ON CUMULATIVE SUMS OF RANDOM VARIABLES

By Abraham Wald

Columbia University

1. Introduction. Let $\{z_i\}$ (i = 1, 2, $\cdots$, ad inf.) be a sequence of independent random variables each having the same distribution. Denote by Z_j the sum of the first j elements of the sequence $\{Z_i\}$, i.e.,

$$Z_j = z_1 + z_2 + \cdots + z_j \qquad (j = 1, 2, \cdots, \text{ad inf.}). \tag{1}$$

Let a be a given positive constant and b a given negative constant. Denote by n the smallest positive integer for which Z_n lies outside the open interval (b, a), i.e., Z_n is either $\leq b$ or $\geq a$. Obviously n is a random variable. If $b < Z_i < a$ for $i = 1, 2, \cdots$, ad inf., we shall say that $n = \infty$.

For any relation R we shall denote the probability that R holds by $P(R)$. It will be shown later that $P(n = \infty) = 0$, provided the variance of z_i is positive.

In this paper we shall deal with the problem of obtaining the value of $P(Z_n \geq a)$[1] and that of finding the probability distribution of n.

The study of such cumulative sums is of interest in various statistical problems. For example, a multiple sampling scheme proposed recently by Walter Bartky[2] makes use of such cumulative sums.

Cumulative sums also play an important role in the theory of the random walk of interest in physics. The results obtained in this paper may have bearing particularly on the theory of the random walk with absorbing barriers. In the presence of an absorbing wall the random walk stops whenever the particle arrives at the wall, i.e., whenever the cumulative sum of the displacements reaches a certain value.[3]

2. Two Lemmas. Lemma 1. *If the variance of z_i is not zero,* $P(n = \infty) = 0$.

Proof: Let $c = |a| + |b|$. If $n = \infty$ then for any positive integer r the following inequalities must hold

$$\left(\sum_{i=kr+1}^{(k+1)r} z_i\right)^2 < c^2 \qquad (k = 0, 1, 2, \cdots, \text{ad inf.}). \tag{2}$$

To prove $P(n = \infty) = 0$, it is sufficient to show that the probability is zero that (2) holds for all integer values of k. Since the variance of z_i is not zero, the ex-

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[1] Since $P(n = \infty) = 0$, we have $P(Z_n \leqq b) = 1 - P(Z_n \geqq a)$.

[2] "Multiple sampling with constant probability", *Annals of Math. Stat.*, Vol. 14 (1943), pp. 363–377.

[3] See in this connection S. Chandrasekhar, "Stochastic problems in physics and astronomy", *Rev. of Modern Physics*, Vol. 15 (1943), p. 5.

pected value of $\left(\sum_{i=1}^{j} z_i\right)^2$ converges to ∞ as $j \to \infty$. Hence there exists a positive integer r such that

$$P\left[\left(\sum_{i=1}^{r} z_i\right)^2 < c^2\right] < 1. \tag{3}$$

From (3) it follows that the probability that (2) is fulfilled for all values of k is equal to zero. Hence $P(n = \infty) = 0$ and Lemma 1 is proved.

LEMMA 2. *Let z be a random variable such that the following four conditions are fulfilled:*

Condition I. Both the expected value Ez of z and the variance of z exist and are unequal to zero.

Condition II. There exists a positive δ such that $P(e^z < 1 - \delta) > 0$ and $P(e^z > 1 + \delta) > 0$.

Condition III. For any real value h the expected value $Ee^{hz} = g(h)$ exists.

Condition IV. The first two derivatives of the function $g(h)$ exist and may be obtained by differentiation under the integral sign, i.e.,

$$g'(h) = \frac{d}{dh} Ee^{hz} = Eze^{hz},$$

and

$$g''(h) = \frac{d^2}{dh^2} Ee^{hz} = Ez^2 e^{hz}.$$

Then there exists one and only one real value $h_0 \neq 0$ such that

$$Ee^{h_0 z} = 1.$$

PROOF: For any positive h we have

$$g(h) > P(e^z > 1 + \delta)(1 + \delta)^h. \tag{4}$$

Hence, since $P(e^z > 1 + \delta) > 0$,

$$\lim_{h=\infty} g(h) = +\infty. \tag{5}$$

Similarly we see that for any negative h

$$g(h) > P(e^z < 1 - \delta)(1 - \delta)^h. \tag{6}$$

Hence, since $P(e^z < 1 - \delta) > 0$, we have

$$\lim_{h=-\infty} g(h) = +\infty. \tag{7}$$

Since $g''(h) = Ez^2e^{hz}$ it follows easily from Condition II that

$$g''(h) > 0, \tag{8}$$

for all real values of h.

The relations (5), (7) and (8) imply that there exists exactly one real value h^* for which $g(h)$ takes its minimum value. Since $g'(0) = Ez$ is unequal to zero by Condition I, we see that $h^* \neq 0$ and $g(h^*) < g(0) = 1$. It is clear that the function $g(h)$ is monotonically decreasing in the strict sense over the interval $(-\infty, h^*)$, and is monotonically increasing in the strict sense over the interval $(h^*, +\infty)$. Since $g(0) = 1$ and $g(h^*) < 1$, there exists exactly one real value $h_0 \neq 0$ such that $g(h_0) = 1$. Hence Lemma 2 is proved.

3. A fundamental identity. Denote by z a random variable whose distribution is equal to the common distribution of $z_i (i = 1, 2, \cdots, \text{ad inf.})$. Let D' be the subset of the complex plane such that $Ee^{zt} = \varphi(t)$ exists and is finite for any point t in D'. Consider the following identity

$$(9)\qquad Ee^{Z_n t + (Z_N - Z_n)t} = Ee^{Z_N t} = [\varphi(t)]^N,$$

where N denotes a positive integer. Let P_N be the probability that $n \leq N$. For any random variable u denote by $E_N(u)$ the conditional expected value of u under the restriction that $n \leq N$, and by $E_N^*(u)$ the conditional expected value of u under the restriction that $n > N$. Then identity (9) can be written as

$$(10)\qquad P_N E_N e^{Z_n t + (Z_N - Z_n)t} + (1 - P_N) E_N^* e^{Z_N t} = [\varphi(t)]^N.$$

Since in the subpopulation defined by any fixed $n \leq N$ the expression $Z_N - Z_n$ is independent of Z_n, we have

$$(11)\qquad E_N e^{Z_n t + (Z_N - Z_n)t} = E_N e^{Z_n t}[\varphi(t)]^{N-n}.$$

From (10) and (11) we obtain the identity

$$(12)\qquad P_N E_N \{e^{Z_n t}[\varphi(t)]^{N-n}\} + (1 - P_N) E_N^* e^{Z_N t} = [\varphi(t)]^N.$$

Dividing both sides by $[\varphi(t)]^N$ we obtain

$$(13)\qquad P_N E_N \{e^{Z_n t}[\varphi(t)]^{-n}\} + (1 - P_N) \frac{E_N^* e^{Z_N t}}{[\varphi(t)]^N} = 1.$$

Let D'' be the subset of the complex plane in which $|\varphi(t)| \geq 1$ and denote by D the common part of the subsets D' and D''. Since $\lim_{N=\infty} (1 - P_N) = 0$, and since $|E_N^*(e^{Z_N t})|$ is a bounded function of N, we have in D

$$(14)\qquad \lim_{N=\infty} (1 - P_N) \frac{E_N^* e^{Z_N t}}{[\varphi(t)]^N} = 0.$$

Since

$$\lim_{N=\infty} P_N E_N \{e^{Z_n t}[\varphi(t)]^{-n}\} = E\{e^{Z_n t}[\varphi(t)^{-n}\},$$

we obtain from (13) and (14) the fundamental identity

$$(15)\qquad E\{e^{Z_n t}[\varphi(t)]^{-n}\} = 1,$$

for any point t in the set D.

4. Derivation of the probability that $Z_n \geq a$. In what follows in this and the subsequent sections we shall always assume that the random variable z satisfies the conditions I–IV of Lemma 2, even if this is not stated explicitly. Since it follows from Condition III that the set D' is the whole complex plane, we see that the identity (15) must hold for all points t for which $|\varphi(t)| \geq 1$.

Let $h_0 \neq 0$ be the real value for which $\varphi(h_0) = 1$. Substituting h_0 for t in (15) we obtain

$$Ee^{Z_n h_0} = 1. \tag{16}$$

Let E_1 be the conditional expected value of $e^{Z_n h_0}$ under the restriction that $Z_n \geq a$ and let E_0 be the conditional expected value of $e^{Z_n h_0}$ under the restriction that $Z_n \leq b$. Furthermore denote $P(Z_n \geq a)$ by α. Then it follows from (16)

$$\alpha E_1 + (1 - \alpha)E_0 = 1. \tag{17}$$

Hence

$$\alpha = \frac{1 - E_0}{E_1 - E_0}. \tag{18}$$

If $h_0 > 0$ then $E_1 > 1$ and $E_0 < 1$. Hence (18) implies the inequality

$$\alpha \leq \frac{1}{E_1} \leq \frac{1}{e^{ah_0}}, \qquad (h_0 > 0). \tag{19}$$

If $h_0 < 0$ then $E_1 < 1$ and $E_0 > 1$. Hence (18) implies the inequality

$$1 - \alpha \leq \frac{1}{E_0} \leq \frac{1}{e^{bh_0}}, \qquad (h_0 < 0). \tag{20}$$

We shall now derive lower and upper limits for E_0 and E_1. We derive these limits under the assumption that $h_0 > 0$. To obtain a lower limit of E_0 consider a real variable ζ which is restricted to values > 1. For any random variable u and any relation R we shall denote by $E(u \mid R)$ the conditional expected value of u under the restriction that R holds. Denote by $P(\zeta)$ the probability that $e^{h_0 Z_{n-1}} < \zeta e^{bh_0}$. Then we have

$$E_0 = \int_1^\infty \left\{\zeta e^{bh_0} E\left[e^{h_0 z} \mid e^{h_0 z} \leq \frac{1}{\zeta}\right]\right\} dP(\zeta). \tag{21}$$

Hence a lower bound of E_0 is given by

$$E_0' = e^{bh_0}\left\{\underset{\zeta}{\text{g.l.b.}}\ \zeta E\left(e^{h_0 z} \mid e^{h_0 z} \leq \frac{1}{\zeta}\right)\right\}, \tag{22}$$

where the symbol $\underset{\zeta}{\text{g.l.b.}}$ stands for greatest lower bound with respect to ζ. Since e^{bh_0} is an upper bound of E_0, we obtain the limits

$$e^{bh_0}\left\{\underset{\zeta}{\text{g.l.b.}}\ \zeta E\left(e^{h_0 z} \mid e^{h_0 z} \leq \frac{1}{\zeta}\right)\right\} \leq E_0 \leq e^{bh_0} \qquad (h_0 > 0). \tag{23}$$

Let ρ be a real ariable restricted to values > 0 and < 1. Denote by $Q(\rho)$ the probability that $e^{h_0 Z_{n-1}} < \rho e^{ah_0}$. Then similarly to (21) we obtain

$$E_1 = \int_0^1 \left\{ \rho e^{ah_0} E\left(e^{h_0 z} \mid e^{h_0 z} \geq \frac{1}{\rho}\right)\right\} dQ(\rho). \tag{24}$$

Hence an upper bound of E_1 is given by

$$e^{ah_0}\left\{ \underset{\rho}{\text{l.u.b.}}\ \rho E\left(e^{h_0 z} \mid e^{h_0 z} \geq \frac{1}{\rho}\right)\right\}. \tag{25}$$

Since e^{ah_0} is a lower bound of E_1, we obtain the following limits for E_1

$$e^{ah_0} \leq E_1 \leq e^{ah_0}\left\{ \underset{\rho}{\text{l.u.b.}}\ \rho E\left(e^{h_0 z} \mid e^{h_0 z} \geq \frac{1}{\rho}\right)\right\}, \qquad (h_0 > 0). \tag{26}$$

In a similar way upper and lower limits can be derived for E_0 and E_1 when $h_0 < 0$. With the help of these limits upper and lower limits for α can be derived on the basis of equation (18). If $h_0 > 0$ then $E_1 > 1$, $E_0 < 1$ and consequently the right hand side of (18) is a monotonically decreasing function of E_0 and E_1. Hence if E_i' is a lower, and E_i'' is an upper bound of $E_i (i = 0, 1)$, then

$$\frac{1 - E_0''}{E_1'' - E_0''} \leq \alpha \leq \frac{1 - E_0'}{E_1' - E_0'}, \qquad (h_0 > 0). \tag{27}$$

In a similar way limits for α can be obtained when $h_0 < 0$. If both the absolute value of Ez and the variance of z are small, E_0 and E_1 will be nearly equal to e^{bh_0} and e^{ah_0}, respectively. Hence, in this case a good approximation to α is given by the expression

$$\bar{\alpha} = \frac{1 - e^{bh_0}}{e^{ah_0} - e^{bh_0}}. \tag{28}$$

The difference $\bar{\alpha} - \alpha$ approaches zero if both the mean and standard deviation of z converge to zero.

5. The characteristic function of n. Let $\bar{Z}_n$ be a random variable defined as follows: $\bar{Z}_n = a$ if $Z_n \geq a$ and $\bar{Z}_n = b$ if $Z_n \leq b$. Denote the difference $\bar{Z}_n - Z_n$ by ϵ. Then ϵ is a random variable.

In what follows we shall neglect ϵ i.e., we shall substitute 0 for ϵ. No error is committed by doing so in the special case when z can take only two values d and $-d$ and the ratios a/d and b/d are integers, since in this case ϵ is exactly zero. Apart from this special case the variate ϵ will not be identical with the constant zero. However, the smaller the values $|Ez|$ and Ez^2, the smaller the error we commit by neglecting ϵ. In fact, for arbitrary small positive numbers δ_1 and δ_2 the inequality $p(|\epsilon| \leq \delta_1) \geq 1 - \delta_2$ will hold if $|Ez|$ and Ez^2 are sufficiently small. Thus in the limiting case when Ez and Ez^2 approach zero the random variable ϵ reduces to the constant zero.

(a). *The characteristic function of n when only one of the quantities a and b is finite.* It will be sufficient to treat the case when a is finite and $b = -\infty$. In this case n is defined as the smallest positive integer for which $Z_n \geq a$. To make the probability of the existence of such a value n to be equal to 1 we have to assume that the expected value μ of z is positive. Since $b = -\infty$, the fundamental identity (15) need not hold for all points t of the set D. However, it follows easily from (13) that (15) holds for all points t in D whose real part is non-negative. Denote by $\psi(\tau)$ the characteristic function of n (τ is a purely imaginary variable). Since $Z_n = a$ (neglecting ϵ), and

$$E[\varphi(t)]^{-n} = \psi[-\log \varphi(t)],$$

identity (15) can be written as

$$e^{at}\psi[-\log \varphi(t)] = 1. \tag{29}$$

Let $t(\tau)$ denote a root (with non-negative real part) of the equation in t

$$\log \varphi(t) + \tau = 0, \tag{30}$$

and substitute $t(\tau)$ for t in (29). Then we obtain

$$\psi(\tau) = e^{-at(\tau)}. \tag{31}$$

As an illustration let us calculate $\psi(\tau)$ in the case when z is normally distributed. In this case

$$\log \varphi(t) = \mu t + \frac{\sigma^2}{2} t^2,$$

where μ is the mean and σ is the standard deviation of z. Hence

$$t(\tau) = \frac{-\mu \pm \sqrt{\mu^2 - 2\sigma^2\tau}}{\sigma^2}. \tag{32}$$

If we take the + sign before the square root sign, the real part of $t(\tau)$ is non-negative, since the real part of $\sqrt{\mu^2 - 2\sigma^2\tau}$ is greater than or equal to μ. Hence the characteristic function of n is given by

$$\psi(\tau) = e^{-a/\sigma^2[-\mu+\sqrt{\mu^2-2\sigma^2\tau}]} \qquad (\mu > 0). \tag{33}$$

(b). *The characteristic function of n when a and b both are finite.* Given the value of n, let p_n be the conditional probability that $\bar{Z}_n = a$. Let p_n^* denote the probability that n is the smallest positive integer for which either $\bar{Z}_n = a$ or $\bar{Z}_n = b$ holds. Neglecting $\bar{Z}_n - Z_n$, identity (15) can be written as

$$\sum_{n=1}^{\infty} [p_n e^{at} + (1 - p_n)e^{bt}][\varphi(t)]^{-n} p_n^* = 1. \tag{34}$$

Let $\psi_1(\tau)$ be the characteristic function of n in the subpopulation where $\bar{Z}_n = a$, and let $\psi_2(\tau)$ be the characteristic function of n in the subpopulation where $\bar{Z}_n =$

b. Furthermore let $\psi(\tau)$ be the characteristic function of n in the total population.

Since we neglect the difference $\bar{Z}_n - Z_n$, it follows from (18) that the probability α that $\bar{Z}_n = a$ is given by

$$\alpha = \frac{1 - e^{bh_0}}{e^{ah_0} - e^{bh_0}}. \tag{35}$$

Putting $1 - p_n = q_n$ the following relations hold

$$\psi_1[-\log \varphi(t)] = \frac{\sum_{n=1}^{\infty} p_n p_n^* [\varphi(t)]^{-n}}{\sum p_n p_n^*} = \frac{\sum p_n p_n^* [\varphi(t)]^{-n}}{\alpha} \tag{36}$$

$$\psi_2[-\log \varphi(t)] = \frac{\sum q_n p_n^* [\varphi(t)]^{-n}}{\sum q_n p_n^*} = \frac{\sum q_n p_n^* [\varphi(t)]^{-n}}{1 - \alpha} \tag{37}$$

$$\begin{aligned} \psi[-\log \varphi(t)] &= \sum p_n^* [\varphi(t)]^{-n} = \sum (p_n + q_n)[\varphi(t)]^{-n} p_n^* \\ &= \alpha\psi_1[-\log \varphi(t)] + (1 - \alpha)\psi_2[-\log \varphi(t)]. \end{aligned} \tag{38}$$

Putting $-\log\varphi(t) = \tau$ we obtain from (34), (36) and (37)

$$\alpha\psi_1(\tau)e^{at} + (1 - \alpha)\psi_2(\tau)e^{bt} = 1. \tag{39}$$

According to Lemma 2 the equation $-\log \varphi(t) = 0$ has two different real roots in t, $t = 0$ and $t = h_0$, and $\varphi'(0)$ and $\varphi'(h_0)$ both are unequal to zero. Hence, if $\varphi(t)$ is not singular at $t = 0$ and $t = h_0$, the equation

$$-\log\varphi(t) = \tau,$$

has two roots $t_1(\tau)$ and $t_2(\tau)$ for sufficiently small values of τ such that $\lim_{\tau \to 0} t_1(\tau) = 0$ and $\lim_{\tau \to 0} t_2(\tau) = h_0$. Since the identity (15) holds for all values of t for which $|\varphi(t)| \geq 1$, and since $|\varphi[t_1(\tau)]| = |\varphi[t_2(\tau)]| = 1$ for all imaginary values of τ, it follows from (39) that both equations hold

$$\alpha\psi_1(\tau)e^{at_1(\tau)} + (1 - \alpha)\psi_2(\tau)e^{bt_1(\tau)} = 1, \tag{39'}$$

$$\alpha\psi_1(\tau)e^{at_2(\tau)} + (1 - \alpha)\psi_2(\tau)e^{bt_2(\tau)} = 1. \tag{39''}$$

Solving these two linear equations we obtain $\psi_1(\tau)$ and $\psi_2(\tau)$. The characteristic function $\psi(\tau)$ is given by

$$\psi(\tau) = \alpha\psi_1(\tau) + (1 - \alpha)\psi_2(\tau).$$

As an illustration we shall determine $\psi_1(\tau)$, $\psi_2(\tau)$ and $\psi(\tau)$ when z has a normal distribution with mean μ and standard deviation σ. We have

$$-\log \varphi(t) = -\mu t - \frac{\sigma^2}{2} t^2 = \tau.$$

Hence

$$t = \frac{-\mu \pm \sqrt{\mu^2 - 2\sigma^2\tau}}{\sigma^2}. \tag{40}$$

Putting $e^a = A$ and $e^b = B$ we obtain from (39) and (40)

$$\alpha\psi_1(\tau)A^{-\mu/\sigma^2+1/\sigma^2\sqrt{\mu^2-2\sigma^2\tau}} + (1 - \alpha)\psi_2(\tau)B^{-\mu/\sigma^2+1/\sigma^2\sqrt{\mu^2-2\sigma^2\tau}} = 1 \tag{41}$$

$$\alpha\psi_1(\tau)A^{-\mu/\sigma^2-1/\sigma^2\sqrt{\mu^2-2\sigma^2\tau}} + (1 - \alpha)\psi_2(\tau)B^{-\mu/\sigma^2-1/\sigma^2\sqrt{\mu^2-2\sigma^2\tau}} = 1. \tag{42}$$

These two equations are valid for any imaginary value of τ. Since $h_0 = \frac{-2\mu}{\sigma^2}$, we obtain from (35)

$$\alpha = \frac{1 - B^{-2\mu/\sigma^2}}{A^{-2\mu/\sigma^2} - B^{-2\mu/\sigma^2}}. \tag{43}$$

Let

$$g_1 = -\frac{\mu}{\sigma^2} + \frac{1}{\sigma^2}\sqrt{\mu^2 - 2\sigma^2\tau}, \tag{44}$$

and

$$g_2 = -\frac{\mu}{\sigma^2} - \frac{1}{\sigma^2}\sqrt{\mu^2 - 2\sigma^2\tau}. \tag{45}$$

Then we obtain from (41) and (42)

$$\alpha\psi_1(\tau) = \frac{B^{g_2} - B^{g_1}}{A^{g_1}B^{g_2} - A^{g_2}B^{g_1}}, \tag{46}$$

and

$$(1 - \alpha)\psi_2(\tau) = \frac{A^{g_1} - A^{g_2}}{A^{g_1}B^{g_2} - A^{g_2}B^{g_1}}. \tag{47}$$

Hence the characteristic function of n is given by

$$\psi(\tau) = \frac{A^{g_1} + B^{g_2} - A^{g_2} - B^{g_1}}{A^{g_1}B^{g_2} - A^{g_2}B^{g_1}}. \tag{48}$$

6. The distribution of n when z is normally distributed. (a) *The case when a is finite and $b = -\infty$.* In this case the characteristic function of n is given by (33). Let

$$m = \frac{\mu^2}{2\sigma^2} n. \tag{49}$$

Then the characteristic function of m is given by

$$\psi^*(t) = e^{c[1-\sqrt{1-t}]}, \tag{50}$$

where

(51) $$c = \frac{a\mu}{\sigma^2} > 0.$$

The distribution of m is given by

(52) $$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{c(1-\sqrt{1-t})-mt}\,dt.$$

Let

(53) $$G(c, m) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-c\sqrt{1-t}-mt}\,dt,$$

and

(54) $$H(c, m) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{\sqrt{1-t}}\, e^{-c\sqrt{1-t}-mt}\,dt.$$

Since

(55) $$\frac{1}{2\pi i}\frac{d}{dt}\, e^{-c\sqrt{1-t}-mt} = \frac{1}{2\pi i}\left(\frac{c}{2\sqrt{1-t}} - m\right) e^{-c\sqrt{1-t}-mt}$$

we have

(56) $$\frac{c}{2}\, H(c, m) - mG(c, m) = \frac{1}{2\pi i}\,[e^{-c\sqrt{1-t}-mt}]_{-i\infty}^{i\infty} = 0.$$

From (53) and (54) we obtain

(57) $$\frac{\partial H(c, m)}{\partial c} + G(c, m) = 0.$$

From (56) and (57) it follows that

(58) $$\frac{c}{2}\, H(c, m) + m\,\frac{\partial H(c, m)}{\partial c} = 0.$$

Hence

(59) $$\log H(c, m) = -\frac{c^2}{4m} + \log \lambda(m)$$

where $\lambda(m)$ is some function of m only. Thus

(60) $$H(c, m) = \lambda(m) e^{-c^2/4m}$$

Now we shall determine $\lambda(m)$. We have

(61) $$\lambda(m) = H(0, m) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{\sqrt{1-t}}\, e^{-mt}\,dt.$$

Since $(1 - t)^{-1/2}$ is the characteristic function of $\frac{1}{2}\chi^2$ where χ^2 has the χ^2-distribution with one degree of freedom, the right hand side of (61) is equal to

$$\frac{1}{\Gamma(\frac{1}{2})\sqrt{m}} e^{-m}.$$

Hence

$$\lambda(m) = \frac{1}{\Gamma(\frac{1}{2})\sqrt{m}} e^{-m}. \tag{62}$$

From (60) and (61) we obtain

$$H(c, m) = \frac{1}{\Gamma(\frac{1}{2})\sqrt{m}} e^{-c^2/4m-m}. \tag{63}$$

From (56) and (63) we obtain

$$G(c, m) = \frac{c}{2\Gamma(\frac{1}{2})m^{3/2}} e^{-c^2/4m-m}. \tag{64}$$

Hence the distribution of m is given by

$$F(m)\, dm = \frac{c}{2\Gamma(\frac{1}{2})m^{3/2}} e^{-c^2/4m-m+c}\, dm, \qquad (0 \leq m < \infty). \tag{65}$$

Let $m = \frac{c}{2} m^*$. Then the distribution of m^* is given by

$$\begin{aligned} D(m^*)\, dm^* &= \frac{c^2/2}{2\Gamma\left(\frac{1}{2}\right)\left(\frac{c}{2}\right)^{3/2} (m^*)^{3/2}} e^{-(c/2)(1/m^*+m^*-2)}\, dm^* \\ &= \frac{\sqrt{c}}{\sqrt{2\pi}(m^*)^{3/2}} e^{-(c/2)(1/m^*+m^*-2)}\, dm^*. \end{aligned} \tag{66}$$

The function $\frac{1}{m^*} + m^* - 2$ is non-negative and is equal to zero only when $m^* = 1$. If c is large, then $D(m^*)$ is exceedingly small for values of m^* not close to 1. Expanding $\frac{1}{m^*} + m^* - 2$ in a Taylor series around $m^* = 1$, we obtain

$$\frac{1}{m^*} + m^* - 2 = (m^* - 1)^2 + \text{higher order terms.} \tag{67}$$

Hence for large c

$$D(m^*)\, dm^* \sim \frac{\sqrt{c}}{\sqrt{2\pi}} e^{-(c/2)(m^*-1)^2}\, dm^*, \tag{68}$$

i.e., if c is large m^* is nearly normally distributed with mean equal to 1 and standard deviation $\frac{1}{\sqrt{c}}$.

(b). *The case when a and b both are finite.* In this case the characteristic function of n is given by (48). Let

$$m = \frac{\mu^2}{2\sigma^2} n \qquad \text{and} \qquad d = -\frac{\mu}{\sigma^2}.$$

Then the characteristic function of m is given by

$$\psi^*(t) = \frac{A^{h_1} + B^{h_2} - A^{h_2} - B^{h_1}}{A^{h_1} B^{h_2} - A^{h_2} B^{h_1}}, \tag{69}$$

where

$$h_1 = d(1 - \sqrt{1 - t}), \qquad h_2 = d(1 + \sqrt{1 - t}), \tag{70}$$

and t is an imaginary variable. Putting $A^d = \bar{A}$, $B^d = \bar{B}$, $da = \bar{a}$ and $db = \bar{b}$, the characteristic function of m can be written as

$$\begin{aligned} \psi^*(t) &= \frac{\bar{A}(e^{-\bar{a}\sqrt{1-t}} - e^{\bar{a}\sqrt{1-t}}) + \bar{B}(e^{\bar{b}\sqrt{1-t}} - e^{-\bar{b}\sqrt{1-t}})}{\bar{A}\bar{B}(e^{(\bar{b}-\bar{a})\sqrt{1-t}} - e^{(\bar{a}-\bar{b})\sqrt{1-t}})} \\ &= \frac{\bar{A}(e^{-\bar{b}\sqrt{1-t}} - e^{(2\bar{a}-\bar{b})\sqrt{1-t}}) + \bar{B}(e^{\bar{a}\sqrt{1-t}} - e^{(\bar{a}-2\bar{b})\sqrt{1-t}})}{\bar{A}\bar{B}(1 - e^{2(\bar{a}-\bar{b})\sqrt{1-t}})}. \end{aligned} \tag{71}$$

It will be sufficient to consider only the case when $\mu > 0$, since the case < 0 can be treated in a similar way. Then $\bar{a} < 0$ and $\bar{b} > 0$. Since the real part of $+\sqrt{1 - t}$ is greater than or equal to one, we have

$$| e^{2(\bar{a}-\bar{b})\sqrt{1-t}} | < 1, \tag{72}$$

for any imaginary value of t. Let

$$T = e^{2(\bar{a}-\bar{b})\sqrt{1-t}}. \tag{73}$$

Then

$$\frac{1}{1 - T} = \sum_{j=0}^{\infty} T^j. \tag{74}$$

From (71) and (74) it follows that $\psi^*(t)$ can be written in the form of an infinite series.

$$\psi^*(t) = \sum_{i=1}^{\infty} r_i e^{-\lambda_i\sqrt{1-t}}, \tag{75}$$

where λ_i and r_i are constants and $\lambda_i > 0$. Each term of this series is a characteristic function of the form given in (50) except for a proportionality factor. Let $F_i(m)$ be the distribution of m corresponding to the characteristic function $e^{\lambda_i - \lambda_i\sqrt{1-t}}$. Then $F_i(m)$ can be obtained from (65) by substituting λ_i for c. Since we may integrate the right hand side member of (75) term by term, the distribution of m is given by

$$F(m)\, dm = \left(\sum_{i=1}^{\infty} \frac{r_i}{e^{\lambda_i}} F_i(m)\right) dm. \tag{76}$$

Since m is a discrete variable, it may seem paradoxical that we obtained a probability density function for m. However, the explanation lies in the fact that we neglected $\epsilon = \bar{Z}_n - Z_n$ and this quantity is zero only in the limiting case when μ and σ approach zero.

If $|\mu|$ and σ are sufficiently small as compared with a and $|b|$, the distribution of m given in (76) will be a good approximation to the exact distribution of m, even if z is not normally distributed. The reason for this can be indicated as follows: Let

$$z_i^* = \sum_{j=(i-1)r+1}^{ir} z_j \qquad (i = 1, 2, \cdots, \text{ad inf.}) \tag{77}$$

where r is a given positive integer. Since the variates z_j are independently distributed each having the same distribution, under some weak conditions the variates $z_i^*(i = 1, 2, \cdots, \text{ad inf.})$ will be nearly normally distributed for large r. Hence, considering the cumulative sums $Z_i^* = z_1^* + z_2^* + \cdots + z_i^*(i = 1, 2, \cdots$, ad inf.), the distribution given in (76) is applicable with good approximation, provided that $r|\mu|$ and $\sqrt{r}\sigma$ are small as compared with a and $|b|$ so that the difference $\epsilon^* = \bar{Z}_n^* - Z_n^*$ can be neglected.

7. The exact probability distribution of Z_n and the exact characteristic function of n when z can take only integral multiples of a given constant d. In the previous sections we derived the probability $P(Z_n \geq a)$ and the characteristic function of n under the assumption that the quantity by which Z_n may differ from a or b is small and can be neglected. This can be done whenever $|Ez|$ and Ez^2 are small. However, if $|Ez|$ or Ez^2 is not small, it is desirable to derive the exact probability distribution of Z_n and the exact characteristic function of n. Both are obtained in the present section for random variables z which can take only a finite number of integral multiples of a given constant d. This is a rather general result, since any distribution of z can be approximated arbitrarily fine by a discrete distribution of the above type if the constant d is chosen sufficiently small.

There is no loss of generality in assuming that $d = 1$, since the quantity d can be chosen as the unit of measurement. Thus, we shall assume that z takes only a finite number of integral values. Let g_1 and g_2 be two positive integers such that $P(z = -g_1)$ and $P(z = g_2)$ are positive and z can take only integral values $\geq -g_1$ and $\leq g_2$. Denote $P(z = i)$ by h_i. Then the characteristic function of z is given by

$$\varphi(t) = \sum_{i=-g_1}^{g_2} h_i e^{ti}. \tag{78}$$

To obtain the roots of the equation $\varphi(t) = 1$, we put $e^t = u$ and solve the equation

$$\sum_{i=-g_1}^{g_2} h_i u^i = 1. \tag{79}$$

Denote $g_1 + g_2$ by g and let the g roots of (79) be $u_1, \cdots, u_g$, respectively. We shall assume that no two roots are equal, i.e., $u_i \neq u_j$ for $i \neq j$. Substituting u_i for e^t in the identity (15) we obtain

$$E(u_i^{Z_n}) = 1 \qquad (i = 1, \cdots, g). \tag{80}$$

Denote by $[a]$ the smallest integer $\geq a$, and by $[b]$ the largest integer $\leq b$. Then Z_n can take only the values

$$[b] - g_1 + 1, \quad [b] - g_1 + 2, \cdots, [b], [a], \quad [a] + 1, \cdots, [a] + g_2 - 1. \tag{81}$$

Denote the g different integers in (81) by $c_1, \cdots, c_g$, respectively. Furthermore, denote $P(Z_n = c_i)$ by ξ_i. Then equations (80) can be written as

$$\sum_{j=1}^{g} \xi_j u_i^{c_j} = 1 \qquad (i = 1, \cdots, g). \tag{82}$$

Let Δ be the determinant value of the matrix $\| u_i^{c_j} \|$ $(i, j = 1, \cdots, g)$ and let Δ_j be the determinant we obtain from Δ by substituting 1 for the elements in the jth column. If $\Delta \neq 0$, it follows from (82) that $P(Z_n = c_j) = \xi_j$ is given by

$$\xi_j = \frac{\Delta_j}{\Delta}. \tag{83}$$

Hence, $P(Z_n \geq a) = \sum_j (\Delta_j/\Delta)$ summed for all values of j for which $c_j \geq a$. From the probability distribution of Z_n we can easily derive the expected value En of n. In fact, differentiating the fundamental identity (15) with respect to t at $t = 0$ we obtain

$$E\left[Z_n - \frac{\varphi'(0)}{\varphi(0)}\, n\right] = 0. \tag{84}$$

Since $\dfrac{\varphi'(0)}{\varphi(0)} = Ez$, we obtain from (84)

$$En = \frac{EZ_n}{Ez} = \frac{1}{Ez} \sum_{j=1}^{g} \frac{c_j \Delta_j}{\Delta}. \tag{85}$$

Now we shall derive the exact characteristic function of n. Denote by $\psi_i(\tau)$ (τ is a purely imaginary variable) the characteristic function of the conditional distribution of n under the restriction that $Z_n = c_i$. Let $t_1(\tau), \cdots, t_g(\tau)$ be g roots of the equation

$$\varphi(t) = e^{-\tau}, \tag{86}$$

such that

$$\lim_{\tau=0} e^{t_i(\tau)} = u_i. \tag{87}$$

Substituting $t_i(\tau)$ for t in the fundamental identity (15) we obtain

$$\sum_{j=1}^{g} \xi_j e^{c_j t_i(\tau)} \psi_j(\tau) = 1 \qquad (i = 1, \cdots, g). \tag{88}$$

These equations are linear in the unknowns $\psi_1(\tau), \cdots, \psi_g(\tau)$ and the determinant of these equations is given by

$$\text{(89)} \qquad \delta(\tau) = \begin{vmatrix} \xi_1 e^{c_1 t_1(\tau)} & \cdots & \xi_g e^{c_g t_1(\tau)} \\ \xi_1 e^{c_1 t_2(\tau)} & \cdots & \xi_g e^{c_g t_2(\tau)} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \xi_1 e^{c_1 t_g(\tau)} & \cdots & \xi_g e^{c_g t_g(\tau)} \end{vmatrix}.$$

Obviously, $\delta(0) = \xi_1 \xi_2 \cdots \xi_g \Delta$. Hence if $\xi_i \neq 0$ $(i = 1, \cdots, g)$ and $\Delta \neq 0$, also $\delta(0) \neq 0$ and consequently $\delta(\tau) \neq 0$ for any τ with sufficiently small absolute value. Thus, $\psi_1(\tau), \cdots, \psi_g(\tau)$ can be obtained by solving the linear equations (88). The characteristic function $\psi(\tau)$ of the unconditional distribution of n is given by

$$\text{(90)} \qquad \psi(\tau) = \sum_{i=1}^{g} \xi_i \psi_i(\tau).$$