SOME COMBINATORIAL FORMULAS WITH APPLICATIONS TO PROBABLE VALUES OF A POLYNOMIAL-PRODUCT AND TO DIFFERENCES OF ZERO

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1. The main purpose of the paper is to establish some combinatorial formulas concerning the mathematical expectations or probable values of a product of n given polynomials. The problem may be stated more definitely as follows:

Let $x_1, \cdots, x_n$ be n non-negative discontinuous variables for which we have assumed that the probability that each x takes a possible value is equally likely, and let $f_1(x), \cdots, f_n(x)$ be n given polynomials. Then we shall ask: What is the probable value of the product $f_1(x_1) \cdots f_n(x_n)$, provided the sum of the variables $x_1, \cdots, x_n$ is known? More generally, we may consider the problem with certain restrictions to x such as $a \leq x_i \leq b$, $(i = 1 \cdots n)$.

By a limiting process[1] it will be found that all the formulas established for the preceding problem can be extended to the case of continuous variables. On this account, it is important to find explicit formulas for the problem merely involving discontinuous variables.

By the definition[2] of MacMahon, we say that a set of numbers $(x_1 \cdots x_n)$ is over all different compositions of m into n parts with each $x \geq k$, if $(x_1 \cdots x_n)$ runs over all different integer solutions of the linear equation $x_1 + \cdots x_n = m$ with each $x \geq k$. We shall use the notation $(m;\ k;\ x_1 \cdots x_n)$, or simply $(m;\ k;\ x)$, to denote that a set of numbers $(x_1 \cdots x_n)$ is over all different compositions of m into n parts with each $x \geq k$.

The notation $E(m; \delta; [f_1(x)] \cdots [f_n(x)])$ will be used to denote the mathematical expectation of the product $f_1(x_1) \cdots f_n(x_n)$ in which the sum of n variable quantities $x_1, \cdots, x_n$ is known, namely $x_1 + \cdots + x_n = m$, and each quantity is a multiple of δ and m is of course a multiple of δ. Thus by the definition[3] of mathematical expectations we have

$$E(m;\ \delta;\ [f_1] \cdots [f_n]) = \Big(\sum_{(m/\delta;\, 1;\, x)} 1 \Big)^{-1} \sum_{(m/\delta;\, 1;\, x)} f_1(x_1 \delta) \cdots f_n(x_n \delta), \tag{1}$$

where the summation on the right-hand side runs over all different compositions of m/δ into n parts with each $x \geq 1$, and the given constant δ is called a "varying unit", that is the least possible difference between two unequal quantities in $(x_1 \cdots x_n)$. If the varying unit approaches zero, $(x_1 \cdots x_n)$ will become a set of continuous variables.

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[1] The limiting process will be illustrated by the proof of corollary 2 of theorem 1 in this paper.

[2] MacMahon, *Combinatory Analysis*, Vol. 1, p. 150.

[3] See for example W. Burnside, *Theory of Probability*, Chap. 4, 13.

If $f_1(x) = \cdots = f_n(x)$, we may write

$$E(m;\delta;[f]^n) \quad \text{instead of} \quad E(m;\delta;[f_1]\cdots[f_n]).$$

A well-known convention for $\binom{m}{n}$ is also adopted here:

$$\binom{m}{n} = \begin{cases} \dfrac{m!}{n!(m-n)!}, & \text{if } 0 \le n \le m, \\ 0 & \text{otherwise.} \end{cases}$$

2. Lemmas. In order to obtain explicit formulas desired we first establish four lemmas as follows:

LEMMA 1. *Let $m, r_1, \cdots, r_n$ be non-negative integers. Then*

$$\sum_{(m;0;x)} \prod_{\nu=1}^{n} \binom{x_\nu}{r_\nu} = \binom{m+n-1}{r_1+\cdots+r_n+n-1}. \tag{2}$$

PROOF. Construct a generating function:

$$\left(\frac{1}{1-x}\right)^{r_1+1} \cdots \left(\frac{1}{1-x}\right)^{r_n+1}, \qquad |x| < 1.$$

It is observed that the coefficient of the term $x^{m-(r_1+\cdots+r_n)}$ in the expansion of the above product is given by

$$\sum_{(m-r_1-\cdots-r_n;0;x)} \binom{r_1+x_1}{x_1} \cdots \binom{r_n+x_n}{x_n} = \sum_{(m;0;x)} \binom{x_1}{r_1} \cdots \binom{x_n}{r_n}.$$

On the other hand we see that the coefficient of the term $x^{m-(r_1+\cdots+r_n)}$ in the expansion of $\left(\frac{1}{1-x}\right)^{r_1+\cdots+r_n+n}$ is given by

$$\binom{r_1+\cdots+r_n+n+m-(r_1+\cdots+r_n)-1}{m-(r_1+\cdots+r_n)} = \binom{m+n-1}{\Sigma r_1+n-1}.$$

Hence the lemma.

LEMMA 2. *Let $a, b, c, \cdots$ be arbitrary constants, and $k_1, k_2, k_3, \cdots$ be positive integers. Then*

$$\begin{aligned} &\sum_{(m;1;x)} \prod_{\nu=1}^{n} \left\{ a\binom{x_\nu}{k_1} + b\binom{x_\nu}{k_2} + c\binom{x_\nu}{k_3} + \cdots \right\} \\ &\quad = n! \sum_{(n;0;\alpha\beta\gamma\cdots)} \binom{m+n-1}{\alpha k_1+\beta k_2+\gamma k_3+\cdots+n-1} \frac{a^\alpha}{\alpha!}\frac{b^\beta}{\beta!}\frac{c^\gamma}{\gamma!}\cdots. \end{aligned} \tag{3}$$

PROOF. Expanding the left-hand side of (3), we see that the coefficient of the term $a^\alpha b^\beta c^\gamma \cdots$ is equal to

$$\frac{n!}{\alpha!\beta!\gamma!\cdots} \sum_{(m;0;x)} \binom{x_1}{k_1} \cdots \binom{x_\alpha}{k_1}\binom{x_{\alpha+1}}{k_2} \cdots \binom{x_{\alpha+\beta}}{k_2}\binom{x_{\alpha+\beta+1}}{k_3} \cdots \binom{x_{\alpha+\beta+\gamma}}{k_3} \cdots.$$

By lemma 1 it is reduced to

$$\frac{n!}{\alpha!\beta!\gamma!\cdots}\binom{m+n-1}{\alpha k_1+\beta k_2+\gamma k_3+\cdots+n-1}.$$

Substituting, we get the lemma.

From now on, we shall frequently write $f^{(x)}$ instead of $f(x)$, so that

$$f^{(\nu)}g^{(0)}+\binom{\nu}{1}f^{(\nu-1)}g^{(1)}+\cdots+\binom{\nu}{\nu}f^{(0)}g^{(\nu)}=(f+g)^{(\nu)}.$$

LEMMA 3. *Let m, n ($\leq m$) be two positive integers. Then for any given polynomial $f(x)$ of the kth degree we have*

$$(4)\qquad \sum_{(m;1;x)} f(x_1)\cdots f(x_n)=n!\sum_{(n;0;p)}\binom{m+n-1}{S(p)+n-1}\prod_{\nu=0}^{k}\frac{[(f-1)^{(\nu)}]^{p_\nu}}{p_\nu!},$$

where

$$f^{(x)}=f(x),\qquad S(p)=1\cdot p_1+\cdots+k\cdot p_k.$$

PROOF. Since $f(x)$ is a polynomial of the kth degree, there exist $(k+1)$ values $\beta_k, \cdots, \beta_0$ such that

$$\beta_k\binom{x}{k}+\cdots+\beta_1\binom{x}{1}+\beta_0=f(x).$$

By putting $x=0, 1, \cdots, k$, we find successively[4]

$$(5)\quad \beta_\nu=f^{(\nu)}-\binom{\nu}{1}f^{(\nu-1)}+\cdots+(-1)^\nu\binom{\nu}{\nu}f^{(0)}=(f-1)^{(\nu)},\qquad (\nu=0, 1, \cdots, k).$$

The lemma is thus obtained by means of (3).

For convenience, we denote the summation

$$\sum_{(m;1;x)} f_1(x_1)\cdots f_n(x_n)\quad\text{by}\quad S(m, [f_1]\cdots[f_n]).$$

Thus formula (4) can be rewritten:

$$(4)'\quad S(m, [f]^n)=n!\sum_{(n;0;p)}\binom{m+n-1}{1\cdot p_1+\cdots+k\cdot p_k+n-1}\frac{\beta_0^{p_0}}{p_0!}\frac{\beta_1^{p_1}}{p_1!}\cdots\frac{\beta_k^{p_k}}{p_k!},$$

where $\beta_0, \beta_1, \cdots, \beta_k$ are given by (5).

LEMMA 4. *Let $f_1(x), \cdots, f_n(x)$ be n given polynomials, Then*

$$(6)\qquad S(m, [f_1]\cdots[f_n])=\frac{1}{n!}\sum_{(\nu_1\cdots\nu_k)\in(1\cdots n)}(-1)^{n-k}S(m, [f_{\nu_1}+\cdots+f_{\nu_k}]^n),$$

[4] Strictly speaking, the relation (5) is established and proved by induction on ν, in which a well known combinatorial equality is applied.

where the summation on the right-hand side runs over all different combinations out of $(1 \cdots n)$, $(k = 1 \cdots n)$.

For example, if $n = 3$, then all the different combinations out of (123) will be: (1), (2), (3), (12), (13), (23), (123).

PROOF. Consider a typical term

$$\frac{n!}{q_1! \cdots q_t!} S(m, [f_{\nu_1}]^{q_1} \cdots [f_{\nu_t}]^{q_t}),$$

where $1 \leq t \leq n$, $q_1 + \cdots + q_t = n$. Now a necessary and sufficient condition that the term will be contained in the expansion of $S(m, [f_{\mu_1} + \cdots + f_{\mu_k}]^n)$ is $(\nu_1 \cdots \nu_t) \epsilon (\mu_1 \cdots \mu_k)$, i.e. $(\nu_1 \cdots \nu_t)$ is a combination out of $(\mu_1 \cdots \mu_k)$, and for a fixed k, there are $\binom{n-t}{k-t}$-different combinations of $(\mu_1 \cdots \mu_k)$ satisfying the condition $(\nu_1 \cdots \nu_t) \epsilon (\mu_1 \cdots \mu_k) \epsilon (1 \cdots n)$. Therefore the number of occurrences of the term in the right-hand side of (6) is given by

$$\sum_{\nu=0}^{n-t} (-1)^\nu \binom{n-t}{\nu} = (1-1)^{n-t} = \begin{cases} 0 & \text{if } t < n, \\ 1 & \text{if } t = n. \end{cases}$$

The term vanishes generally except $q_1 = \cdots = q_t = 1$. Hence the right-hand side gives $S(m, [f_1] \cdots [f_n])$.

3. Theorems and corollaries. In the following statement of theorems and corollaries the notation $(x_1 \cdots x_n)$ will be always used to denote a set of undetermined quantities as specified.

THEOREM 1. *Suppose that $(x_1 \cdots x_n)$ is a set of natural numbers in which only the sum of the numbers is known, namely $x_1 + \cdots + x_n = m$. Then, for any given polynomial $f(x)$ of the kth degree, the mathematical expectation of $f(x_1) \cdots f(x_n)$ is given by*

$$E(m, 1, [f]^n) = n! \binom{m-1}{n-1}^{-1} \sum_{(n;0;p)} \binom{m+n-1}{S(p)+n-1} \frac{[(f-1)^{(0)}]^{p_0} \cdots [(f-1)^{(k)}]^{p_k}}{p_0! \cdots p_k!}. \tag{7}$$

PROOF. Let $m' = m + nr$. By lemma 1 we have

$$\sum_{(m;0;x)} \binom{x_1}{0} \cdots \binom{x_n}{0} = \sum_{(m;0;x)} 1 = \sum_{(m';r;x)} 1 = \binom{m' - nr + n - 1}{n-1}.$$

This is the number of compositions of m' into n parts with each part $\geq r$. In particular, if $r = 1$, we find that the number of compositions of m into n parts is $\binom{m-1}{n-1}$. Thus by (1), the required value is equal to

$$\{S(m, [1]^n)\}^{-1} S(m, [f]^n), \quad \text{i.e.} \quad \binom{m-1}{n-1}^{-1} S(m, [f]^n).$$

The theorem is therefore proved by lemma 3.

COROLLARY 1. *Let $(x_1 \cdots x_n)$ be a set of positive quantities, of which the varying unit is δ, and the sum is m. Then, for any given polynomial $f(x)$ of the kth degree, we have*

$$E(m, \delta, [f]^n) \tag{8}$$
$$= n!\begin{pmatrix}\left(\frac{m}{\delta}\right) - 1\\ n-1\end{pmatrix}^{-1} \sum_{(n;0;p)} \begin{pmatrix}\left(\frac{m}{\delta}\right) + n - 1\\ S(p) + n - 1\end{pmatrix} \frac{[(g-1)^{(0)}]^{p_0} \cdots [(g-1)^{(k)}]^{p_k}}{p_0! \cdots p_k!},$$

where

$$g(x) = f(x\delta), \qquad S(p) = p_1 + \cdots + kp_k .$$

PROOF. It follows immediately by the relation:

$$E(m, \delta, [f(x)]^n) = E\left(\left(\frac{m}{\delta}\right), 1, [f(\delta x)]^n\right).$$

COROLLARY 2. *Let $(x_1 \cdots x_n)$ be a set of non-negatively real numbers, of which the sum is known, namely $x_1 + \cdots + x_n = m$, m being a known real number. Then, for any given polynomial $f(x) = a_0 + \cdots + a_kx^k$, $(a_k \neq 0)$ we have*

$$E(m, 0, [f]^n) \tag{9}$$
$$= \frac{(n!)^2}{n} \sum_{(n;0;p)} \frac{m^{1\cdot q_1+\cdots+k\cdot q_k}}{(1\cdot q_1 + \cdots + kq_k + n - 1)!} \frac{(0!a_0)^{q_0}}{q_0!} \cdots \frac{(k!a_k)^{q_k}}{q_k!}.$$

PROOF. The proof of the corollary depends essentially on the concept that two unequal real numbers may differ by an arbitrary small number h.

Let h be an arbitrary positive number and write $f(xh)/h^k = g(x, h)$, where the number k is the degree of $f(x)$. Then, since

$$\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu}(n-\nu)^p = \begin{cases} 0 & \text{if } p < n \\ n! & \text{if } p = n \\ \binom{n+1}{2} n! & \text{if } p = n+1, \end{cases}$$

we may write

$$g(\nu, h) - \binom{\nu}{1} g(\nu - 1, h) + \cdots + (-1)^\nu \binom{\nu}{\nu} g(0, h) = h^{\nu-k}\{\nu!a_\nu + h\cdot R_\nu(h)\},$$
$$\nu = 0, 1, \cdots k,$$

where

$$\lim_{h\to 0} R_\nu(h) = \left\{\binom{\nu}{0}\nu^{\nu+1} - \binom{\nu}{1}(\nu-1)^{\nu+1} + \cdots + (-1)^{\nu+1}\binom{\nu}{\nu-1}\right\} a_{\nu+1}$$
$$= \binom{\nu+1}{2} \nu! a_{\nu+1} .$$

Now we pass to the limit $h \to 0$ in which we assume that h runs through a sequence of real numbers of the form $\frac{m}{N}$, N being a natural number. Thus by corollary 1 we have

$$\lim_{h\to 0} E(m, h, [f]^n)$$

$$= n! \lim_{h\to 0} \sum_{(n;0;p)} \left(\binom{\left(\frac{m}{h}\right) + n - 1}{S(p) + n - 1}\binom{\left(\frac{m}{h}\right) - 1}{n - 1}\right)^{-1} h^{nk} \prod_{\nu=0}^{k} \frac{\{\nu! a_\nu + h\cdot R_\nu(h)\}^{p_\nu}}{h^{(k-\nu)p_\nu}\cdot p_\nu!},$$

$$= n! \sum_{(n;0;p)} \left\{\lim_{h\to 0} \frac{\left(\left(\frac{m}{h}\right) + n - 1\right)!(n-1)!\left(\left(\frac{m}{h}\right) - n\right)! h^{S(p)}}{(S(p) + n - 1)!\left(\left(\frac{m}{h}\right) - S(p)\right)!\left(\left(\frac{m}{h}\right) - 1\right)!}\right\}$$

$$\left\{\lim_{h\to 0} \prod_{\nu=0}^{k} \frac{[\nu! a_\nu + h\cdot R_\nu(h)]^{p_\nu}}{p_\nu!}\right\},$$

$$= \sum_{(n;0;p)} \frac{n!(n-1)! m^{S(p)}}{(S(p) + n - 1)!} \prod_{\nu=0}^{k} \frac{(\nu! a_\nu)^{p_\nu}}{p_\nu!}, \qquad (S(p) = 1\cdot p_1 + \cdots + k\cdot p_k).$$

Hence the corollary.[5]

COROLLARY 3. *Let $(x_1 \cdots x_n)$ be a set of positive real numbers under a known condition $a \leq x_1 + \cdots + x_n \leq b$, where a $(< b)$, b are two positive real numbers. Then, for any given polynomial $f(x) = a_0 + \cdots + a_k x^k$, $(a_k \neq 0)$, the mathematical expectation of the product $f(x_1) \cdots f(x_n)$ which we denote by $E((ab), 0, [f]^n)$ is given by the formula*

$$(10) \quad E((ab), 0, [f]^n) = \frac{n!(n-1)!}{b-a} \sum_{(n;0;q)} \frac{b^{1+S(q)} - a^{1+S(q)}}{(1 + S(q))\cdot(n - 1 + S(q))!}$$

$$\cdot \frac{a_0^{q_0} \cdots (k!\, a_k)^{q_k}}{q_0! \cdots q_k!}, \quad (S(q) = q_1 + \cdots + kq_k).$$

PROOF. Since the probability that the sum of $x_1, \cdots, x_n$ takes a value between a and b is equally likely, we see that the required mathematical expectation will be the mean of $\int_a^b E(u, 0, [f]^n)du$, that is

$$E((ab), 0, [f]^n) = \frac{1}{b-a}\int_a^b E(u, 0, [f]^n)\, du.$$

[5] This corollary can also be proved by means of Dirichlet's integral. In fact, the right-hand side of (9) is given by the quotient of the two integrals:

$$(\int \ldots \int f(x_1) \ldots f(x_n)\, dx_1 \ldots dx_n)/(\int \ldots \int dx_1 \ldots dx_n),$$

the integrals being taken over the region: $x_1 + \ldots + x_n = m$, $x_1 \geq 0, \ldots, x_n \geq 0$.

The formula (10) is therefore obtained by integrating the right-hand side of (9) and dividing it by $(b - a)$.

On the other hand we find that

$$\lim_{h \to 0} E((a, a + h), 0, [f]^n) = E(a, 0, [f]^n).$$

This shows that the corollary 2 can be also deduced from 3.

THEOREM 2. (A generalization of theorem 1). *Let $f_1(x), \cdots, f_n(x)$ be n given polynomials whose degrees do not exceed k. Then we have*

$$E(m, 1, [f_1] \cdots [f_n]) = \sum_{(\nu_1 \cdots \nu_t) \in (1 \cdots n)} \sum_{(n;0;p)} (-1)^{n-t} \frac{\binom{m+n-1}{S(p)+n-1}}{\binom{m-1}{n-1}} \prod_{j=0}^{k} \frac{[(f_{\nu_1 \cdots \nu_t} - 1)^{(j)}]^{p_j}}{p_j!}, \tag{11}$$

where

$$f_{\nu_1 \cdots \nu_t} = f_{\nu_1}(x) + \cdots + f_{\nu_t}(x), \qquad S(p) = 1 \cdot p_1 + \cdots + k \cdot p_j.$$

PROOF. In the proof of theorem 1 we have shown that

$$E(m, 1, [f]^n) = \binom{m-1}{n-1}^{-1} S(m, [f]^n).$$

Hence, by similar reasoning and lemma 4 we obtain

$$\begin{aligned} E(m, 1, [f_1] \cdots [f_n]) &= \binom{m-1}{n-1}^{-1} S(m, [f_1] \cdots [f_n]) \\ &= \sum_{(\nu_1 \cdots \nu_t) \in (1 \cdots n)} \frac{(-1)^{n-t}}{n! \binom{m-1}{n-1}} S(m, [f_{\nu_1} + \cdots + f_{\nu_t}]^n) \\ &= \sum_{(\nu_1 \cdots \nu_t) \in (1 \cdots n)} \sum_{(n;0;p)} (-1)^{n-t} \frac{\binom{m+n-1}{S(p)+n-1}}{\binom{m-1}{n-1}} \\ &\qquad \prod_{j=0}^{k} \frac{[(f_{\nu_1 \cdots \nu_t} - 1)^{(j)}]^{p_j}}{p_j!}. \end{aligned}$$

Theorem 2 is proved.

COROLLARY 1. *Let δ be a varying unit. Then*

$$\begin{aligned} E(m, \delta, [f_1] \cdots [f_n]) = \sum_{(\nu_1 \cdots \nu_t) \in (1 \cdots n)} \sum_{(n;0;p)} (-1)^{n-t} \binom{\binom{m}{\delta} + n - 1}{S(p) + n - 1} \\ \cdot \binom{m-1}{n-1}^{-1} \prod_{j=0}^{k} \frac{[(g_{\nu_1 \cdots \nu_t} - 1)^{(j)}]^{p_j}}{p_j!}, \end{aligned} \tag{12}$$

where

$$g_\nu(x) = f_\nu(x\delta), \ (\nu = 1 \cdots n) \quad \text{and} \quad g_{\nu_1 \cdots \nu_t} = g_{\nu_1}(x) + \cdots + g_{\nu_t}(x).$$

PROOF. This is almost trivial because of

$$E(m, \ \delta, \ [f_1(x)] \cdots [f_n(x)]) = E\left(\left(\frac{m}{\delta}\right), \ 1, \ [f_1(x\delta)] \cdots [f_n(x\delta)]\right).$$

COROLLARY 2.[6] *For any given positive real number m, we have*

$$E(m, 0, [x^{p_1}] \cdots [x^{p_n}]) = \frac{p_1! \cdots p_n! \, (n-1)!}{(p_1 + \cdots + p_n + n - 1)!} m^{p_1 + \cdots + p_n}. \tag{13}$$

PROOF. By a passage to the limit $\delta \to 0$, we get

$$\lim_{\delta \to 0} E(m, \delta, [f_1] \cdots [f_n]) = \lim_{\delta \to 0} \sum \frac{(-1)^{n-t}}{n!} E(m, \delta, [f_{\nu_1} + \cdots + f_{\nu_t}]^n)$$

$$= \sum \frac{(-1)^{n-t}}{n!} \lim_{\delta \to 0} E(m, \delta, [f_{\nu_1 \cdots \nu_t}]^n).$$

i.e.

$$E(m, 0, [f_1] \cdots [f_n]) = \sum \frac{(-1)^{n-t}}{n!} E(m, 0, [f_{\nu_1 \cdots \nu_t}]^n).$$

The corollary is then deduced by (9).

THEOREM 3. (Further generalization of theorem 1). *Let $(x_1 \cdots x_n)$ be a set of arbitrary integers restricted to the conditions:*

$$x_1 + \cdots + x_n = m, \qquad a \leq x_i \leq b,$$

where m, a, b are all known integers. Then for any given polynomial $f(x)$, the mathematical expectation of the product $f(x_1) \cdots f(x_n)$ denoted by $\underset{(a,b)}{E}(m, 1, [f]^n)$ is given by the following

$$\underset{(a,b)}{E}(m, 1, [f]^n) = \frac{\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} S(m', [g]^\nu [h]^{n-\nu})}{\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \binom{m'-1}{n-1}}, \tag{14}$$

where

$$g(x) = f(x+b), \quad h(x) = f(x+a-1), \quad m' = m - (a-1)n + (a-b-1)\nu.$$

PROOF. Define

$$S(m, [f]^n) = 0 \quad \text{for } m < n \text{ and} \quad S(m, [f]^0) = \begin{cases} 0 & \text{for} \quad m > 0, \\ 1 & \text{for} \quad m = 0. \end{cases}$$

[6] This can also be deduced by Dirichlet's integrals.

We shall show that

$$\sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} S(m', [g]^{\nu}[h]^{n-\nu}) = \sum_{\substack{(x_1 \cdots x_n) \\ a \leq x \leq b}} f(x_1) \cdots f(x_n),$$

where on the right-hand side of the expression the set $(x_1 \cdots x_n)$ under the summation runs over all different compositions of m into n parts with each $a \leq x_i \leq b$, $(i = 1 \cdots n)$.

We denote the left-hand side of the expression by $\mathfrak{S}$, then by decomposing $S(m', [g]^{\nu}[h]^{n-\nu})$ we have

$$\mathfrak{S} = \sum_{\nu=0}^{n} \sum_{\bar{m}=\nu}^{m'} (-1)^{\nu} \binom{n}{\nu} S(\bar{m}, [f(x + b)]^{\nu}) S(m' - \bar{m}, [f(x + a - 1)]^{n-\nu}).$$

Let $f(\bar{x}_1) \cdots f(\bar{x}_n)$ be a term contained in $\mathfrak{S}$, i.e. $\bar{x}_1 + \cdots + \bar{x}_n = m$, $\bar{x}_1 \geq a, \cdots, \bar{x}_n \geq a$. And we suppose that $\bar{x}_{\nu_1} \geq b + 1, \cdots, \bar{x}_{\nu_t} \geq b + 1$, where $\nu_i \neq \nu_j$ if $i \neq j$. Then it is found that the number of occurrences of the term in $\mathfrak{S}$ is given by

$$\sum_{s=0}^{t} (-1)^{s} \binom{t}{s} = (1 - 1)^{t} = \begin{cases} 0 & \text{if} \quad t \geq 1, \\ 1 & \text{if} \quad t = 0. \end{cases}$$

This shows that the term $f(\bar{x}_1) \cdots f(\bar{x}_n)$ of $\mathfrak{S}$ generally vanishes except $a \leq x_i \leq b$. Hence we have

$$\mathfrak{S} = \sum_{\substack{(x_1 \cdots x_n) \\ a \leq x \leq b}} f(x_1) \cdots f(x_n).$$

Next, we shall find the number of compositions of m into n parts with each $a \leq x_i \leq b$, i.e. the number of terms in $\mathfrak{S}$. By the result just obtained we see that the number is given by

$$\sum_{\nu=0}^{n} \sum_{\bar{m}=0}^{m'} (-1)^{\nu} \binom{n}{\nu} \sum_{(\bar{m};1;x)} 1 \sum_{(m'-\bar{m};1;x)} 1$$

$$= \sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} \left\{ \sum_{\bar{m}=0}^{m'} \binom{\bar{m} - 1}{\nu - 1} \binom{m' - \bar{m} - 1}{n - \nu - 1} \right\}$$

$$= \sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} \binom{m' - 1}{n - 1}.$$

Hence the theorem.

The theorem just proved shows that the mathematical expectation $\underset{(a,b)}{E}(m, 1, [f]^{n})$ can be expressed by $S(\bar{m}, [g]^{\nu})$ and is therefore expressible[7] in terms of the linear combinations of the coefficients of the polynomial $f(x)$.

COROLLARY 1. *Let δ be a varying unit for which $\frac{m}{\delta}, \frac{a}{\delta}, \frac{b}{\delta}$ are all integers. Then*

$$\underset{(a,b)}{E}(m, \delta, [f(x)]^{n}) = \underset{\left(\frac{a}{\delta}\frac{b}{\delta}\right)}{E}\left(\frac{m}{\delta}, 1, [f(\delta x)]^{n}\right).$$

[7] See lemmas 3 and 4.

COROLLARY 2. *Let $f_1(x), \cdots, f_n(x)$ be n given polynomials. Then*

$$\underset{(a,b)}{E}(m, \delta, [f_1] \cdots [f_n]) = \sum_{(\nu_1 \cdots \nu_t) \epsilon (1 \cdots n)} \frac{(-1)^{n-t}}{n!} \underset{(a,b)}{E}(m, \delta, [f_{\nu_1} + \cdots + f_{\nu_t}]^n).$$

COROLLARY 3. *The number of compositions of m into n parts with each $a_i \leq x_i \leq b_i$, $(i = 1 \cdots n)$ is equal to*

$$\sum_{\nu_1=0, \cdots \nu_n=0}^{1,\cdots,1} (-1)^{\nu_1+\cdots+\nu_n} \cdot \binom{m + n - (a_1 + \cdots + a_n) + (a_1 - b_1 - 1)\nu_1 + \cdots + (a_n - b_n - 1)\nu_n - 1}{n - 1}$$

PROOF. We have shown that the number of compositions of m into n parts with each $a \leq x \leq b$ is given by

$$\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \binom{m - (a - 1)n + (a - b - 1)\nu - 1}{n - 1}.$$

Hence the number of integer solutions of the equation

$$x_{11} + \cdots + k_{1n_1} + \cdots + x_{s1} + \cdots + x_{sn_s} = m$$

with each $a_\nu \leq x_{\nu\mu} \leq b_\nu (\nu = 1 \cdots s; \mu = 1 \cdots n_\nu)$ is given by

$$\sum_{(m;1;m_1 \cdots m_s)} \sum_{\nu_1=0}^{n_1} \cdots \sum_{\nu_s=0}^{n_s} (-1)^{\nu_1+\cdots+\nu_s} \prod_{i=1}^{s} \binom{n_i}{\nu_i} \cdot \binom{m_i - (a_i - 1)n_i + (a_i - b_i - 1)\nu_i - 1}{n_i - 1}$$

$$= \sum_{\nu_1=0}^{n_1} \cdots \sum_{\nu_s=0}^{n_s} (-1)^{\nu_1+\cdots+\nu_s} \prod_{i=1}^{s} \binom{n_i}{\nu_i} \cdot \left\{ \sum_{(m;1;m_i)} \prod_{i=1}^{s} \binom{m_i - (a_i - 1)n_i + (a_i - b_i - 1)\nu_i - 1}{n_i - 1} \right\}$$

$$= \sum_{\nu_1=0, \cdots, \nu_s=0}^{1,\cdots,1} (-1)^{\nu_1+\cdots+\nu_s} \binom{n_1}{\nu_1} \cdots \binom{n_s}{\nu_s} \cdot \binom{m - 1 - \sum (a_1 - 1)n_1 + \sum (a_1 - b_1 - 1)\nu_1}{n_1 + \cdots + n_s - 1}.$$

The corollary follows at once by putting $n_1 = \cdots = n_s = 1$, $s = n$.

The last corollary may be restated as follows: Let there be n stores, $b_1, \cdots, b_n$ being the numbers of stocks contained in 1st, 2nd, $\cdots$, nth store respectively. Then m stocks containing at least a_i stocks of the ith store can be taken from these stores in

$$\sum_{\nu_1=0, \cdots, \nu_n=0}^{1,\cdots,1} (-1)^{\nu_1+\cdots+\nu_n} \binom{m + n - 1 - \sum a_1 + \sum (a_1 - b_1 - 1)\nu_1}{n - 1}$$

— different ways.

We have now established several combinatorial formulas concerning the mathematical expectations of the product $f_1(x_1) \cdots f_n(x_n)$ under certain conditions. Apparently, there are many examples which can be solved by means of the results just obtained. For brevity, we may state a general criterion as follows: The mathematical expectation of a function, $F(x_1, \cdots, x_n)$ say, can be estimated by the above mentioned formulas, if and only if 1) the sum of $x_1, \cdots, x_n$ is known, and 2) there exist n polynomials $f_1(x), \cdots, f_n(x)$ such that $F(x_1, \cdots x_n)$ is proportional to $f_i(x_i)$, $(i = 1 \cdots n)$. The undetermined quantities in $(x_1, \cdots, x_n)$ may or may not be continuous, if the quantities are discontinuous, the varying unit is necessarily known.

4. Convenient formulas for differences of zero.[8]

Given $f(x) = \beta_0 + \beta_1 x + \cdots + \beta_k x^k (\beta_k \neq 0)$ we may write

$$(f-1)^{(\nu)} = \sum_{s=0}^{k} \nu!\beta_s S_{\nu,s} = \sum_{s=0}^{k} \beta_s \Delta^\nu 0^s,$$

where $S_{\nu,s}$ is a Stirling number of the second kind, as used by Jordan and defined by

$$\nu! S_{\nu,s} = \Delta^\nu 0^s = \sum_{x=0}^{\nu} (-1)^{\nu-x} \binom{\nu}{x} x^s,$$

$\Delta^\nu 0^s$ being in the language of the calculus of finite differences, "a difference of zero".

In terms of the differences of zero, the formulas (7) and (11) may also be restated as follows:

$$(7)' \quad E(m, 1, [f]^n) = \sum_{(n;0;p)} \frac{(m+n-1)!(m-n)!n!(n-1)!}{(m-S(p))!(S(p)+n-1)!(m-1)!} \times \prod_{\nu=0}^{k} \frac{1}{p_\nu!} (\beta_\nu \Delta^\nu 0^\nu + \cdots + \beta_k \Delta^\nu 0^k)^{p_\nu}.$$

$$(11)' \quad E(m, 1, [f_1] \cdots [f_n]) = \sum_{(\nu_1 \cdots \nu_t) \epsilon (1 \cdots n)} (-1)^{n-t} \sum_{(n;0;p)} \frac{(m+n-1)!(m-n)!(n-1)!}{(m-S(p))!(S(p)+n-1)!(m-1)!} \times \prod_{\nu=0}^{k} \frac{1}{p_\nu!} (B_\nu \Delta^\nu 0^\nu + \cdots + B_k \Delta^\nu 0^k)^{p_\nu},$$

where

$$f_\nu(x) = \beta_{\nu 0} + \cdots + \beta_{\nu k} x^k, \quad B_\mu = \beta_{1\mu} + \cdots + \beta_{n\mu}, \quad S(p) = 1 \cdot p_1 + \cdots + k p_k.$$

[8] The methods for obtaining convenient formulas for differences of zero as stated in the first part of this paragraph are similar to those used by Paul S. Dwyer in his paper "The computation of moments with cumulative totals," *Annals of Math. Stat.*, Vol. 9 (1938), pp. 288–303.

The formulas (7)′ and (11)′ tell us that the difference of zero plays an important rôle in the calculation of mathematical expectations of a polynomial product under known conditions. On account of this fact, we are now going to investigate some recurrence relations and approximations for the differences of zero.

As m is larger than t, we may find a convenient recurrence relation as follows:

(15)
$$\frac{\Delta^m 0^{m+t}}{m!} = S_{m,m+t} = \lambda_1(t)\binom{m+t}{t+1} + \lambda_2(t)\binom{m+t}{t+2} + \cdots + \lambda_t(t)\binom{m+t}{t+t},$$
$$\frac{\Delta^m 0^{m+t+1}}{m!} = S_{m,m+t+1} = \lambda_1(t+1)\binom{m+t+1}{t+1+1} + \lambda_2(t+1)\binom{m+t+1}{t+1+2} + \cdots + \lambda_{t+1}(t+1)\binom{m+t+1}{2t+2},$$

where

$$(t+j)\lambda_{j-1}(t) + j\cdot\lambda_j(t) \equiv \lambda_j(t+1), \quad \lambda_0 \equiv \lambda_{t+1}(t) \equiv 0, \quad \lambda_1(t) \equiv \lambda_1(t+1) \equiv 1,$$

and $\lambda_2(t), \cdots, \lambda_t(t)$ are all independent of m but depending on t.

Starting with the first equation of (15) and using a well known relation (due to Jordan) $S_{m,n+1} = S_{m-1,n} + m\cdot S_{m,n}$, successively, we get

$$\begin{aligned}
S_{m,m+t+1} &= \sum_{\nu=1}^{m} (m-\nu+1) S_{m-\nu+1,m+t+1-\nu} \\
&= \sum_{j=1}^{t} \lambda_j(t) \sum_{\nu=1}^{m} \binom{m+t+1-\nu}{t+j}(m-\nu+1) \\
&= \sum_{j=1}^{t} \lambda_j(t) \left\{ \sum_{\nu=1}^{m} \binom{m+t+1-\nu}{t+j+1}(t+j+1) + \sum_{\nu=1}^{m} \binom{m+t+1-\nu}{t+j}\cdot j \right\} \\
&= \sum_{j=1}^{t} \lambda_j(t) \left\{ \binom{m+t+1}{t+j+2}(t+j+1) + \binom{m+t+1}{t+j+1}\cdot j \right\} \\
&= \sum_{j=1}^{t+1} \{(t+j)\lambda_{j-1}(t) + j\lambda_j(t)\} \binom{m+t+1}{t+j+1}.
\end{aligned}$$

The recurrence relation thus follows.

By successive applications of the relation $\lambda_j(t+1) = (t+j)\cdot\lambda_{j-1}(t) + j\cdot\lambda_j(t)$, after nth time say, we may express $\lambda_j(t)$ as a linear combination of $\lambda_j(t-n), \cdots, \lambda_{j-n}(t-n)$, but the coefficients are too complicated.

For $t \leq 9$, by applying the recurrence relation as obtained above, the coefficients may be exhibited as follows:

t	$\lambda_2(t)$	$\lambda_3(t)$	$\lambda_4(t)$	$\lambda_5(t)$	$\lambda_6(t)$	$\lambda_7(t)$	$\lambda_8(t)$	$\lambda_9(t)$
1								
2	3							
3	10	15						
4	25	105	105					
5	56	490	1260	945				
6	119	1918	9450	17325	10395			
7	246	6825	56980	190575	270270	135135		
8	501	22935	302995	1636635	4099095	4729725	2027025	
9	1012	74316	1487200	12122110	47507460	94594500	91891800	34459425

For example, when $t = 4$ we have, according to the table:

$$\Delta^m 0^{m+4} = \left\{\binom{m+4}{5} + 25\binom{m+4}{6} + 105\binom{m+4}{7} + 105\binom{m+4}{8}\right\} m!.$$

We shall now proceed to find some approximations for $S_{n,n+t}$ and $\Delta^n 0^{n+t}$. Firstly, we may write

$$S_{n,n+t} \equiv \frac{\Delta^n 0^{n+t}}{n!} \equiv \lambda_1(t)\binom{n+t}{t+1} + \cdots + \lambda_t(t)\binom{n+t}{2t}.$$

According to the recurrence relation we have

(i) $\lambda_t(t) = (2t-1)\lambda_{t-1}(t-1),$

(ii) $\lambda_{t-1}(t) = 2(t-1)\cdot\lambda_{t-2}(t-1) + (t-1)\cdot\lambda_{t-1}(t-1),$

(iii) $\lambda_{t-2}(t) = (2t-3)\cdot\lambda_{t-3}(t-1) + (t-2)\cdot\lambda_{t-2}(t-1).$

Hence

$$\lambda_t(t) = \frac{(2t)!}{t!\cdot 2^t}; \qquad \lambda_{t-1}(t) = (t-1)!2^{t-1}\sigma(t);$$

$$\lambda_{t-2}(t) = 2^{t-2}\sum_{j=0}^{t-3}(t-2-j)!(t-2-j)\cdot(t-1.5)_j\cdot\sigma(t-1-j),$$

where

$$\sigma(k) = \sum_{x=1}^{k-1}\frac{x}{2^{2x}}\binom{2x}{x}, \qquad (t-1.5)_j = (t-1.5)(t-2.5)\cdots(t-j-0.5).$$

Evidently, the orders of $\binom{n+t}{2t-1}, \binom{n+t}{2t-2}, \cdots, \binom{n+t}{t+1}$ are all less than $2t$ as n tends to infinity.

Now, it can be easily found that

$$\lambda_t(t)\binom{n+t}{2t} = \frac{(2t)!}{t!2^t}\binom{n+t}{2t} = \left(\frac{n^2}{2}\right)^t \frac{1}{t!}\left(1+\frac{t}{n}\right)\cdots\left(1-\frac{t-1}{n}\right)$$

$$= \frac{1}{t!}\left(\frac{n^2}{2}\right)^t\left(1+\frac{t}{n}\right)\left(1-\frac{1}{n^2}\right)\cdots\left(1-\frac{(t-1)^2}{n^2}\right)$$

$$= \frac{1}{t!}\left(\frac{n^2}{2}\right)^t\left\{1+\frac{t}{n}-\frac{(t-1)t(2t-1)}{6n^2}+0\left(\frac{1}{n^3}\right)\right\};$$

$$\lambda_{t-1}(t)\binom{n+t}{2t-1} = \frac{2^t\cdot t!}{n-t+1}\binom{n+t}{2t}\sigma(t) = \frac{2^{2t}\,t!\,t!}{(n-t+1)\cdot(2t)!}\sigma(t)\binom{n+t}{2t}\lambda_t(t)$$

$$= \frac{1}{t!}\left(\frac{n^2}{2}\right)^t\left\{\frac{2^{2t}\sigma(t)}{n\binom{2t}{t}}+\frac{(2t-1)\cdot 2^{2t}\sigma(t)}{n^2\binom{2t}{t}}+0\left(\frac{1}{n^3}\right)\right\};$$

$$\lambda_{t-2}(t)\binom{n+t}{2t-2} = \frac{1}{t!}\left(\frac{n^2}{2}\right)^t\left\{\frac{2^t\cdot t!\cdot\lambda_{t-2}(t)}{n^2\cdot(2t-2)!}+0\left(\frac{1}{n^3}\right)\right\}.$$

Hence, we may write

$$S_{n,n+t} = \frac{1}{t!}\left(\frac{n^2}{2}\right)^t\left\{1+\frac{\rho_1}{n}+\frac{\rho_2}{n^2}+0\left(\frac{1}{n^3}\right)\right\}, \tag{16}$$

where

$$\rho_1 = t + 2^{2t}\binom{2t}{t}^{-1}\sigma(t) = t + \frac{4^t\cdot t!\,t!\,\sigma(t)}{(2t)!},$$

$$\rho_2 = -\tfrac{1}{6}t(t-1)(2t-1) + \frac{4^t\cdot t!\,t!\,(2t-1)}{(2t)!}\sigma(t) + \frac{2^t\cdot t!}{(2t-2)!}\lambda_{t-2}(t).$$

Moreover, it can be shown by Wallis' formula that

$$\sqrt{\frac{x-1}{\pi}} < \frac{x}{2^{2x}}\binom{2x}{x} < \sqrt{\frac{x}{\pi}}, \qquad (x = 1, 2, 3, \cdots).$$

Thus we have

$$\sigma(t) = \sum_{x=1}^{t-1}\frac{x}{2^{2x}}\binom{2x}{x} < \sum_{x=1}^{t-1}\sqrt{\frac{x}{\pi}} < \int_1^t\sqrt{\frac{x}{\pi}}\,dx = \frac{2}{3\sqrt{\pi}}(t^{3/2}-1);$$

$$\sigma(t) = \sum_{x=1}^{t-1}\frac{x}{2^{2x}}\binom{2x}{x} > \sum_{x=0}^{t-2}\sqrt{\frac{x}{\pi}} > \int_0^{t-2}\sqrt{\frac{x}{\pi}}\,dx = \frac{2}{3\sqrt{\pi}}(t-2)^{3/2}.$$

Again, by Wallis' formula we have

$$\sqrt{\pi t} < 4^t\binom{2t}{t}^{-1} < \sqrt{\frac{\pi t^2}{t-1}}.$$

Combining these inequalities we get

$$\tfrac{2}{3}\sqrt{t}\,(t-2)^{\frac{3}{2}} < 4^t \binom{2t}{t}^{-1} \sigma(t) < \frac{2}{3}\sqrt{\frac{t^2}{t-1}}\,(t^{\frac{3}{2}}-1),$$

where

$$\tfrac{2}{3}\sqrt{t}\,(t-2)^{\frac{3}{2}} \sim \tfrac{2}{3}t^2, \qquad \frac{2}{3}\sqrt{\frac{t^2}{t-1}}\,(t^{\frac{3}{2}}-1) \sim \tfrac{2}{3}t^2.$$

Therefore,

$$t + \tfrac{2}{3}\sqrt{t}\sqrt{(t-2)^3} < \rho_1 < t + \tfrac{2}{3}\sqrt{t+1}\,(\sqrt{t^3}-1), \qquad \rho_1 \sim \tfrac{2}{3}t^2.$$

Next, by Stirling's formula

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left\{1 + \frac{1}{12n} + \frac{1}{288n^2} + 0\left(\frac{1}{n^3}\right)\right\}$$

we obtain

$$\Delta^n 0^{n+t} = \left(\frac{n}{e}\right)^n \left(\frac{n^2}{2}\right)^t \frac{\sqrt{2\pi n}}{t!}\left\{1 + \frac{\theta_1}{n} + \frac{\theta_2}{n^2} + 0\left(\frac{1}{n^3}\right)\right\} \sim \frac{n^{n+2t}\sqrt{2\pi n}}{e^n \cdot 2^t \cdot t!}\left(1 + \frac{2t^2}{3n}\right),$$

where

$$\theta_1 = \rho_1 + \tfrac{1}{12} \sim \tfrac{2}{3}t^2,$$

$$\theta_2 = \rho_2 + \tfrac{1}{12}\rho_1 + \tfrac{1}{288}.$$

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