INVERSE TABLES OF PROBABILITIES OF ERRORS OF THE SECOND KIND

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1. Introduction. The problem of testing linear hypotheses was discussed by Kolodziejczyk [1], and later in greater detail by Tang [2], who computed a table giving the probabilities of errors of the second kind P_{II} for a range of values of the two degrees of freedom f_1 and f_2, ($f_1 = 1(1)8, f_2 = 2, 4, 6(1)30, 60, \infty$)[1] and for two fixed levels $P_I = .01$ and $.05$ of the probability of errors of the first kind. These tables are in terms of a parameter φ, ($\varphi = 1(.5)3(1)8$) whose statistical significance, or rather that of

$$\lambda = (f_1 + 1)\varphi^2/2$$

is discussed in Tang's paper. A restatement of the problem of testing linear hypotheses in a more canonical form, giving an interpretation of λ, will also be found in a recent paper by Wald [3].

Professor Neyman has felt for some time that a table giving $\varphi = \varphi(a, b, \alpha, \beta)$ as a function of the two degrees of freedom $f_1 = 2a$, and $f_2 = 2b$, and of the two probability levels $\alpha = P_I$ and $\beta = 1 - P_{II}$ would be more useful for statistical purposes, where β is the probability of detecting the falsehood of the hypothesis tested. A paper by Professor Neyman explaining this point of view and giving applications of the present tables to some statistical problems will appear shortly. These tables were computed in the Statistical Laboratory of the University of California,[2] and give values of φ for the following range of parameters:

$$(\alpha, \beta) = (.01, .7), (.01, .8), (.05, .7), (.05, .8)$$

$$f_1 = 1(1)10, 12, 15, 20, 24, 30, 40, 60, 80, 120, \infty.$$

$$f_2 = 2(2)20, 24, 30, 40, 60, 80, 120, 240, \infty.$$

2. Analytic definitions. The statistical parameter

$$\lambda = \lambda(a, b, \alpha, \beta) = (a + \tfrac{1}{2})\, \varphi^2(a, b, \alpha, \beta) \tag{1}$$

can be thought of as an inverse function connected with the hypergeometric distribution. Inverse functions $y(\alpha)$, $u(a, \alpha)$ and $x(a, b, \alpha)$ of the better known normal, Gamma and Beta distributions respectively have all been tabulated,

[1] The notation $m = r(s)t$ is equivalent to $m = r, r + s, r + 2s, \ldots, t$.

[2] These tables were begun by Miss Leone Gintzler, and were carried on by Mark Eudy under a University of California Research Grant. The bulk of the computing was done, however, by the author and by Mrs. Julia Robinson under a grant of the American Philosophical Society.

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and are sometimes called "percentage points" of the distribution. To begin with the simplest, the normal distribution, we may define $y(\alpha)$ as the solution of

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y(\alpha)} e^{-x^2/2}\, dx = \alpha. \tag{2}$$

This function has been recently tabulated by Truman Lee Kelley [4] for $\alpha = p = 0(.0001)\ 1$ to 8 decimal places.

The function $u(a, \alpha)$ is the solution of

$$\frac{1}{\Gamma(a)} \int_0^{u(a,\alpha)} t^{a-1} e^{-t}\, dt = \Gamma_u(a)/\Gamma(a) = \alpha. \tag{3}$$

This is connected with the well known percentage points of the χ_ν^2 distribution with $\nu = 2a$ degrees of freedom as follows:

$$\chi_\nu^2(\alpha) = 2u(a, 1 - \alpha). \tag{4}$$

Catherine Thomson [5] has tabulated $\chi_\nu^2(\alpha)$ for $\alpha = .005, .01, .025, .05, .1, .25, .5, .75, .9, .95, .975, .99, .995$, and for $\nu = 1(1)30(10)100$. She has also tabulated [6] the corresponding parameter $x = x(a, b, \alpha)$ of the Beta distribution with $\nu_1 = 2b$, $\nu_2 = 2a$ degrees of freedom defined by

$$\frac{1}{\mathrm{B}(a, b)} \int_0^{x(a,b,\alpha)} t^{a-1}(1 - t)^{b-1}\, dt = \frac{\mathrm{B}_x(a, b)}{\mathrm{B}(a, b)} = \alpha \tag{5}$$

for $\alpha = .005, .01, .025, .05, .1, .25$ and $.5$ and $\nu_1 = 1(1)10, 12, 15, 20, 24, 30, 40, 60, 120$, $\nu_2 = 1(1)30, 40, 60, 120, \infty$ to five significant places.

Similarly $\lambda(a, b, \alpha, \beta)$ can be defined as the solution of

$$\frac{1}{\mathrm{B}(a, b)} \int_0^{x(a,b,1-\alpha)} e^{-\lambda(1-t)} t^{a-1}(1 - t)^{b-1} F(-b, a, -\lambda t)\, dt = 1 - \beta \tag{6}$$

where

$$F(\gamma, \delta, z) = 1 + \frac{\gamma}{\delta} z + \frac{\gamma(\gamma + 1)}{\delta(\delta + 1)} z^2 + \cdots$$

is the confluent hypergeometric function.

3. Limiting cases. It is well known that as a tends to infinity

$$\frac{\chi_\nu^2(\alpha)}{\nu} = \frac{u(a, 1 - \alpha)}{a} = 1 + \frac{y(\alpha)}{\sqrt{a}} + 0\left(\frac{1}{a}\right). \tag{7}$$

There are many approximations [7] to χ^2. In a recent paper Peiser [8] gave a rigorous derivation of an asymptotic formula for χ^2.

Similarly, the limiting cases of $x(a, b, \alpha)$ as a and b tend to infinity are known

TABLE OF φ

Level of Significance, $\alpha = .05$. Probability of Detecting the Falsehood of the Hypothesis Tested, $\beta = .7$

f_2 \ f_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞	$240/f_2$
2	3.438	3.920	4.141	4.267	4.350	4.408	4.451	4.484	4.510	4.532	4.564	4.598	4.632	4.650	4.667	4.685	4.704	4.722	4.740	
4	2.341	2.418	2.436	2.441	2.442	2.442	2.441	2.440	2.439	2.438	2.437	2.434	2.432	2.43	2.429	2.43	2.426	2.424	2.421	
6	2.104	2.091	2.059	2.031	2.010	1.992	1.978	1.966	1.956	1.948	1.935	1.921	1.905	1.90	1.888	1.88	1.870	1.860	1.850	
8	2.003	1.950	1.895	1.852	1.818	1.792	1.771	1.754	1.739	1.727	1.707	1.685	1.662	1.65	1.636	1.62	1.608	1.593	1.576	
10	1.948	1.872	1.803	1.751	1.710	1.679	1.653	1.632	1.614	1.599	1.575	1.548	1.519	1.50	1.487	1.47	1.451	1.431	1.411	
12	1.913	1.823	1.745	1.687	1.641	1.606	1.577	1.553	1.533	1.516	1.488	1.457	1.424	1.41	1.386	1.37	1.344	1.321	1.297	
14	1.888	1.789	1.705	1.642	1.593	1.554	1.523	1.497	1.475	1.457	1.426	1.393	1.356	1.34	1.314	1.29	1.267	1.240	1.212	
16	1.871	1.764	1.675	1.609	1.557	1.517	1.483	1.456	1.432	1.413	1.380	1.344	1.304	1.28	1.259	1.23	1.206	1.178	1.146	
18	1.857	1.744	1.653	1.584	1.530	1.487	1.453	1.424	1.400	1.378	1.344	1.306	1.264	1.24	1.215	1.19	1.158	1.127	1.093	
20	1.846	1.729	1.635	1.563	1.508	1.464	1.428	1.398	1.373	1.351	1.316	1.276	1.231	1.21	1.180	1.15	1.120	1.086	1.048	12
24	1.831	1.707	1.608	1.534	1.476	1.430	1.392	1.360	1.333	1.310	1.273	1.230	1.182	1.16	1.13	1.09	1.06	1.02	.978	10
30	1.815	1.685	1.582	1.505	1.444	1.396	1.356	1.322	1.294	1.269	1.229	1.183	1.131	1.10	1.070	1.03	.996	.951	.902	8
40	1.80	1.66	1.56	1.48	1.41	1.36	1.32	1.28	1.25	1.23	1.19	1.14	1.08	1.05	1.01	.97	.93	.88	.816	6
60	1.786	1.643	1.532	1.447	1.381	1.328	1.284	1.247	1.215	1.187	1.141	1.088	1.026	.99	.951	.90	.854	.792	.714	4
80	1.78	1.63	1.52	1.43	1.37	1.31	1.27	1.23	1.20	1.17	1.12	1.06	1.00	.96	.92	.87	.81	.75	.653	3
120	1.77	1.62	1.51	1.42	1.35	1.29	1.25	1.21	1.18	1.15	1.10	1.04	.97	.93	.87	.83	.77	.692	.577	2
240	1.76	1.61	1.50	1.41	1.33	1.28	1.23	1.19	1.16	1.13	1.07	1.01	.94	.90	.85	.79	.72	.63	.473	1
∞	1.757	1.602	1.483	1.392	1.320	1.262	1.213	1.170	1.135	1.104	1.051	.988	.913	.87	.815	.75	.673	.557	0	0

TABLE OF φ

Level of Significance, $\alpha = .05$. Probability of Detecting the Falsehood of the Hypothesis Tested, $\beta = .8$

f_2 \ f_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞	$240/f_2$
2	3.898	4.558	4.814	4.961	5.057	5.125	5.175	5.213	5.244	5.269	5.307	5.346	5.386	5.406	5.427	5.447	5.469	5.490	5.511	
4	2.659	2.736	2.752	2.755	2.755	2.754	2.752	2.750	2.749	2.747	2.745	2.742	2.739	2.74	2.735	2.73	2.730	2.728	2.725	
6	2.381	2.351	2.309	2.274	2.248	2.226	2.209	2.195	2.183	2.174	2.158	2.141	2.122	2.11	2.103	2.09	2.081	2.069	2.058	
8	2.263	2.188	2.119	2.066	2.026	1.995	1.970	1.949	1.932	1.918	1.894	1.869	1.841	1.83	1.811	1.80	1.779	1.761	1.742	
10	2.199	2.098	2.014	1.951	1.903	1.865	1.835	1.810	1.789	1.771	1.743	1.712	1.678	1.66	1.640	1.62	1.599	1.577	1.553	
12	2.159	2.041	1.947	1.877	1.824	1.782	1.748	1.720	1.696	1.676	1.644	1.609	1.570	1.55	1.526	1.50	1.478	1.452	1.423	
14	2.131	2.002	1.901	1.826	1.769	1.723	1.687	1.656	1.631	1.609	1.574	1.535	1.493	1.47	1.444	1.42	1.390	1.360	1.328	
16	2.110	1.974	1.868	1.789	1.730	1.681	1.642	1.610	1.583	1.560	1.522	1.481	1.434	1.41	1.382	1.35	1.322	1.289	1.253	
18	2.095	1.952	1.842	1.760	1.698	1.647	1.607	1.574	1.545	1.521	1.482	1.438	1.389	1.36	1.333	1.30	1.268	1.233	1.193	
20	2.083	1.935	1.822	1.737	1.673	1.621	1.580	1.545	1.516	1.490	1.449	1.404	1.352	1.32	1.294	1.26	1.225	1.186	1.143	12
24	2.065	1.910	1.792	1.704	1.636	1.583	1.539	1.501	1.471	1.444	1.401	1.352	1.297	1.27	1.23	1.20	1.16	1.11	1.065	10
30	2.047	1.886	1.763	1.671	1.601	1.544	1.498	1.460	1.427	1.399	1.352	1.300	1.240	1.21	1.171	1.13	1.087	1.037	.979	8
40	2.03	1.86	1.73	1.64	1.57	1.51	1.46	1.42	1.38	1.35	1.30	1.25	1.18	1.15	1.11	1.06	1.01	.95	.885	6
60	2.014	1.838	1.706	1.607	1.531	1.469	1.418	1.375	1.339	1.307	1.254	1.194	1.124	1.08	1.039	.99	.930	.857	.773	4
80	2.01	1.83	1.69	1.59	1.51	1.45	1.40	1.35	1.32	1.28	1.23	1.17	1.09	1.05	1.00	.95	.88	.81	.705	3
120	2.00	1.81	1.68	1.58	1.50	1.43	1.38	1.33	1.30	1.26	1.21	1.14	1.06	1.02	.97	.91	.84	.750	.623	2
240	1.99	1.80	1.67	1.56	1.48	1.41	1.36	1.31	1.27	1.24	1.18	1.11	1.03	.98	.93	.87	.79	.68	.509	1
∞	1.981	1.792	1.654	1.545	1.462	1.395	1.339	1.292	1.251	1.215	1.155	1.084	.997	.95	.889	.82	.731	.603	0	0

TABLE OF φ

Level of Significance, $\alpha = .01$. Probability of Detecting the Falsehood of the Hypothesis Tested, $\beta = .8$

f_2 \ f_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞	240/f_2
2	8.965	10.326	10.943	11.297	11.527	11.689	11.809	11.901	11.974	12.034	12.125	12.219	12.314	12.363	12.412	12.462	12.512	12.563	12.613	
4	4.157	4.311	4.353	4.372	4.380	4.383	4.385	4.386	4.386	4.386	4.386	4.386	4.384	4.383	4.383	4.381	4.380	4.379	4.378	
6	3.371	3.335	3.282	3.239	3.205	3.179	3.157	3.140	3.125	3.113	3.093	3.072	3.050	3.037	3.023	3.012	2.998	2.984	2.969	
8	3.068	2.959	2.866	2.798	2.746	2.706	2.674	2.647	2.625	2.607	2.577	2.545	2.511	2.492	2.473	2.453	2.432	2.410	2.388	
10	2.910	2.763	2.649	2.566	2.503	2.455	2.416	2.384	2.358	2.335	2.299	2.260	2.218	2.195	2.171	2.146	2.120	2.092	2.063	
12	2.813	2.642	2.515	2.423	2.353	2.300	2.256	2.220	2.191	2.166	2.125	2.081	2.032	2.006	1.978	1.950	1.919	1.886	1.851	
14	2.748	2.561	2.425	2.327	2.252	2.194	2.148	2.109	2.077	2.049	2.005	1.957	1.904	1.875	1.844	1.812	1.778	1.741	1.701	
16	2.701	2.503	2.359	2.257	2.179	2.118	2.069	2.028	1.994	1.965	1.918	1.866	1.809	1.778	1.745	1.710	1.672	1.632	1.588	
18	2.666	2.460	2.312	2.205	2.124	2.060	2.009	1.966	1.931	1.901	1.851	1.796	1.736	1.703	1.668	1.630	1.590	1.546	1.498	
20	2.638	2.426	2.274	2.164	2.080	2.014	1.962	1.918	1.881	1.850	1.798	1.741	1.679	1.644	1.607	1.567	1.524	1.477	1.425	12
24	2.60	2.38	2.22	2.10	2.02	1.95	1.89	1.85	1.81	1.77	1.72	1.66	1.59	1.56	1.52	1.47	1.42	1.37	1.312	10
30	2.559	2.328	2.164	2.046	1.955	1.884	1.826	1.778	1.737	1.702	1.644	1.580	1.507	1.47	1.421	1.37	1.321	1.261	1.193	8
40	2.52	2.28	2.11	1.99	1.89	1.82	1.76	1.71	1.67	1.63	1.57	1.50	1.42	1.38	1.33	1.27	1.21	1.14	1.064	6
60	2.485	2.236	2.062	1.935	1.837	1.760	1.696	1.643	1.598	1.559	1.494	1.420	1.335	1.29	1.232	1.17	1.102	1.019	.917	4
80	2.47	2.21	2.04	1.91	1.81	1.73	1.66	1.61	1.56	1.52	1.46	1.38	1.29	1.24	1.18	1.12	1.04	.95	.830	3
120	2.45	2.19	2.01	1.88	1.78	1.70	1.63	1.58	1.53	1.49	1.42	1.34	1.25	1.20	1.13	1.06	.98	.875	.727	2
240	2.43	2.17	1.99	1.86	1.75	1.67	1.60	1.55	1.50	1.46	1.38	1.30	1.20	1.15	1.08	1.00	.91	.79	.587	1
∞	2.417	2.150	1.966	1.830	1.726	1.642	1.572	1.514	1.464	1.422	1.346	1.256	1.158	1.10	1.026	.94	.839	.687	0	0

TABLE OF φ

Level of Significance, $\alpha = .01$. Probability of Detecting the Falsehood of the Hypothesis Tested, $\beta = .7$

f_2 \ f_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞	$240/f_2$
2	7.746	8.922	9.455	9.761	9.960	10.094	10.204	10.282	10.346	10.398	10.476	10.557	10.640	10.681	10.724	10.767	10.810	10.855	10.900	
4	3.714	3.858	3.900	3.917	3.923	3.929	3.931	3.932	3.933	3.933	3.934	3.933	3.932	3.932	3.931	3.930	3.929	3.928	3.928	
6	3.037	3.015	2.972	2.935	2.907	2.884	2.865	2.850	2.838	2.827	2.810	2.790	2.771	2.761	2.750	2.738	2.726	2.713	2.700	
8	2.773	2.686	2.608	2.548	2.503	2.468	2.440	2.416	2.398	2.381	2.356	2.327	2.297	2.281	2.264	2.246	2.227	2.208	2.188	
10	2.634	2.512	2.415	2.343	2.289	2.246	2.212	2.184	2.160	2.141	2.109	2.074	2.037	2.016	1.995	1.973	1.950	1.925	1.899	
12	2.548	2.406	2.296	2.216	2.155	2.108	2.070	2.038	2.012	1.990	1.954	1.914	1.871	1.848	1.824	1.798	1.770	1.740	1.710	
14	2.490	2.334	2.216	2.130	2.064	2.013	1.972	1.938	1.910	1.886	1.847	1.804	1.756	1.730	1.703	1.674	1.645	1.611	1.575	
16	2.448	2.282	2.158	2.068	1.999	1.945	1.901	1.865	1.834	1.807	1.768	1.722	1.671	1.643	1.614	1.582	1.548	1.512	1.473	
18	2.417	2.243	2.114	2.020	1.949	1.893	1.847	1.810	1.778	1.752	1.708	1.659	1.605	1.577	1.545	1.511	1.474	1.435	1.392	
20	2.392	2.212	2.080	1.984	1.910	1.852	1.805	1.767	1.734	1.706	1.660	1.610	1.553	1.525	1.489	1.453	1.414	1.372	1.326	12
24	2.356	2.167	2.030	1.930	1.853	1.79	1.74	1.70	1.67	1.64	1.59	1.54	1.48	1.44	1.40	1.36	1.32	1.28	1.223	10
30	2.321	2.124	1.982	1.877	1.797	1.734	1.682	1.639	1.603	1.571	1.520	1.462	1.397	1.36	1.321	1.28	1.230	1.176	1.115	8
40	2.29	2.08	1.94	1.83	1.74	1.68	1.62	1.58	1.54	1.50	1.45	1.39	1.32	1.28	1.24	1.19	1.13	1.07	.996	6
60	2.255	2.041	1.889	1.776	1.690	1.621	1.564	1.517	1.476	1.441	1.383	1.316	1.239	1.20	1.147	1.09	1.029	.954	.860	4
80	2.24	2.02	1.87	1.75	1.66	1.59	1.54	1.49	1.44	1.41	1.35	1.28	1.20	1.16	1.10	1.04	.97	.89	.780	3
120	2.22	2.00	1.84	1.73	1.64	1.57	1.51	1.46	1.41	1.38	1.32	1.24	1.16	1.11	1.05	.99	.91	.820	.684	2
240	2.21	1.98	1.82	1.70	1.61	1.54	1.48	1.43	1.38	1.35	1.28	1.21	1.12	1.07	1.01	.94	.85	.74	.560	1
∞	2.193	1.963	1.801	1.678	1.587	1.513	1.451	1.400	1.353	1.313	1.247	1.169	1.076	1.02	.956	.882	.785	.647	0	0

[9] and follow readily from (5), although no attempt has been made, as far as I know, to find better approximations. These limiting values are as follows:

$$\lim_{a\to\infty} a[1 - x(a, b, \alpha)] = u(b, 1 - \alpha) \tag{8}$$

$$\lim_{b\to\infty} bx(a, b, \alpha) = u(a, \alpha). \tag{9}$$

The two corresponding limiting cases for λ are not at first glance so symmetric. When a tends to infinity, we have from (1)

$$\lim_{a\to\infty} \frac{\lambda}{a} = \varphi^2 \tag{10}$$

while

$$\lim_{a\to\infty} F(-b, a, -a\varphi^2 t) = (1 + \varphi^2 t)^b. \tag{11}$$

Substituting these in (6) and letting $t = 1 - z/a(1 + \varphi^2)$ and passing to the limit we get with the help of (8)

$$\frac{1}{\Gamma(b)} \int_{(1+\varphi^2)u(b,\alpha)}^{\infty} e^{-z} z^{b-1}\, dz = 1 - \beta. \tag{12}$$

In other words

$$(1 + \varphi^2)u(b, \alpha) = u(b, \beta)$$

or

$$\varphi(\infty, b, \alpha, \beta) = \sqrt{\frac{u(b, \beta)}{u(b, \alpha)} - 1}\,. \tag{13}$$

This is the only case, except for $b = 1$ in which φ can be given explicitly. For $b = 1$, we have from (5)

$$x(a, 1, \alpha) = \sqrt[a]{\alpha}$$

and (6) can be easily integrated to give

$$\varphi(a, 1, \alpha, \beta) = \left[\log\left(\frac{1-\alpha}{1-\beta}\right) \Big/ \left(a + \frac{1}{2}\right)(1 - \sqrt[a]{1-\alpha})\right]^{\frac{1}{2}}.$$

In all other cases it was found impractical to attempt an inversion of (6) to get φ.

When b becomes infinite (6) becomes with $t = z/b$, and with the help of (9),

$$e^{-\lambda} \int_0^{u(a,1-\alpha)} e^{-z} \left(\frac{z}{\lambda}\right)^{(a-1)/2} I_{a-1}(2\sqrt{\lambda z})\, dz = 1 - \beta \tag{14}$$

where

$$I_{a-1}(2\sqrt{\lambda z}) = \frac{(\lambda z)^{-(a-1)/2}}{\Gamma(a)} \lim_{b\to\infty} F\left(-b, a, -\frac{\lambda z}{b}\right) \tag{15}$$

is the Bessel function of a purely imaginary argument, which is usually defined by

$$I_n(x) = \sum_{\nu=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2\nu}}{\nu!\Gamma(n+\nu+1)}.$$

Expression (14) was also obtained for this limiting case of the hypergeometric distribution by Wishart [10]. This integral, however, does not give λ any more explicitly than the general integral (6), and since the calculation of λ increases in difficulty as a increases, an attempt was made to derive an approximate formula for $\varphi(a, \infty, \alpha, \beta)$ for large a. To this end an asymptotic formula [11] was developed for $I_n(nx)$, the principal term of which is

$$I_n(nx) = \frac{(1+x^2)^{1/4}}{\sqrt{2\pi n}} \left(\frac{\sqrt{1+x^2}-1}{x} e^{\sqrt{1+x^2}}\right)^n. \tag{16}$$

Substituting this into (14) with $x = 2\sqrt{\lambda z}/a - 1$ and $n = a - 1$ we get for large a

$$\frac{e^{-\lambda}}{\Gamma(a)} \int_0^{u(a,1-\alpha)} t^{a-1} e^{-t(1-\varphi^2+t\varphi^4/2a)}\, dt = 1 - \beta \tag{17}$$

If we assume that φ is sufficiently small to neglect the term in φ^4 we get as a first approximation

$$\frac{1}{\Gamma(a)} \int_0^{(1-\varphi^2)u(a,1-\alpha)} t^{a-1} e^{-t}\, dt = 1 - \beta \tag{18}$$

or

$$\varphi(a, \infty, \alpha, \beta) \sim \sqrt{1 - \frac{u(a, 1-\beta)}{u(a, 1-\alpha)}} \tag{19}$$

a formula very similar to (11). In fact since a is large this formula can be reduced one more step using (7). This gives

$$\lim_{a\to\infty} \sqrt{a}\varphi^2(a, \infty, \alpha, \beta) = y(\alpha) - y(\beta). \tag{20}$$

Similarly (13) becomes

$$\lim_{b\to\infty} \sqrt{a}\varphi^2(\infty, b, \alpha, \beta) = y(\alpha) - y(\beta).$$

If instead of neglecting the term in φ^4, we multiply it by the value of t at its upper limit, we get

$$\varphi^2 \sim \frac{1 - \sqrt{1 - 2[u(a, 1-\alpha) - u(a, 1-\beta)]/a}}{u(a, 1-\alpha)/a}. \tag{21}$$

Professor Neyman derived another approximation for $\varphi(a, \infty, \alpha, \beta)$ by assuming that the distribution (14) approaches a normal distribution for large a. He obtained:

$$\sqrt{a}\varphi^2 \sim y(\alpha) + \frac{y^2(\beta)}{\sqrt{a}} - y(\beta)\left(1 + \frac{2y(\alpha)}{\sqrt{a}} + \frac{y^2(\beta)}{a}\right)^{1/2}. \tag{22}$$

Both (21) and (22) obviously reduce to (20) in the limit. The following table shows the efficiency of these formulas for $a = 60$

	Table	(21)	(22)	(20)
$\alpha = .01$, $\beta = .8$	.687	.695	.668	.640
$\alpha = .01$, $\beta = .7$	.647	.642	.622	.607
$\alpha = .05$, $\beta = .8$	.603	.585	.593	.566
$\alpha = .05$, $\beta = .7$	.557	.540	.544	.529

A rigorous derivation of some such formula giving the actual order of approximation of φ would of course be of interest, but is likely to be quite complicated.

4. Calculation of tables. It is fairly obvious that the integral (6), although very useful theoretically is not well adapted to actual calculations. It can easily be integrated by parts to produce the infinite series.

$$e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \frac{B_{1-x}(a+i, b)}{B(a+i, b)} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \frac{B_x(b, a+i)}{B(a+i, b)} = 1 - \beta. \tag{23}$$

This series can be used effectively for calculation purposes only if λ is comparatively small. If b is an integer, however, this series can be replaced by a finite series of b terms, which was also used by Tang [2] in calculating his tables. This series is as follows with $x = x(a, b, 1 - \alpha) = x(b, a, \alpha)$:

$$e^{-\lambda x}(1 - x)^{a+b-1} \sum_{i=0}^{b-1} T_i = 1 - \beta \tag{24}$$

where

$$T_0 = 1, \quad T_1 = x[\lambda(1 - x) + a + b - 1]/(1 - x)$$

and[3]

$$T_n = x\{[\lambda(1 - x) + a + b - n]\, T_{n-1} + \lambda x T_{n-2}\}/n(1 - x). \tag{25}$$

The subjoined tables can be thought of as inverses of Tang's tables, and could have been obtained from tables such as Tang's by inverse interpolation, had the interval of tabulation been sufficiently fine. The interval of tabulation of .5 for φ allowed only a crude approximation or trial value of φ, the corresponding probability was then calculated for this point, and then corrected with the help

[3] It will be noticed in comparing these formulas with those given by Tang, that x is used for $1 - x$. This is done to conform with Miss Thomson's table for x.

of derivatives. In the beginning of the work, a recalculation was usually made for the corrected value of φ, and the tabulated value of φ was then obtained by inverse interpolation between these very close values. As the work progressed, and difference tables were calculated in several directions, the guesses improved considerably so that a correction could be made using the first derivative to give a tabulated value of φ correct to three decimal places. Such corrections never exceeded .004, and therefore it is hoped that the tables are correct to the last place. The derivative in question is given by

$$\frac{d\beta}{d\varphi} = (2a + 1)\varphi x(1 - x)^{a+b-1}e^{-\lambda x}T_{b-1}, \tag{26}$$

and was obtained as a by-product of the calculation of (24). This method was used for all finite a's and for all b's less than 30. For $b = 30$ or more it was found more expeditious to use the infinite series (23), about 20 terms of which sufficed.

The values of $x(a, b, 1 - \alpha) = x(b, a, \alpha)$ used in these calculations were obtained to five significant places from Miss Thomson's table [6] for $\alpha = .01$ and .05 with $\nu_1 = 2a$, $\nu_2 = 2b$.

No calculations were made for non-integer b's since for small values of b, λ was too large to make the infinite series (23) practicable, while for $b \geqq 7$, φ can be easily obtained by interpolation. The only available method for calculating φ for $b = 1/2, 3/2$ or $5/2$ would be by numerical integration of (6), which would be rather lengthy. Furthermore, the interest seems to be in large rather than small values of a and b.

5. Calculations for infinite cases. The case $a = \infty$ was readily disposed of using (13), as for $b = \infty$, the integral (14) was again integrated by parts to give

$$e^{-(\lambda+u)} \sum_{\nu=0}^{\infty} \left(\frac{u}{\lambda}\right)^{(a+\nu)/2} I_{a+\nu}(2\sqrt{u\lambda}) = 1 - \beta. \tag{27}$$

This was found to be effective, especially when $u < \lambda$, which is the case for small values of a. When u exceeded λ, the complementary series was used, namely

$$e^{-(\lambda+u)} \sum_{\nu=0}^{\infty} \left(\frac{u}{\lambda}\right)^{(a-1-\nu)/2} I_{a-1-\nu}(2\sqrt{u\lambda}) = \beta. \tag{28}$$

The calculations proceeded in much the same manner as in the finite case. The values of $2u$ were obtained from Miss Thomson's table of percentage points of $\chi^2_\nu(\alpha)$ distribution with $\nu = 2a$ degrees of freedom, while the values of I_0 and I_1 were obtained from the tables of Bessel Functions [12] for $2\sqrt{\lambda z} \leq 20$, and from the tables of Anding [13] for larger values of the variable. The values of I_ν for $\nu > 1$, were computed from the recurrence relation[4]

[4] If a is an odd integer, the values of $I_\nu(z)$ can be built up using (29) from $I_{-\frac{1}{2}}(z) = \frac{\cosh z}{\sqrt{\pi z}}$ and $I_{\frac{1}{2}}(z) = \frac{\sinh z}{\sqrt{\pi z}}$, which of course are tabled.

$$I_{\nu+1} = -\frac{2\nu}{z} I_\nu(z) + I_{\nu-1}(z). \tag{29}$$

Future calculations of this sort could make use of the forthcoming tables [14] of $I_\nu(z)$ for $\nu \leq 20$, and $z \leq 20$ to ten significant places. As before, the tabulated values were obtained by correcting trial values of φ by means of the derivative

$$\frac{d\beta}{d\varphi} = \sqrt{2u(2a+1)}e^{-(\lambda+u)} \left(\frac{u}{\lambda}\right)^{(a-1)/2} I_a(2\sqrt{u\lambda}), \tag{30}$$

which again was obtained as a by-product of the calculations. This method becomes impractical when a is too large, because a great deal of accuracy is lost in applying recurrence (29) many times. For some of the larger values of a it was found preferable to use the series

$$1 - \beta = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \frac{\Gamma_u(a+i)}{\Gamma(a+i)} \tag{31}$$

although it converges rather slowly. In other words the upper limits for a and b were pushed as far as was practicable.

The values which are tabled to two instead of three decimals were interpolate using second differences, all other values were computed in the manner described above. Difference tables were made by rows, columns and between tables, as a final check on the work. Difference tables using harmonic interpolation were also made for both rows and columns, and found very effective, with the exception of the lower right hand corner, where φ drops rapidly to zero. The last column of each table is to be used for harmonic interpolation.

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