methods used are applicable to many other distributions as well.

The following results are obtained. Let y_p be the p per cent point of the normal distribution, that is, the distribution defined by

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2}\, dv, \tag{1.1}$$

so that

$$\Phi(y_p) = 1 - p. \tag{1.2}$$

If $\chi^2_{p,n}$ and $t_{p,n}$ denote the p per cent points of the Chi-Square and Student's t distributions with n degrees of freedom respectively, then

$$\chi^2_{p,n} = n + y_p\sqrt{2n} + \frac{2}{3}(y_p^2 - 1) + \frac{y_p^3 - 7y_p}{9\sqrt{2n}} + o\left(\frac{1}{\sqrt{n}}\right), \tag{1.3}$$

and

$$t_{p,n} = y_p + \frac{y_p^3 + y_p}{4n} + o\left(\frac{1}{n}\right). \tag{1.4}$$

These formulas approximate the true values of $\chi^2_{p,n}$ and $t_{p,n}$ to a high degree of accuracy. Tables of comparative values for several values of p and n are given in Section 4.

We shall need the following theorem due to Cramér [3, p. 81; see also pp. 86–87].

Theorem 1: *Let $X_1, X_2, \cdots$ be a sequence of independent, identically distributed random variables having an absolutely continuous distribution function with mean value zero, dispersion σ and finite fifth absolute moment. Let $H_n(x)$ be the distribution function of $(X_1 + \cdots + X_n)/(\sigma\sqrt{n})$, and let $n^{-\frac{1}{2}(r-2)}\lambda_r$ denote the r-th semi-invariant of $H_n(x)$. Then*

$$\Phi(x) - H_n(x) = \frac{\lambda_3}{3!\sqrt{n}}\Phi^{(3)}(x) - \frac{\lambda_4}{4!\,n}\Phi^{(4)}(x) - \frac{10\lambda_3^2}{6!\,n}\Phi^{(6)}(x) + 0(n^{-3/2}). \tag{1.5}$$

[1] This problem was proposed to the author by J. H. Curtiss.

2. The Chi-Square distribution. A random variable X is said to be distributed according to Chi-Square with n degrees of freedom ($X = \chi_n^2$) if its distribution function is

$$F_n(x) = \begin{cases} \int_0^x \frac{t^{\frac{1}{2}n-1}e^{-\frac{1}{2}t}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\,dt, & x \geqq 0, \\ 0 \quad , & x < 0. \end{cases} \tag{2.1}$$

The variable $(\chi_n^2 - n)/\sqrt{2n}$ then has the distribution function

$$H_n(x) = F_n(n + x\sqrt{2n}). \tag{2.2}$$

If we write

$$\chi_{p,n}^2 = n + y_p\sqrt{2n} + a_n , \tag{2.3}$$

so that

$$F_n(\chi_{p,n}^2) = 1 - p, \tag{2.4}$$

and let $z_{pn} = y_p + a_n/\sqrt{2n}$, then $H_n(z_{pn}) = 1 - p$, and it follows from (1.1) and (1.2) that

$$\Phi(z_{pn}) - H_n(z_{pn}) = \Phi(z_{pn}) - \Phi(y_p) = \frac{1}{\sqrt{2\pi}}\frac{a_n}{\sqrt{2n}}e^{-\frac{1}{2}(y_p+\theta a_n/\sqrt{2n})^2}, \tag{2.5}$$

where $0 < \theta < 1$. Then by a theorem of Liapounoff's [3, p. 77],

$$\frac{|a_n|}{\sqrt{2n}}e^{-\frac{1}{2}(y_p+\theta a_n/\sqrt{2n})^2} \leq \frac{K \log n}{\sqrt{n}},$$

where K denotes a constant independent of n. But if $\lim_{n\to\infty} |a_n|/\sqrt{2n} = \infty$, then $\lim_{n\to\infty} H_n(z_{pn})$ is either 0 or 1. Hence $a_n = o(\sqrt{n})$.

Fisher [1, p. 81] has suggested the use of

$$\chi_{p,n}^2 \doteq \tfrac{1}{2}[y_p + \sqrt{2n-1}]^2.$$

A closer approximation,

$$\chi_{p,n}^2 \doteq n\left[1 - \frac{2}{9n} + y_p\sqrt{\frac{2}{9n}}\right]^3,$$

has been obtained by Wilson and Hilferty [2].. It is interesting to note that, according to (1.3), this last approximation is correct to terms of the zero-th order in n.

We apply Theorem 1 to the variables $X_j = (\chi_1^2 - 1)/\sqrt{2}$, $j = 1, 2, \cdots$. Then $\sigma = 1$, and, by the additive property of the Chi-Square distribution [3, p. 45], $H_n(x)$, the distribution function of the variable $(X_1 + \cdots + X_n)/\sqrt{n}$, is related to $F_n(x)$ by (2.2). Thus, $\lambda_3 = 2\sqrt{2}$. It follows from (1.5) and (2.3) that

$$\lim_{n\to\infty} \sqrt{2n}\,[\Phi(z_{pn}) - H_n(z_{pn})] = \lim_{n\to\infty} \frac{2}{3\sqrt{2\pi}} (z_{pn}^2 - 1)e^{-\frac{1}{2}z_{pn}^2} \tag{2.6}$$
$$= \frac{2}{3\sqrt{2\pi}} (y_p^2 - 1)e^{-\frac{1}{2}y_p^2},$$

since $a_n = o(\sqrt{n})$. Then by (2.5) and (2.6)

$$\lim_{n\to\infty} a_n = \tfrac{2}{3}(y_p^2 - 1).$$

According to (2.3) we may now write

$$\chi_{p,2n}^2 = 2n + 2y_p\sqrt{n} + 2r_p + 2b_n,$$

where

$$r_p = \tfrac{1}{3}(y_p^2 - 1), \tag{2.7}$$

and $b_n = o(1)$. A simple change of variables in (2.1) yields

$$F_{2n}(\chi_{p,2n}^2) = \int_{-\frac{n+r_p}{\sqrt{n}}}^{y_p+\frac{b_n}{\sqrt{n}}} \frac{\sqrt{n}e^{-n}n^n}{\Gamma(n+1)} e^{-v\sqrt{n}-r_p}\left[1 + \frac{v}{\sqrt{n}} + \frac{r_p}{n}\right]^{n-1} dv. \tag{2.8}$$

If we let

$$J_n = \int_{y_p}^{y_p+\frac{b_n}{\sqrt{n}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2}\, dv, \tag{2.9}$$

then

$$nJ_n = \frac{b_n\sqrt{n}}{\sqrt{2\pi}} e^{-\frac{1}{2}(y_p+\delta_n)^2}, \tag{2.10}$$

where $\delta_n = o(1)$. By (1.2) and (2.4),

$$J_n = \Phi\left(y_p + \frac{b_n}{\sqrt{n}}\right) - F_{2n}(\chi_{p,2n}^2). \tag{2.11}$$

Using Stirling's formula for $\Gamma(n+1)$ in (2.8), (2.11) becomes

$$J_n = \int_{y_p+\frac{b_n}{\sqrt{n}}}^{\infty} \frac{e^{-\frac{1}{2}v^2}}{\sqrt{2\pi}}\left[\exp\left\{-\frac{1}{12n} + 0(n^{-3})\right.\right.$$
$$\left.\left.+ (n-1)\log\left(1 + \frac{v}{\sqrt{n}} + \frac{r_p}{n}\right) + \frac{v^2}{2} - v\sqrt{n} - r_p\right\} - 1\right] dv$$
$$= \int_{y+\frac{b_n}{\sqrt{n}}}^{\infty} \frac{e^{-\frac{1}{2}v^2}}{\sqrt{2\pi}} [e^{A_n} - 1]\, dv,$$

where $A_n = \frac{1}{n}\left(-\frac{1}{12} - r_p - \frac{r_p^2}{2} + \frac{v^2}{2} + v^2 r_p - \frac{v^4}{4}\right) + \frac{1}{\sqrt{n}}\left(\frac{v^3}{3} - v - vr_p\right) + f_p(v)$, $nf_n(v)$ being dominated by $P(|v|)$, where P is a polynomial in v independent of n, and $f_n(v) = 0(n^{-3/2})$. Then

$$J_n = \int_{y_p+\frac{b_n}{\sqrt{n}}}^{\infty} \frac{e^{-\frac{1}{2}v^2}}{n\sqrt{2\pi}}\left[-\frac{1}{12} - r_p - \frac{r_p^2}{2} + v^2\left(1 + 2r_p + \frac{r_p^2}{2}\right)\right.$$

$$\tag{2.12} \left. -v^4\left(\frac{7}{12} + \frac{r_p}{3}\right) + \frac{v^6}{18}\right] dv + \int_{y_p+\frac{b_n}{\sqrt{n}}}^{\infty} \frac{e^{-\frac{1}{2}v^2}}{\sqrt{2\pi n}}\left[\frac{v^3}{3} - v - vr_p\right] dv$$

$$+ \int_{y_p+\frac{b_n}{\sqrt{n}}}^{\infty} \frac{e^{-\frac{1}{2}v^2}}{\sqrt{2\pi}} g_n(v)\, dv + \int_{y_p+\frac{b_n}{\sqrt{n}}}^{\infty} \frac{e^{-\frac{1}{2}v^2}}{\sqrt{2\pi}}\left(\sum_{j=3}^{\infty} \frac{A_n^j}{j!}\right) dv,$$

where $g_n(v)$ has the same properties given above for $f_n(v)$. If we call these last integrals K_1, K_2, K_3 and K_4 respectively, we see that

$$\tag{2.13} \lim_{n\to\infty} nK_3 = \int_{y_p}^{\infty} \frac{e^{-\frac{1}{2}v^2}}{\sqrt{2\pi}} \lim_{n\to\infty} ng_n(v)\, dv = 0.$$

Also, since A_n^j, $j = 1, 2, \cdots$ is dominated by $P_j(|v|)$, $P_j(v)$ being a polynomial in v independent of n, we see that

$$\sum_{j=3}^{\infty} \int_{y_p+\frac{b_n}{\sqrt{n}}}^{\infty} \frac{e^{-\frac{1}{2}v^2}}{\sqrt{2\pi}} \frac{|A_n|^j}{j!}\, dv \leq \sum_{j=3}^{\infty} e^{-\frac{1}{2}\left(y_p+\frac{b_n}{\sqrt{n}}\right)^2} \frac{Q_j\left(y_p + \frac{b_n}{\sqrt{n}}\right)}{j!},$$

where Q_j is a polynomial. Since this last sum converges, we have

$$\tag{2.14} nK_4 = \sum_{j=3}^{\infty} \int_{y_p+\frac{n}{\sqrt{n}}}^{\infty} \frac{e^{-\frac{1}{2}v^2}}{\sqrt{2\pi}} \frac{nA_n^j}{j!}\, dv,$$

and by the uniform convergence of (2.14),

$$\tag{2.15} \lim_{n\to\infty} nK_4 = \sum_{j=3}^{\infty} \int_{y_p}^{\infty} \frac{e^{-\frac{1}{2}v^2}}{j!\sqrt{2\pi}} \lim_{n\to\infty} nA_n^j\, dv = 0,$$

since $A_n^j = 0(n^{-j/2})$.

Integrating by parts we obtain

$$nK_2 = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(y_p+\frac{b_n}{\sqrt{n}}\right)^2}\left(\frac{2}{3} y_p b_n + \frac{b_n^2}{3\sqrt{n}}\right),$$

and since $b_n = o(1)$,

$$\tag{2.16} \lim_{n\to\infty} nK_2 = 0.$$

Further integration by parts and the use of (2.7) yields

$$\lim_{n\to\infty} nK_1 = \frac{e^{-\frac{1}{2}y_p^2}}{36\sqrt{2\pi}}(y_p^3 - 7y_p). \tag{2.17}$$

Then, by (2.10), (2.12), (2.13), (2.15), (2.16) and (2.17),

$$\lim_{n\to\infty} b_n\sqrt{n} = \frac{1}{36}(y_p^3 - 7y_p),$$

so that

$$\chi^2_{p,2n} = 2n + 2y_p\sqrt{n} + \frac{2}{3}(y_p^2 - 1) + \frac{y_p^3 - 7y_p}{18\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right).$$

Equation (1.3) now follows at once.

3. Student's t. If the random variable Y_n has the distribution function $\Phi(x/\sqrt{n})$, then $t_n = Y_n/\chi_n$ is distributed according to Student's distribution for n degrees of freedom and has the distribution function

$$G_n(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{n\pi}} \frac{\Gamma[\frac{1}{2}(n+1)]}{\Gamma[\frac{1}{2}n]} \left(1 + \frac{t^2}{n}\right)^{-\frac{1}{2}(n+1)} dt.$$

If $\sigma = \sqrt{n/(n-2)}$, the variable t_n/σ then has the distribution function

$$H_n(x) = G_n(x\sigma). \tag{3.1}$$

If we write

$$t_{p,n} = y_p + a_n, \tag{3.2}$$

so that

$$G_n(t_{p,n}) = 1 - p, \tag{3.3}$$

and let $z_{pn} = t_{p,n}/\sigma$, then $H_n(z_{pn}) = 1 - p$, and it follows from (1.1) and (1.2) that

$$\begin{aligned} \Phi(z_{pn}) - H_n(z_{pn}) &= \Phi(z_{pn}) - \Phi(y_p) \\ &= \frac{1}{\sqrt{2\pi}}\left[y_p\left(\frac{1}{\sigma} - 1\right) + \frac{a_n}{\sigma}\right] e^{-\frac{1}{2}\left[y_p + \theta\left(y_p\left(\frac{1}{\sigma}-1\right)+a_n\right)\right]^2}, \end{aligned} \tag{3.4}$$

where $0 < \theta < 1$. Then by Liapounoff's Theorem [3, p. 77],

$$\left| y_p\left(\frac{1}{\sigma} - 1\right) + \frac{a_n}{\sigma}\right| e^{-\frac{1}{2}\left[y_p+\theta\left(y_p\left(\frac{1}{\sigma}-1\right)+a_n\right)\right]^2} \leqslant \frac{K \log n}{\sqrt{n}},$$

where K denotes a constant independent of n. But if $\lim_{n\to\infty} |a_n| = \infty$, then $\lim_{n\to\infty} H_n(z_{pn})$ is either 0 or 1.

Hence $a_n = o(1)$.

We apply Theorem 1 to the variables $X_j = Y_n/\chi_n$, $j = 1, 2, \cdots$. Then $\sigma = \sqrt{n/(n-2)}$, and by the additive property of the normal distribution, $H_n(x)$, the distribution function of $(X_1 + \cdots + X_n)/(\sigma\sqrt{n})$, satisfies the relation (3.1). Thus $\lambda_3 = 0$ and $\lambda_4 = 6n/(n-4)$. It follows from (1.5) and (3.2) that

$$\begin{aligned}\lim_{n\to\infty} n[\Phi(z_{pn}) - H_n(z_{pn})] &= \lim_{n\to\infty} \frac{n}{4(n-4)\sqrt{2\pi}} (z_{pn}^3 - 3z_{pn})e^{-\frac{1}{2}z_{pn}^2} \\ &= \frac{1}{4\sqrt{2\pi}} (y_p^3 - 3y_p)e^{-\frac{1}{2}y_p^2},\end{aligned} \tag{3.5}$$

since $a_n = o(1)$. By (3.4) and (3.5) we have

$$\lim_{n\to\infty} n\left[y_p\left(\frac{1}{\sigma} - 1\right) + \frac{a_n}{\sigma}\right] = \frac{y_p^3 - 3y_p}{4}.$$

But $\lim_{n\to\infty} n(1-\sigma)/\sigma = -1$, so that

$$\lim_{n\to\infty} na_n = \frac{y_p^3 + y_p}{4}.$$

Hence

$$a_n = \frac{y_p^3 + y_p}{4n} + o\left(\frac{1}{n}\right),$$

and equation (1.4) follows at once.

4. Tables. The following tables compare the true values of $\chi^2_{p,n}$ and $t_{p,n}$ with those obtained from (1.3) and (1.4). The true values [4], [5], (to three decimal places) are shown in *italics*.

TABLE 1

$\chi^2_{p,n}$

n \ p	.01	.05	.1	.5	.9
10	23.253 *23.209*	18.318 *18.307*	15.989 *15.987*	9.333 *9.342*	4.875 *4.865*
30	50.908 *50.892*	43.777 *43.773*	40.257 *40.256*	29.333 *29.336*	20.600 *20.599*
50	76.163 *76.154*	67.507 *67.505*	63.168 *63.167*	49.333 *49.335*	37.689 *37.689*
100	135.811 *135.807*	124.343 *124.342*	118.499 *118.498*	99.333 *99.334*	82.358 *82.358*

TABLE II

$t_{p,n}$

n \ p	.0125	.025	.05	.125	.25
10	2.579 *2.634*	2.197 *2.228*	1.797 *1.813*	1.212 *1.221*	0.700 *0.700*
30	2.354 *2.360*	2.039 *2.042*	1.696 *1.697*	1.171 *1.173*	0.683 *0.683*
60	2.298 *2.299*	2.000 *2.000*	1.670 *1.671*	1.161 *1.162*	0.679 *0.679*
120	2.270 *2.270*	1.980 *1.980*	1.658 *1.658*	1.156 *1.156*	0.677 *0.677*

REFERENCES

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